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LK, LJ, Dual Intuitionistic Logic, and Quantum Logic

Hiroshi Aoyama

Abstract In this paper, we study the relationship among classical logic, intuitionistic logic, and quantum logic (orthologic and orthomodular logic). These logics are related in an interesting way and are not far apart from each other, as is widely believed. The results in this paper show how they are related with each other through a dual intuitionistic logic (a kind of paraconsistent logic). Our study is completely syntactical.

1 Introduction

The aim of this paper is to show the relationship among the classical logic, the intuitionistic logic, and the quantum logic. Takeuti says in his paper [15], "quantum logic is drastically different from the classical logic or the intuitionistic logic." It is, however, unclear how drastically different these logics are from each other. To answer this question syntactically, we use the formalization of sequent calculus. For the axioms and inference rules of the classical logic LK and those of the intuitionistic logic LJ, we refer the reader to Takeuti [16].

Sequent calculi are attractive in the sense that we can obtain various pieces of logical information from their structures. If we regard Gentzen's LK as the most basic sequent calculus, we can obtain a number of logically interesting sequent calculi from it, for example, by dropping some of the inference rules, by changing the forms of some or all of its inference rules, or by introducing new logical connectives together with inference rules associated with them. These changes usually result in a great diversity of nonclassical logics; see Paoli [10], Restall [12], and Schroeder-Heister and Došen [14].

The sequent calculi to be studied in the present paper are also obtained from LK by making some changes or others. The so-called dual intuitionistic logic, a kind of paraconsistent logic, plays a crucial role in this paper. A few logicians have considered it; see Czermak [4], Goodman [6], and Urbas [17]. Their systems are all

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in the form of sequent calculus. Our dual intuitionistic logic DI is different from the three systems they considered. It is completely dual of LJ. LJ is obtained from LK by restricting sequents so that they contain at most one formula in their succedents. DI is obtained from LK by restricting sequents so that they contain at most one formula in their antecedents. Such restrictions on sequents naturally yield an interesting translation mapping between extensions of LJ and DI. This mapping also plays an important role in this paper. In Section 2, we will define the system DI more specifically and show some important properties about it.

We first fix the language for the sequent calculi to be studied below.

Definition 1.1 The language for the sequent calculi to be studied in this paper consists of the following symbols:

1. Predicate constants with n argument-places ($n \geqq 0$) : $p_0^n, p_1^n, p_2^n, \ldots$
2. Individual constants: $c_0, c_1, c_2, \ldots$
3. Free variables: $a_0, a_1, a_2, \ldots$
4. Bound variables: $x_0, x_1, x_2, \ldots$
5. Logical symbols: $\wedge, \vee, \rightarrow, \forall, \exists$
6. Auxiliary symbols: (,) , ,(comma)

Terms consist of individual constants and free variables. Well-formed formulas (wffs) are defined as usual. In a sequent $\Gamma \Rightarrow \Delta$, the antecedent Γ and the succedent Δ are both finite sequences of zero or more formulas unless otherwise stated. Proofs (formal proofs) are also defined in the usual way. If a sequent $\Gamma \Rightarrow \Delta$ is provable in a sequent calculus S, we write '$\mathrm{S} \vdash \Gamma \Rightarrow \Delta$'. We use Γ, Δ, Λ, and Π to express sequences of wffs and α, β, γ, δ, λ, and π to express wffs.

2 The System DI

Definition 2.1 DI has the following axioms and rules of inference:

1. Axioms $\alpha \Rightarrow \alpha$

2. Inference Rules (Γ consists of at most one wff.)

Structural Rules

WL $$\frac{\Rightarrow \Delta}{\alpha \Rightarrow \Delta}$$ WR $$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}$$

CR $$\frac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}$$ ER $$\frac{\Gamma \Rightarrow \Delta, \alpha, \beta, \Lambda}{\Gamma \Rightarrow \Delta, \beta, \alpha, \Lambda}$$

Cut $$\frac{\Gamma \Rightarrow \Delta, \alpha \qquad \alpha \Rightarrow \Lambda}{\Gamma \Rightarrow \Delta, \Lambda}$$

Logical Rules (t is a term and a is a free variable.)

$\neg$L $\dfrac{\Rightarrow \Delta, \alpha}{\neg\alpha \Rightarrow \Delta}$ $\neg$R $\dfrac{\alpha \Rightarrow \Delta}{\Rightarrow \Delta, \neg\alpha}$

$\wedge$L $\dfrac{\alpha \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta}$ $\dfrac{\beta \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta}$ $\wedge$R $\dfrac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$

$\vee$L $\dfrac{\alpha \Rightarrow \Delta \quad \beta \Rightarrow \Delta}{\alpha \vee \beta \Rightarrow \Delta}$ $\vee$R $\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$ $\dfrac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$

$\rightarrow$L $\dfrac{\Rightarrow \Delta, \alpha \quad \beta \Rightarrow \Lambda}{\alpha \rightarrow \beta \Rightarrow \Delta, \Lambda}$ $\rightarrow$R $\dfrac{\alpha \Rightarrow \Delta, \beta}{\Rightarrow \Delta, \alpha \rightarrow \beta}$

$\forall$L $\dfrac{\alpha(t) \Rightarrow \Delta}{\forall x \alpha(x) \Rightarrow \Delta}$ (t is an arbitrary term.) $\forall$R $\dfrac{\Gamma \Rightarrow \Delta, \alpha(a)}{\Gamma \Rightarrow \Delta, \forall x \alpha(x)}$ (a does not appear in the lower sequent.)

$\exists$L $\dfrac{\alpha(a) \Rightarrow \Delta}{\exists x \alpha(x) \Rightarrow \Delta}$ (a does not appear in the lower sequent.) $\exists$R $\dfrac{\Gamma \Rightarrow \Delta, \alpha(t)}{\Gamma \Rightarrow \Delta, \exists x \alpha(x)}$ (t is an arbitrary term.)

DI is very close to Urbas's system LDJ in Urbas [17]. In fact, these two systems are the same except that the latter has the following two rules of inference in place of the rule $\rightarrow$R above: (Γ consists of at most one wff.)

$\rightarrow$R(U) : 1 $\dfrac{\alpha \Rightarrow \Delta}{\Rightarrow \Delta, \alpha \rightarrow \beta}$ $\rightarrow$R(U) : 2 $\dfrac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta}$

We list some of the important sequents which are provable in DI.

Proposition 2.2 *The following sentential sequents are provable in* DI*:*

1. $\Rightarrow \alpha \vee \neg\alpha$
2. $\neg\neg\alpha \Rightarrow \alpha$
3. $\neg\alpha \Rightarrow \neg\neg\neg\alpha$
4. $\alpha \rightarrow \beta \Rightarrow \neg\alpha \vee \beta$
5. $\alpha \rightarrow \neg\beta \Rightarrow \beta \rightarrow \neg\alpha$
6. $\neg\alpha \rightarrow \neg\beta \Rightarrow \beta \rightarrow \alpha$
7. $\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$
8. $\neg\alpha \vee \neg\beta \Rightarrow \neg(\alpha \wedge \beta)$
9. $\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
10. $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \Rightarrow \alpha \wedge (\beta \vee \gamma)$
11. $\alpha \vee (\beta \wedge \gamma) \Rightarrow (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
12. $(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \Rightarrow \alpha \vee (\beta \wedge \gamma)$.

Thus the law of excluded middle and the distributive laws hold in DI. Similarly we have the following two propositions.

Proposition 2.3 *The following sequents are provable in* DI*: (‘$\Gamma \Leftrightarrow \Delta$’ indicates that $\Gamma \Rightarrow \Delta$ and $\Delta \Rightarrow \Gamma$. In (6), (8), (10), and (11), the variable x does not appear in β.)*

1. $\forall x \alpha(x) \Rightarrow \exists x \alpha(x)$
2. $\neg \forall x \alpha(x) \Leftrightarrow \exists x \neg \alpha(x)$
3. $\neg \forall x \neg \alpha(x) \Rightarrow \exists x \alpha(x)$
4. $\neg \exists x \alpha(x) \Rightarrow \forall x \neg \alpha(x)$
5. $\exists x \neg\neg \alpha(x) \Rightarrow \neg\neg \exists x \alpha(x)$
6. $\forall x (\alpha(x) \wedge \beta) \Leftrightarrow \forall x \alpha(x) \wedge \beta$
7. $\forall x (\alpha(x) \wedge \beta(x)) \Leftrightarrow \forall x \alpha(x) \wedge \forall x \beta(x)$
8. $\exists x (\alpha(x) \vee \beta) \Leftrightarrow \exists x \alpha(x) \vee \beta$
9. $\exists x (\alpha(x) \vee \beta(x)) \Leftrightarrow \exists x \alpha(x) \vee \exists x \beta(x)$
10. $\forall x (\alpha(x) \vee \beta) \Leftrightarrow \forall x \alpha(x) \vee \beta$
11. $\exists x (\alpha(x) \wedge \beta) \Rightarrow \exists x \alpha(x) \wedge \beta$.

Proposition 2.4 *The following derived rules of inference hold in* DI*:*

1. $$\frac{\alpha \Rightarrow \beta}{\neg \beta \Rightarrow \neg \alpha}$$
2. $$\frac{\Rightarrow \alpha \qquad \Rightarrow \alpha \to \beta}{\Rightarrow \beta}$$
3. $$\frac{\alpha \Rightarrow \Delta}{\neg\neg \alpha \Rightarrow \Delta}$$
4. $$\frac{\Rightarrow \Delta, \alpha}{\Rightarrow \Delta, \neg\neg \alpha} \quad .$$

As a sequent calculus, DI satisfies the cut elimination theorem.

Theorem 2.5 *The rule Cut is eliminable from* DI.

Proof The proof is routine. □

Corollary 2.6 DI *has the subformula property and is consistent.*

The following sequents are not in general provable in DI:

1. $\alpha \Rightarrow \neg\neg \alpha$
2. $\alpha \wedge \neg \alpha \Rightarrow$
3. $\alpha \Rightarrow (\beta \to \alpha)$
4. $\alpha \wedge (\alpha \to \beta) \Rightarrow \beta$.

In LJ, the sequents '$\alpha \Rightarrow \neg\neg\alpha$' and '$\alpha \wedge \neg\alpha \Rightarrow$ ' are provable but the sequents '$\neg\neg\alpha \Rightarrow \alpha$' and ' $\Rightarrow \alpha \vee \neg\alpha$' are not. On the other hand, the latter two are provable in DI but the former are not. This kind of duality suggests a translation between the two systems. Czermak studied such a translation between LJ and his dual intuitionistic system DJ in [4]. Urbas refined it in [17] and presented a translation between LJ and his dual intuitionistic system LDJ. His method was to extend LDJ to $\mathrm{LDJ}^{\dot{-}}$ and LJ to $\mathrm{LJ}^{\dot{-}}$. These new systems contain additional rules of inference for a new logical symbol $\dot{-}$. $\mathrm{LDJ}^{\dot{-}}$ is the system LDJ with the following two inference rules (Γ contains at most one wff):

(a) $$\frac{\alpha \Rightarrow \Delta, \beta}{\alpha \dot{-} \beta \Rightarrow \Delta}$$ (b) $$\frac{\Gamma \Rightarrow \Delta, \alpha \qquad \beta \Rightarrow \Lambda}{\Gamma \Rightarrow \Delta, \Lambda, \alpha \dot{-} \beta}$$

$\mathrm{LJ}^{\dot{-}}$ is the system LJ with the following three inference rules (Δ contains at most one wff):

$$\text{(c)}\quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \mathbin{\dot{-}} \beta, \Gamma \Rightarrow \Delta} \qquad \text{(d)}\quad \frac{\Gamma \Rightarrow \beta}{\alpha \mathbin{\dot{-}} \beta, \Gamma \Rightarrow} \qquad \text{(e)}\quad \frac{\Gamma \Rightarrow \alpha \qquad \beta, \Pi \Rightarrow}{\Gamma, \Pi \Rightarrow \alpha \mathbin{\dot{-}} \beta}$$

He then defined a translation mapping between $\mathrm{LDJ}^{\dot{-}}$ and $\mathrm{LJ}^{\dot{-}}$. Goodman uses the symbol $\dot{-}$ as the pseudo-difference operator in [6] and suggests reading it as "but not." Urbas uses it so that '$\alpha \mathbin{\dot{-}} \beta$' expresses that α excludes β.

DI is a kind of paraconsistent logic because sequents of the form '$\alpha \wedge \neg\alpha \Rightarrow$ ' are not in general provable in it. Thus it is not always provable that $\alpha \wedge \neg\alpha \Rightarrow \beta$, for every wff β. We can, however, find sequents of the form '$\alpha \wedge \neg\alpha \Rightarrow$ ' which are provable in DI. To mention a few: '$(\alpha \to \alpha) \wedge \neg(\alpha \to \alpha) \Rightarrow$ ' and '$(\alpha \vee \neg\alpha) \wedge \neg(\alpha \vee \neg\alpha) \Rightarrow$ '.

We now consider a translation mapping between DI and LJ. We first extend DI and LJ by adding the logical symbol $\dot{-}$ and inference rules associated with it as Urbas did. The resulting systems are DI^{+} and LJ^{+}, respectively.

Definition 2.7 DI^{+} is obtained from DI by adding the two inference rules:

$$\dot{-}\mathrm{L}\quad \frac{\alpha \Rightarrow \Delta, \beta}{\alpha \mathbin{\dot{-}} \beta \Rightarrow \Delta} \qquad \dot{-}\mathrm{R}\quad \frac{\Gamma \Rightarrow \Delta, \alpha \qquad \beta \Rightarrow \Lambda}{\Gamma \Rightarrow \Delta, \Lambda, \alpha \mathbin{\dot{-}} \beta}$$

(Γ consists of at most one wff.)

Definition 2.8 LJ^{+} is obtained from LJ by adding the two inference rules:

$$\dot{-}\mathrm{L}\quad \frac{\alpha, \Gamma \Rightarrow \beta}{\alpha \mathbin{\dot{-}} \beta, \Gamma \Rightarrow} \qquad \dot{-}\mathrm{R}\quad \frac{\Gamma \Rightarrow \alpha \qquad \beta, \Pi \Rightarrow}{\Gamma, \Pi \Rightarrow \alpha \mathbin{\dot{-}} \beta}$$

The following is the translation mapping due to Czermak and Urbas.

Definition 2.9 The translation mapping * between DI^{+} and LJ^{+} is defined as in (1)–(4) below:

1. Wffs:

$A^* = A$ for each atomic wff A, $\quad (\neg\alpha)^* = \neg\alpha^*$,

$(\alpha \wedge \beta)^* = \beta^* \vee \alpha^*$, $\quad (\alpha \vee \beta)^* = \beta^* \wedge \alpha^*$,

$(\alpha \to \beta)^* = \beta^* \mathbin{\dot{-}} \alpha^*$, $\quad (\alpha \mathbin{\dot{-}} \beta)^* = \beta^* \to \alpha^*$,

$(\forall x \alpha(x))^* = \exists x (\alpha(x))^*$, $\quad (\exists x \alpha(x))^* = \forall x (\alpha(x))^*$

2. Sequences of wffs:

$$\text{For each sequence } \Gamma = \alpha_1, \alpha_2, \ldots, \alpha_n (n \geqq 0),$$
$$\Gamma^* = (\alpha_n)^*, \ldots, (\alpha_2)^*, (\alpha_1)^*$$

3. Sequents:

$$\text{For each sequent } \Gamma \Rightarrow \Delta,\ (\Gamma \Rightarrow \Delta)^* = \Delta^* \Rightarrow \Gamma^*$$

4. Inference rules:

$$\text{For each inference rule } I = \frac{S_1, \ldots, S_{n-1}}{S_n} (n = 2, 3),$$

$$I^* = \frac{(S_{n-1})^*, \ldots, (S_1)^*}{(S_n)^*}$$

Then we can easily show the following.

Proposition 2.10 *For any wff* α*, sequence* Γ*, sequent* $\Gamma \Rightarrow \Delta$*, and inference rule* I *of* DI^+ *(and of* LJ^+*), we have* $\alpha^{**} = \alpha$, $\Gamma^{**} = \Gamma$, $(\Gamma \Rightarrow \Delta)^{**} = \Gamma \Rightarrow \Delta$*, and* $I^{**} = I$.

Theorem 2.11 $\mathrm{DI}^+ \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{LJ}^+ \vdash (\Gamma \Rightarrow \Delta)^*$*, where* Γ *contains at most one wff.*

We now consider the relation between DI and LDJ. In our formal proofs below, we will omit writing applications of the inference rule ER (and EL, if any). Also, we often indicate successive applications of WR (WL) and combinations of applications of $\vee$R and CR ($\wedge$L and CL) by writing a double line between the upper sequent and the lower sequent.

Proposition 2.12 *The rule* $\to$*R(U):1 of* LDJ *holds in* DI.

Proof

$$\dfrac{\dfrac{\alpha \Rightarrow \Delta}{\Rightarrow \Delta, \neg\alpha} \qquad \dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \Rightarrow \beta, \alpha}}{\Rightarrow \alpha \to \beta, \alpha}}{\neg\alpha \Rightarrow \alpha \to \beta}}{\Rightarrow \Delta, \alpha \to \beta}$$

□

As Czermak ([4], p. 473) and Urbas ([17], p. 442) remark, the sequent '$\Rightarrow \alpha \to (\beta \to \alpha)$' is not provable in DI and thus '$\alpha \Rightarrow (\beta \to \alpha)$' is not either, as we mentioned above. Therefore, we can easily obtain this proposition.

Proposition 2.13 *The rule* $\to$*R(U):2 of* LDJ *does not hold in* DI.

The following proposition is proved in [17], p. 443.

Proposition 2.14 (Urbas) *The rule* $\to$*R of* DI *holds in* LDJ.

Then we can obtain the following theorem.

Theorem 2.15 *Let* $\mathrm{DI}^\#$ *be the system obtained from* DI *by adding axioms of the form '*$\alpha \Rightarrow (\beta \to \alpha)$*'. Then* LDJ = $\mathrm{DI}^\#$*, that is,* LDJ $\vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{DI}^\#$ $\vdash \Gamma \Rightarrow \Delta$*, where* Γ *consists of at most one wff.*

3 DI and LK

Sequents of the forms '$\alpha \Rightarrow \neg\neg\alpha$' and '$\alpha \wedge (\alpha \to \beta) \Rightarrow \beta$' are not in general provable in DI. These, however, play important roles in connecting DI to LK. To this end, we first make some definitions.

Definition 3.1 $\mathrm{DI_{dnr}}$ is defined to be the system obtained from DI by adding axioms of the form '$\alpha \Rightarrow \neg\neg\alpha$'. $\mathrm{DI_{mp}}$ is the system obtained from DI by adding axioms of the form '$\alpha \wedge (\alpha \to \beta) \Rightarrow \beta$'. $\mathrm{DI_{+}}$ is the system obtained from DI by adding the following rule of inference, which we will call 'RT':

$$\frac{\alpha \Rightarrow \neg\beta}{\beta \Rightarrow \neg\alpha}\ .$$

The three systems defined above are all equivalent to LK. We first need to do some preliminary work.

Proposition 3.2 *The following hold:*

1. $\mathrm{DI_{dnr}} \vdash \neg\alpha \wedge \beta \Rightarrow \neg(\alpha \vee \neg\beta)$*;*
2. $\mathrm{DI_{dnr}} \vdash \beta \vee \neg(\alpha \wedge \gamma) \Rightarrow \alpha \to \beta, \neg\gamma$*;*
3. $\mathrm{DI_{dnr}} \vdash (\alpha \to \beta) \wedge \gamma \wedge \pi \Rightarrow \neg(\alpha \vee \neg\gamma) \vee (\beta \wedge \pi)$*;*
4. $\mathrm{DI_{dnr}} \vdash \forall x \neg\alpha(x) \Rightarrow \neg\exists x \alpha(x)$.

Given a sequent $\Gamma \Rightarrow \Delta$, where both Γ and Δ are finite, possibly empty, sequences of wffs, it is a simple fact that $\mathrm{LK} \vdash \Gamma \Rightarrow \Delta$ if and only if $\mathrm{LK} \vdash \bigwedge\!\!\bigwedge \Gamma \Rightarrow \Delta$, where '$\bigwedge\!\!\bigwedge \Gamma$' indicates the conjunction of all the wffs in Γ. When Γ is empty, $\bigwedge\!\!\bigwedge \Gamma$ is also empty. Now we can prove the following theorem.

Theorem 3.3 $\mathrm{DI_{dnr}} = \mathrm{LK}$*, that is,* $\mathrm{DI_{dnr}} \vdash \bigwedge\!\!\bigwedge \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{LK} \vdash \bigwedge\!\!\bigwedge \Gamma \Rightarrow \Delta$.

Proof ($\Rightarrow$) This is trivial since DI is a subsystem of LK and since $\mathrm{LK} \vdash \alpha \Rightarrow \neg\neg\alpha$.

($\Leftarrow$) We need to show that all the inference rules of LK hold in $\mathrm{DI_{dnr}}$. We only check the rule $\to$L. We set $\gamma = \bigwedge\!\!\bigwedge \Gamma$ and $\pi = \bigwedge\!\!\bigwedge \Pi$.

$$(\to\text{L of LK}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \qquad \beta, \Pi \Rightarrow \Lambda}{\alpha \to \beta, \Gamma, \Pi \Rightarrow \Delta, \Lambda}\ .$$

We only need to show that the following equivalent rule holds in $\mathrm{DI_{dnr}}$:

$$\frac{\gamma \Rightarrow \Delta, \alpha \qquad \beta \wedge \pi \Rightarrow \Lambda}{(\alpha \to \beta) \wedge \gamma \wedge \pi \Rightarrow \Delta, \Lambda}\ .$$

Using Proposition 3.2(3), we have

$$\dfrac{(\alpha \to \beta) \wedge \gamma \wedge \pi \Rightarrow \neg(\alpha \vee \neg\gamma) \vee (\beta \wedge \pi) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \Delta, \alpha}{\Rightarrow \Delta, \alpha, \neg\gamma}}{\Rightarrow \Delta, \alpha \vee \neg\gamma}}{\neg(\alpha \vee \neg\gamma) \Rightarrow \Delta}}{\neg(\alpha \vee \neg\gamma) \Rightarrow \Delta, \Lambda} \qquad \dfrac{\beta \wedge \pi \Rightarrow \Lambda}{\beta \wedge \pi \Rightarrow \Delta, \Lambda}}{\neg(\alpha \vee \neg\gamma) \vee (\beta \wedge \pi) \Rightarrow \Delta, \Lambda}}{(\alpha \to \beta) \wedge \gamma \wedge \pi \Rightarrow \Delta, \Lambda}$$

□

Theorem 3.4 $\mathrm{DI_{dnr}} = \mathrm{DI_{mp}} = \mathrm{DI_{+}}$.

Proof We show that

1. $\mathrm{DI_{mp}} \vdash \alpha \Rightarrow \neg\neg\alpha$,
2. $\mathrm{DI_{+}} \vdash \alpha \wedge (\alpha \to \beta) \Rightarrow \beta$, and
3. the inference rule RT holds in $\mathrm{DI_{dnr}}$.

We freely use Proposition 2.2.

(1)

$$
\begin{prooftree}
\AxiomC{$\alpha \Rightarrow \alpha$}
\AxiomC{$\neg\alpha \Rightarrow \neg\alpha$}
\UnaryInfC{$\Rightarrow \neg\alpha \to \neg\alpha$}
\UnaryInfC{$\alpha \Rightarrow \neg\alpha \to \neg\alpha$}
\AxiomC{$\neg\alpha \to \neg\alpha \Rightarrow \alpha \to \neg\neg\alpha$}
\BinaryInfC{$\alpha \Rightarrow \alpha \to \neg\neg\alpha$}
\BinaryInfC{$\alpha \Rightarrow \alpha \wedge (\alpha \to \neg\neg\alpha)$}
\AxiomC{$\alpha \wedge (\alpha \to \neg\neg\alpha) \Rightarrow \neg\neg\alpha$}
\BinaryInfC{$\alpha \Rightarrow \neg\neg\alpha$}
\end{prooftree}
$$

(2)

$$
\begin{prooftree}
\AxiomC{$\alpha \Rightarrow \alpha$}
\UnaryInfC{$\Rightarrow \alpha, \neg\alpha$}
\AxiomC{$\beta \Rightarrow \beta$}
\BinaryInfC{$\alpha \to \beta \Rightarrow \beta, \neg\alpha$}
\UnaryInfC{$\Rightarrow \beta, \neg\alpha, \neg(\alpha \to \beta)$}
\UnaryInfC{$\neg\beta \Rightarrow \neg\alpha, \neg(\alpha \to \beta)$}
\doubleLine
\UnaryInfC{$\neg\beta \Rightarrow \neg\alpha \vee \neg(\alpha \to \beta)$}
\AxiomC{$\neg\alpha \vee \neg(\alpha \to \beta) \Rightarrow \neg(\alpha \wedge (\alpha \to \beta))$}
\BinaryInfC{$\neg\beta \Rightarrow \neg(\alpha \wedge (\alpha \to \beta))$}
\UnaryInfC{$\alpha \wedge (\alpha \to \beta) \Rightarrow \neg\neg\beta$}
\AxiomC{$\neg\neg\beta \Rightarrow \beta$}
\BinaryInfC{$\alpha \wedge (\alpha \to \beta) \Rightarrow \beta$}
\end{prooftree}
$$

(3)

$$
\begin{prooftree}
\AxiomC{$\beta \Rightarrow \neg\neg\beta$}
\AxiomC{$\alpha \Rightarrow \neg\beta$}
\UnaryInfC{$\Rightarrow \neg\beta, \neg\alpha$}
\UnaryInfC{$\neg\neg\beta \Rightarrow \neg\alpha$}
\BinaryInfC{$\beta \Rightarrow \neg\alpha$}
\end{prooftree}
$$

□

It is interesting to notice that DI $\vdash\ \ \Rightarrow \alpha \to \neg\neg\alpha$, as the following proof shows:

$$
\begin{prooftree}
\AxiomC{$\Rightarrow \neg\alpha \vee \neg\neg\alpha$}
\AxiomC{$\alpha \Rightarrow \alpha$}
\UnaryInfC{$\alpha \Rightarrow \neg\neg\alpha, \alpha$}
\UnaryInfC{$\Rightarrow \alpha \to \neg\neg\alpha, \alpha$}
\UnaryInfC{$\neg\alpha \Rightarrow \alpha \to \neg\neg\alpha$}
\AxiomC{$\neg\alpha \Rightarrow \neg\alpha$}
\UnaryInfC{$\Rightarrow \neg\neg\alpha, \neg\alpha$}
\UnaryInfC{$\alpha \Rightarrow \neg\neg\alpha, \neg\alpha$}
\UnaryInfC{$\Rightarrow \alpha \to \neg\neg\alpha, \neg\alpha$}
\UnaryInfC{$\neg\neg\alpha \Rightarrow \alpha \to \neg\neg\alpha$}
\BinaryInfC{$\neg\alpha \vee \neg\neg\alpha \Rightarrow \alpha \to \neg\neg\alpha$}
\BinaryInfC{$\Rightarrow \alpha \to \neg\neg\alpha$}
\end{prooftree}
$$

This shows that it is not always true that DI $\vdash\ \ \Rightarrow \alpha \to \beta$ implies DI $\vdash \alpha \Rightarrow \beta$, although the converse holds trivially.

In passing, we here mention the well-known relation between LK and LJ using the following definition.

Definition 3.5 $\mathrm{LJ}_{\mathrm{dnl}}$ is defined to be the system obtained from LJ by adding axioms of the form '$\neg\neg\alpha \Rightarrow \alpha$'. LJ_{+} is the system obtained from LJ by adding the following rule of inference, which we will call 'IT':

$$
\frac{\neg\alpha \Rightarrow \beta}{\neg\beta \Rightarrow \alpha}
$$

As before, we note a simple fact that given a sequent $\Gamma \Rightarrow \Delta$, where both Γ and Δ contain a finite number of wffs, LK $\vdash \Gamma \Rightarrow \Delta$ if and only if LK $\vdash \Gamma \Rightarrow \bigvee\!\!\bigvee \Delta$, where '$\bigvee\!\!\bigvee \Delta$' indicates the disjunction of all the wffs in Δ. When Δ is empty, $\bigvee\!\!\bigvee \Delta$ is also empty. Then the following two theorems are easy to obtain.

Theorem 3.6 $\mathrm{LJ_{dnl}} = \mathrm{LK}$, *that is,* $\mathrm{LJ_{dnl}} \vdash \Gamma \Rightarrow \bigvee\!\!\bigvee \Delta$ *if and only if* LK $\vdash \Gamma \Rightarrow \bigvee\!\!\bigvee \Delta$.

Theorem 3.7 $\mathrm{LJ_{dnl}} = \mathrm{LJ_{+}}$.

4 DI and GO

Nishimura [9] presents a sequent calculus version of Goldblatt's orthologic O [5], which is a propositional logic. The following is his system GO, where Γ and Δ in a sequent $\Gamma \Rightarrow \Delta$ are (possibly infinite or empty) sets of wffs. In this section and the next, we assume that the language is propositional and does not contain the logical symbol '$\rightarrow$'.

Definition 4.1 GO consists of the following axioms and rules of inference:

1. Axioms $\alpha \Rightarrow \alpha$

2. Inference Rules

Structural Rules

Extension $\dfrac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Lambda}$ Cut $\dfrac{\Gamma \Rightarrow \Delta, \alpha \qquad \alpha, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$

Logical Rules

$\neg$L $\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}$ $\neg$R $\dfrac{\alpha \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\alpha}$ $(\neg\Delta =_{\mathrm{df}} \{\neg\beta \mid \beta \in \Delta\})$

$\neg\neg$L $\dfrac{\alpha, \Gamma \Rightarrow \Delta}{\neg\neg\alpha, \Gamma \Rightarrow \Delta}$ $\neg\neg$R $\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg\neg\alpha}$

$\wedge$L $\dfrac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}$ $\dfrac{\beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}$ $\wedge$R $\dfrac{\Gamma \Rightarrow \Delta, \alpha \qquad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$

In GO, '$\alpha \vee \beta$' is defined to be '$\neg(\neg\alpha \wedge \neg\beta)$'. As usual, we use the symbol '$\nvdash$' to express the unprovability of a sequent. We then have the following proposition.

Proposition 4.2 *The following hold:*

1. GO $\vdash \alpha \wedge \neg\alpha \Rightarrow$;
2. GO $\vdash \ \Rightarrow \alpha \vee \neg\alpha$;
3. GO $\nvdash \alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$;
4. GO $\nvdash (\alpha \vee \beta) \wedge (\alpha \vee \gamma) \Rightarrow \alpha \vee (\beta \wedge \gamma)$.

The converse sequents of (3) and (4) are provable in GO. As Nishimura remarks, if we replace ¬R of GO with the following rule ¬Rc,

$$\neg\mathrm{Rc} \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha} \ ,$$

the resulting system is the classical propositional logic. Cutland and Gibbins [3] also consider adding to GO the two rules of inference $\neg\mathrm{R}^{\dagger}$ and $\vee\mathrm{L}^{\dagger}$:

$$\neg\mathrm{R}^{\dagger} \quad \frac{\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\Gamma} \qquad\qquad \vee\mathrm{L}^{\dagger} \quad \frac{\alpha \Rightarrow \Delta \quad \beta \Rightarrow \Delta}{\alpha \vee \beta \Rightarrow \Delta}$$

They prove the following theorem in [3], p. 227.

Theorem 4.3 (Cutland and Gibbins) *The following hold:*

1. GO + '$\alpha \vee \beta \Rightarrow \alpha, \beta$' *is equivalent to the classical propositional logic;*
2. GO + $\neg R^{\dagger}$ *is equivalent to the classical propositional logic;*
3. GO + $\vee L^{\dagger}$ *is equivalent to the classical propositional logic.*

In order to investigate the relationship between DI and the orthologic, we reformulate GO to the following sequent calculus LO, where Γ and Δ in a sequent $\Gamma \Rightarrow \Delta$ express finite sequences (not sets) of formulas and $\neg\Delta$ is defined to be the same sequence of formulas as Δ except that each formula in Δ is negated.

Definition 4.4 LO consists of the following axioms and rules of inference:

1. Axioms $\alpha \Rightarrow \alpha, \quad \neg\neg\alpha \Rightarrow \alpha\ , \quad \alpha \Rightarrow \neg\neg\alpha$

2. Inference Rules

Structural Rules

$$\mathrm{WL} \quad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \qquad\qquad \mathrm{WR} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}$$

$$\mathrm{CL} \quad \frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \qquad\qquad \mathrm{CR} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}$$

$$\mathrm{EL} \quad \frac{\Gamma, \alpha, \beta, \Pi \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Pi \Rightarrow \Delta} \qquad\qquad \mathrm{ER} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta, \Lambda}{\Gamma \Rightarrow \Delta, \beta, \alpha, \Lambda}$$

$$\mathrm{Cut} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \qquad \alpha, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

Logical Rules

$$\neg\mathrm{L} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta} \qquad\qquad \neg\mathrm{R} \quad \frac{\alpha \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\alpha}$$

$$\wedge\mathrm{L} \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \quad \frac{\beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \qquad\qquad \wedge\mathrm{R} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

$$\vee\mathrm{L} \quad \frac{\alpha \Rightarrow \Delta \qquad \beta \Rightarrow \Delta}{\alpha \vee \beta \Rightarrow \Delta} \qquad\qquad \vee\mathrm{R} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \quad \frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

(Δ contains at most one wff.)

For LO, we may use the rules $\neg\neg$L and $\neg\neg$R of GO instead of the pair of double negation axioms '$\neg\neg\alpha \Rightarrow \alpha$' and '$\alpha \Rightarrow \neg\neg\alpha$', respectively. Now, since Nishimura shows that if GO $\vdash \Gamma \Rightarrow \Delta$, then there exists a finite subsequent $\Gamma' \Rightarrow \Delta'$ of $\Gamma \Rightarrow \Delta$ such that GO $\vdash \Gamma' \Rightarrow \Delta'$, it is easy to show the following.

Proposition 4.5 GO = LO.

In LO, the restriction on Δ in the rule $\vee$L that it contains at most one wff is essential; otherwise, LO would become the classical logic as Theorem 4.3(3) shows. We now have this proposition.

Proposition 4.6 *The following hold for* LO*:*

1. LO $\vdash \Gamma \Rightarrow \Delta$ *iff* LO $\vdash \bigwedge\!\!\!\bigwedge \Gamma \Rightarrow \Delta$.
2. *If the rule $\neg R$ of* LO *is replaced by the following rule $\neg Rs$, the resulting system will be the classical propositional logic.*

$$\neg\text{Rs} \quad \frac{\alpha \Rightarrow \Delta}{\Rightarrow \Delta, \neg\alpha}$$

Proof (1) is trivial. For (2), we first set $\gamma = \bigwedge\!\!\!\bigwedge \Gamma$ for Γ in $\neg$Rc. Then, by using (1), it is enough to show that the following version of the rule $\neg$Rc holds in the resulting system:

$$\neg\text{Rc} \quad \frac{\alpha \wedge \gamma \Rightarrow \Delta}{\gamma \Rightarrow \Delta, \neg\alpha}$$

It is simple:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Rightarrow \alpha, \neg\alpha}}{\gamma \Rightarrow \neg\alpha, \alpha} \qquad \dfrac{\gamma \Rightarrow \gamma}{\gamma \Rightarrow \neg\alpha, \gamma}}{\gamma \Rightarrow \neg\alpha, \alpha \wedge \gamma} \qquad \alpha \wedge \gamma \Rightarrow \Delta}{\gamma \Rightarrow \Delta, \neg\alpha}$$

□

We now prove the disjunction theorem for LO.

Definition 4.7 A sequent $\Gamma \Rightarrow \Delta$ is called normal if its succedent Δ contains at most one wff. A formal proof in LO is called normal if each sequent appearing in it is normal.

The disjunction theorem is an easy corollary of the following normal form theorem, whose proof is similar to that for GO in Nishimura [9].

Theorem 4.8 *If* LO $\vdash \Gamma \Rightarrow \Delta$*, then there is a normal subsequent* $\Gamma \Rightarrow \Delta'$ *of* $\Gamma \Rightarrow \Delta$ *such that* $\Gamma \Rightarrow \Delta'$ *has a normal proof in* LO.

Corollary 4.9 *Let* $\Gamma \Rightarrow \Delta$ *be a sequent such that* Δ *is not empty. Then* LO $\vdash \Gamma \Rightarrow \Delta$ *if and only if for some* α *in* Δ, LO $\vdash \Gamma \Rightarrow \alpha$.

In order to connect LO with DI, we put the restriction on LO that the antecedents of sequents contain at most one formula, and the resulting system will be denoted as 'LO^-'. It is this.

Definition 4.10 LO^{-} consists of the following axioms and rules of inference:

1. Axioms $\alpha \Rightarrow \alpha, \quad \neg\neg\alpha \Rightarrow \alpha, \quad \alpha \Rightarrow \neg\neg\alpha$

2. Inference Rules (Γ contains at most one wff.)

Structural Rules The same as those of DI.

Logical Rules

$$\neg\mathrm{L} \quad \frac{\Rightarrow \Delta, \alpha}{\neg\alpha \Rightarrow \Delta} \qquad \neg\mathrm{R} \quad \frac{\alpha \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\alpha}$$

(Δ contains at most one wff.)

$$\wedge\mathrm{L} \quad \frac{\alpha \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta} \quad \frac{\beta \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta} \qquad \wedge\mathrm{R} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

$$\vee\mathrm{L} \quad \frac{\alpha \Rightarrow \Delta \quad \beta \Rightarrow \Delta}{\alpha \vee \beta \Rightarrow \Delta} \qquad \vee\mathrm{R} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \quad \frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

(Δ contains at most one wff.)

Definition 4.11 $\mathrm{DI_P}^{-}$ is the propositional part of DI which lacks the rules $\rightarrow$L and $\rightarrow$R. Thus the language of $\mathrm{DI_P}^{-}$ is propositional and lacks the logical symbol '$\rightarrow$'.

The difference of inference rules between LO^{-} and $\mathrm{DI_P}^{-}$ is that the rules $\neg$R and $\vee$L of the former system can only be applied to those sequents whose succedent contains at most one wff, although there is no such restriction on the corresponding two rules of the latter system. Now we have the following.

Proposition 4.12 *The following sequents are provable in* LO *and in* LO^{-} *:*

1. $\alpha \wedge \beta \Leftrightarrow \neg(\neg\alpha \vee \neg\beta)$,
2. $\alpha \vee \beta \Leftrightarrow \neg(\neg\alpha \wedge \neg\beta)$,
3. $\neg(\alpha \wedge \beta) \Leftrightarrow \neg\alpha \vee \neg\beta$,
4. $\neg(\alpha \vee \beta) \Leftrightarrow \neg\alpha \wedge \neg\beta$.

Proposition 4.13 *The following hold for* LO^{-} *:*

1. $\mathrm{LO}^{-} \vdash \quad \Rightarrow \alpha \vee \neg\alpha$ *iff* $\mathrm{LO}^{-} \vdash \alpha \wedge \neg\alpha \Rightarrow \quad$;
2. *The distributive laws hold in* LO^{-} $+$'$\alpha \vee \beta \Rightarrow \alpha, \beta$';
3. $\mathrm{LO}^{-} \nvdash \alpha \vee \beta \Rightarrow \alpha, \beta$;
4. $\mathrm{LO}^{-} \vdash \quad \Rightarrow \alpha, \neg\alpha$ *iff* ($\mathrm{LO}^{-} \vdash \quad \Rightarrow \alpha \vee \neg\alpha$ *and* $\mathrm{LO}^{-} \vdash \alpha \vee \beta \Rightarrow \alpha, \beta$);
5. $\mathrm{LO}^{-} \nvdash \quad \Rightarrow \alpha, \neg\alpha$;
6. $\mathrm{LO}^{-} \vdash \alpha \wedge \neg\alpha \Rightarrow \quad$ *iff the inference rule* $\dfrac{\gamma \Rightarrow \Delta, \alpha}{\neg\alpha \wedge \gamma \Rightarrow \Delta}$ *holds in* LO^{-}.

Proof (1) ($\Rightarrow$)

$$\dfrac{\dfrac{\dfrac{\neg\alpha \Rightarrow \neg\alpha}{\alpha \wedge \neg\alpha \Rightarrow \neg\alpha} \quad \dfrac{\alpha \Rightarrow \neg\neg\alpha}{\alpha \wedge \neg\alpha \Rightarrow \neg\neg\alpha}}{\alpha \wedge \neg\alpha \Rightarrow \neg\alpha \wedge \neg\neg\alpha} \quad \neg\alpha \wedge \neg\neg\alpha \Rightarrow \neg(\alpha \vee \neg\alpha)}{\alpha \wedge \neg\alpha \Rightarrow \neg(\alpha \vee \neg\alpha)} \quad \dfrac{\Rightarrow \alpha \vee \neg\alpha}{\neg(\alpha \vee \neg\alpha) \Rightarrow}$$
$$\overline{\alpha \wedge \neg\alpha \Rightarrow}$$

(⇐)

$$\dfrac{\dfrac{\dfrac{\neg(\alpha \vee \neg\alpha) \Rightarrow \neg\alpha \wedge \neg\neg\alpha \qquad \dfrac{\dfrac{\neg\neg\alpha \Rightarrow \alpha}{\neg\alpha \wedge \neg\neg\alpha \Rightarrow \alpha} \qquad \dfrac{\neg\alpha \Rightarrow \neg\alpha}{\neg\alpha \wedge \neg\neg\alpha \Rightarrow \neg\alpha}}{\neg\alpha \wedge \neg\neg\alpha \Rightarrow \alpha \wedge \neg\alpha}}{\neg(\alpha \vee \neg\alpha) \Rightarrow \alpha \wedge \neg\alpha} \qquad \alpha \wedge \neg\alpha \Rightarrow}{\dfrac{\neg(\alpha \vee \neg\alpha) \Rightarrow}{\Rightarrow \neg\neg(\alpha \vee \neg\alpha)}} \qquad \neg\neg(\alpha \vee \neg\alpha) \Rightarrow \alpha \vee \neg\alpha}{\Rightarrow \alpha \vee \neg\alpha}$$

(2) Add sequents of the form '$\alpha \vee \beta \Rightarrow \alpha, \beta$' as axioms to LO^-. Then we can prove the distributive law '$\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$':

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \wedge (\beta \vee \gamma) \Rightarrow \alpha}}{\alpha \wedge (\beta \vee \gamma) \Rightarrow \alpha, \alpha \wedge \gamma} \qquad \dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \Rightarrow \beta, \alpha}}{\alpha \wedge (\beta \vee \gamma) \Rightarrow \beta, \alpha} \qquad \dfrac{\beta \vee \gamma \Rightarrow \beta, \gamma}{\alpha \wedge (\beta \vee \gamma) \Rightarrow \beta, \gamma}}{\alpha \wedge (\beta \vee \gamma) \Rightarrow \beta, \alpha \wedge \gamma}}{\alpha \wedge (\beta \vee \gamma) \Rightarrow \alpha \wedge \beta, \alpha \wedge \gamma}}{\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}$$

Similarly, we can obtain the rest of the distributive laws in $\mathrm{LO}^- +$ '$\alpha \vee \beta \Rightarrow \alpha, \beta$'.

(3) Since some of the distributive laws do not hold in LO, they do not, either, in the subsystem LO^-. This implies from (2) that $\mathrm{LO}^- \nvdash \alpha \vee \beta \Rightarrow \alpha, \beta$.

(4) (⇒)

$$\dfrac{\alpha \vee \beta \Rightarrow \neg(\neg\alpha \wedge \neg\beta) \qquad \dfrac{\dfrac{\dfrac{\Rightarrow \alpha, \neg\alpha}{\Rightarrow \alpha, \beta, \neg\alpha} \qquad \dfrac{\Rightarrow \beta, \neg\beta}{\Rightarrow \alpha, \beta, \neg\beta}}{\Rightarrow \alpha, \beta, \neg\alpha \wedge \neg\beta}}{\neg(\neg\alpha \wedge \neg\beta) \Rightarrow \alpha, \beta}}{\alpha \vee \beta \Rightarrow \alpha, \beta}$$

Showing that $\mathrm{LO}^- \vdash \ \Rightarrow \alpha \vee \neg\alpha$ is trivial.

(⇐)

$$\dfrac{\Rightarrow \alpha \vee \neg\alpha \qquad \alpha \vee \neg\alpha \Rightarrow \alpha, \neg\alpha}{\Rightarrow \alpha, \neg\alpha}$$

(5) This is from (3) and (4).

(6) (⇒)

$$\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \Delta, \alpha}{\neg\alpha \wedge \gamma \Rightarrow \Delta, \alpha} \qquad \dfrac{\dfrac{\neg\alpha \Rightarrow \neg\alpha}{\neg\alpha \wedge \gamma \Rightarrow \neg\alpha}}{\neg\alpha \wedge \gamma \Rightarrow \Delta, \neg\alpha}}{\neg\alpha \wedge \gamma \Rightarrow \Delta, \alpha \wedge \neg\alpha} \qquad \alpha \wedge \neg\alpha \Rightarrow}{\neg\alpha \wedge \gamma \Rightarrow \Delta}$$

(⇐) Trivial. □

Although it seems unlikely that the sequent '$\alpha \wedge \neg\alpha \Rightarrow$ ' is provable in LO^{-}, we do not know for sure that $\mathrm{LO}^{-} \nvdash \alpha \wedge \neg\alpha \Rightarrow$. We leave this problem open for now. In what follows, we assume that '$\alpha \wedge \neg\alpha \Rightarrow$ ' is unprovable in LO^{-}. We now make a few more definitions.

Definition 4.14

1. $\mathrm{LO}^{-\mathrm{e}}$ is the system LO^{-} + ' $\Rightarrow \alpha, \neg\alpha$'.
2. $\mathrm{LO}^{-\mathrm{c}}$ is the system LO^{-} + '$\alpha \wedge \neg\alpha \Rightarrow$ '.
3. $\mathrm{DI_P}^{-\mathrm{d}}$ is the system $\mathrm{DI_P}^{-}$ + '$\alpha \Rightarrow \neg\neg\alpha$'.

We can then obtain the following theorem.

Theorem 4.15 *If* $\mathrm{LO} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathrm{LO}^{-\mathrm{e}} \vdash \bigwedge\!\!\bigwedge \Gamma \Rightarrow \Delta$.

The converse of the above theorem does not hold since all the distributive laws hold in $\mathrm{LO}^{-\mathrm{e}}$ but some of them do not in LO. We can also show the next two theorems easily.

Theorem 4.16 $\mathrm{LO} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{LO}^{-c} \vdash \bigwedge\!\!\bigwedge \Gamma \Rightarrow \Delta$.

Theorem 4.17 $\mathrm{LO}^{-\mathrm{e}} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{DI_P}^{-\mathrm{d}} \vdash \Gamma \Rightarrow \Delta$, *where* Γ *contains at most one wff.*

Thus LO is equivalent to $\mathrm{LO}^{-\mathrm{c}}$ and so is $\mathrm{LO}^{-\mathrm{e}}$ to $\mathrm{DI_P}^{-\mathrm{d}}$. Moreover, we can show that $\mathrm{LO}^{-\mathrm{e}}$ is equivalent to the propositional part of LK, which we will denote by '$\mathrm{LK_P}$'. We need a preliminary work to show this.

Proposition 4.18 *The following hold:*

1. *Let* Γ *consist of at most one wff. Then,* $\mathrm{LO}^{-} \vdash \Gamma \Rightarrow \bigvee\!\!\bigvee \Delta$ *if*
   $\mathrm{LO}^{-} \vdash \Gamma \Rightarrow \Delta$*;*
2. LO^{-} + '$\alpha \vee \beta \Rightarrow \alpha, \beta$' $\vdash \bigvee\!\!\bigvee \Delta \Rightarrow \Delta$, *where* Δ *is not empty;*
3. LO^{-} + '$\alpha \vee \beta \Rightarrow \alpha, \beta$' $\vdash (\neg\alpha \vee \beta) \wedge \gamma \wedge \pi \Rightarrow (\neg\alpha \wedge \gamma) \vee (\beta \wedge \pi)$.

We now define '$\alpha \rightarrow \beta$' by '$\neg\alpha \vee \beta$' in $\mathrm{LO}^{-\mathrm{e}}$. Then we can prove the equivalence of $\mathrm{LO}^{-\mathrm{e}}$ and $\mathrm{LK_P}$.

Theorem 4.19 $\mathrm{LO}^{-\mathrm{e}} \vdash \bigwedge\!\!\bigwedge \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{LK_P} \vdash \Gamma \Rightarrow \Delta$.

Proof ($\Rightarrow$) Trivial.

($\Leftarrow$) We need to show that all the inference rules of $\mathrm{LK_P}$ hold in $\mathrm{LO}^{-\mathrm{e}}$. We here consider the rules $\rightarrow$L and $\rightarrow$R of $\mathrm{LK_P}$. $\neg$L of $\mathrm{LK_P}$ holds in $\mathrm{LO}^{-\mathrm{e}}$ because of Proposition 4.13(6). We set $\gamma = \bigwedge\!\!\bigwedge \Gamma$ and $\pi = \bigwedge\!\!\bigwedge \Pi$.

$$(\rightarrow \mathrm{L}) \qquad \frac{\gamma \Rightarrow \Delta, \alpha \qquad \beta \wedge \pi \Rightarrow \Lambda}{(\neg\alpha \vee \beta) \wedge \gamma \wedge \pi \Rightarrow \Delta, \Lambda}$$

Using the rule ¬L of LK_P, we can obtain '$\neg\alpha \wedge \gamma \Rightarrow \Delta$' from the upper sequent '$\gamma \Rightarrow \Delta, \alpha$'. We also have '$\bigvee\!\!\!\bigvee \Delta \vee \bigvee\!\!\!\bigvee \Lambda \Rightarrow \Delta, \Lambda$' by Proposition 4.18(2). Using Proposition 4.18, we have

$$
\begin{prooftree}
\AxiomC{$(\neg\alpha\vee\beta)\wedge\gamma\wedge\pi \Rightarrow (\neg\alpha\wedge\gamma)\vee(\beta\wedge\pi)$}
\AxiomC{$\neg\alpha \wedge \gamma \Rightarrow \Delta$}
\doubleLine
\UnaryInfC{$\neg\alpha \wedge \gamma \Rightarrow \Delta, \Lambda$}
\doubleLine
\UnaryInfC{$\neg\alpha\wedge\gamma \Rightarrow \bigvee\!\!\!\bigvee\Delta\vee\bigvee\!\!\!\bigvee\Lambda$}
\AxiomC{$\beta \wedge \pi \Rightarrow \Lambda$}
\doubleLine
\UnaryInfC{$\beta \wedge \pi \Rightarrow \Delta, \Lambda$}
\doubleLine
\UnaryInfC{$\beta\wedge\pi \Rightarrow \bigvee\!\!\!\bigvee\Delta\vee\bigvee\!\!\!\bigvee\Lambda$}
\BinaryInfC{$(\neg\alpha\wedge\gamma)\vee(\beta\wedge\pi) \Rightarrow \bigvee\!\!\!\bigvee\Delta\vee\bigvee\!\!\!\bigvee\Lambda$}
\BinaryInfC{$(\neg\alpha\vee\beta)\wedge\gamma\wedge\pi \Rightarrow \bigvee\!\!\!\bigvee\Delta\vee\bigvee\!\!\!\bigvee\Lambda$}
\AxiomC{$\bigvee\!\!\!\bigvee\Delta\vee\bigvee\!\!\!\bigvee\Lambda \Rightarrow \Delta, \Lambda$}
\BinaryInfC{$(\neg\alpha \vee \beta) \wedge \gamma \wedge \pi \Rightarrow \Delta, \Lambda$}
\end{prooftree}
$$

$$
(\rightarrow \mathrm{R}) \qquad \frac{\alpha \wedge \gamma \Rightarrow \Delta, \beta}{\gamma \Rightarrow \Delta, \neg\alpha \vee \beta}
$$

This is simple:

$$
\begin{prooftree}
\AxiomC{$\Rightarrow \alpha, \neg\alpha$}
\doubleLine
\UnaryInfC{$\gamma \Rightarrow \alpha, \neg\alpha, \beta$}
\doubleLine
\UnaryInfC{$\gamma \Rightarrow \alpha, \neg\alpha \vee \beta$}
\AxiomC{$\gamma \Rightarrow \gamma$}
\UnaryInfC{$\gamma \Rightarrow \gamma, \neg\alpha \vee \beta$}
\BinaryInfC{$\gamma \Rightarrow \neg\alpha \vee \beta, \alpha \wedge \gamma$}
\AxiomC{$\alpha \wedge \gamma \Rightarrow \Delta, \beta$}
\UnaryInfC{$\alpha \wedge \gamma \Rightarrow \Delta, \neg\alpha \vee \beta$}
\BinaryInfC{$\gamma \Rightarrow \Delta, \neg\alpha \vee \beta, \neg\alpha \vee \beta$}
\UnaryInfC{$\gamma \Rightarrow \Delta, \neg\alpha \vee \beta$}
\end{prooftree}
$$

□

5 Orthomodular Logic and Dual Orthomodular Logic

In the first half of this section, we will consider $LJ_P{}^-$, a subsystem of LJ, and DO^-, the dual system of LO^-. In the second half, we will consider the dual orthologic DO and the dual orthomodular logic DOM, which can be obtained from DO by adding an inference rule for the dual orthomodular law. DOM might be called the "dual quantum logic."

In this section, when we use the translation mapping * from Definition 2.9, we will omit the clauses for '$\alpha \rightarrow \beta$', '$\alpha \dot{-} \beta$', '$\forall x\alpha(x)$', and '$\exists x\alpha(x)$' in (1) thereof. Since most of the results of this section are the duals of those in Section 4 and can be proved dually, we will often omit their proofs.

5.1 $LJ_P{}^-$

We first consider intuitionistic analogs of $DI_P{}^-$ and LO^-. $LJ_P{}^-$ is the system obtained from the propositional part of LJ by eliminating the two rules for the implication symbol '→', that is, →L and →R. We list it here for convenience.

Definition 5.1 $LJ_P{}^-$ consists of the following axioms and rules of inference:

1. Axioms $\alpha \Rightarrow \alpha$

2. Inference Rules (Δ contains at most one wff.)

Structural Rules

$$\text{WL}\quad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \qquad \text{WR}\quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}$$

$$\text{CL}\quad \frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \qquad \text{EL}\quad \frac{\Gamma, \alpha, \beta, \Pi \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Pi \Rightarrow \Delta}$$

$$\text{Cut}\quad \frac{\Gamma \Rightarrow \alpha \qquad \alpha, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}$$

Logical Rules

$$\neg\text{L}\quad \frac{\Gamma \Rightarrow \alpha}{\neg\alpha, \Gamma \Rightarrow} \qquad \neg\text{R}\quad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\alpha}$$

$$\wedge\text{L}\quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \quad \frac{\beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \qquad \wedge\text{R}\quad \frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$$

$$\vee\text{L}\quad \frac{\alpha, \Gamma \Rightarrow \Delta \qquad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta} \qquad \vee\text{R}\quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}$$

We introduce a few more definitions.

Definition 5.2 DO^- is the system obtained from $\text{LJ}_\text{P}{}^-$ as follows:

1. Add axioms of the forms: $\neg\neg\alpha \Rightarrow \alpha, \quad \alpha \Rightarrow \neg\neg\alpha$
2. (i) Replace $\neg$L of $\text{LJ}_\text{P}{}^-$ with $\dfrac{\Gamma \Rightarrow \alpha}{\neg\alpha \Rightarrow \neg\Gamma}$, where Γ contains at most one wff.
   (ii) $\wedge$R of $\text{LJ}_\text{P}{}^-$ is so restricted that Γ contains at most one wff.

Definition 5.3

1. $\text{LJ}_\text{P}{}^{-\text{d}}$ is the system $\text{LJ}_\text{P}{}^-$ + '$\neg\neg\alpha \Rightarrow \alpha$'.
2. $\text{DO}^{-\text{c}}$ is the system DO^- + '$\alpha, \neg\alpha \Rightarrow$ '.
3. $\text{DO}^{-\text{e}}$ is the system DO^- + ' $\Rightarrow \alpha \vee \neg\alpha$'.

As we have been assuming that $\text{LO}^- \nvdash \alpha \wedge \neg\alpha \Rightarrow$, we also assume dually that $\text{DO}^- \nvdash \ \Rightarrow \alpha \vee \neg\alpha$. Using the translation mapping *, we can easily establish the following theorem.

Theorem 5.4 *Assume that Γ contains at most one wff. Then*

1. $\text{DI}_\text{P}{}^- \vdash \Gamma \Rightarrow \Delta$ *iff* $\text{LJ}_\text{P}{}^- \vdash (\Gamma \Rightarrow \Delta)^*$;
2. $\text{DI}_\text{P}{}^{-\text{d}} \vdash \Gamma \Rightarrow \Delta$ *iff* $\text{LJ}_\text{P}{}^{-\text{d}} \vdash (\Gamma \Rightarrow \Delta)^*$;
3. $\text{LO}^- \vdash \Gamma \Rightarrow \Delta$ *iff* $\text{DO}^- \vdash (\Gamma \Rightarrow \Delta)^*$;
4. $\text{LO}^{-\text{c}} \vdash \Gamma \Rightarrow \Delta$ *iff* $\text{DO}^{-\text{e}} \vdash (\Gamma \Rightarrow \Delta)^*$;
5. $\text{LO}^{-\text{e}} \vdash \Gamma \Rightarrow \Delta$ *iff* $\text{DO}^{-\text{c}} \vdash (\Gamma \Rightarrow \Delta)^*$.

Thus, $\text{LJ}_\text{P}{}^-$, $\text{LJ}_\text{P}{}^{-\text{d}}$, DO^- , $\text{DO}^{-\text{e}}$, and $\text{DO}^{-\text{c}}$ are the dual systems of, respectively, $\text{DI}_\text{P}{}^-$, $\text{DI}_\text{P}{}^{-\text{d}}$, LO^- , $\text{LO}^{-\text{c}}$, and $\text{LO}^{-\text{e}}$. The following is an easy proposition for $\text{LJ}_\text{P}{}^-$.

Proposition 5.5 *The following hold:*

1. $\mathrm{LJ_P}^- \vdash \alpha \Rightarrow \neg\neg\alpha$;
2. $\mathrm{LJ_P}^- \vdash \alpha, \beta \Rightarrow \alpha \wedge \beta$;
3. $\mathrm{LJ_P}^- \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LJ_P}^- \vdash \bigwedge\!\!\!\bigwedge \Gamma \Rightarrow \Delta$, *where* Δ *contains at most one wff.*

We also have the dual of Proposition 4.13.

Proposition 5.6 *The following hold:*

1. $\mathrm{DO}^- \vdash\ \Rightarrow \alpha \vee \neg\alpha$ *iff* $\mathrm{DO}^- \vdash \alpha \wedge \neg\alpha \Rightarrow\ $;
2. *The distributive laws hold in* DO^- + '$\alpha, \beta \Rightarrow \alpha \wedge \beta$';
3. $\mathrm{DO}^- \nvdash \alpha, \beta \Rightarrow \alpha \wedge \beta$;
4. $\mathrm{DO}^- \vdash \alpha, \neg\alpha \Rightarrow\ $ *iff* ($\mathrm{DO}^- \vdash \alpha \wedge \neg\alpha \Rightarrow\ $ *and* $\mathrm{DO}^- \vdash \alpha, \beta \Rightarrow \alpha \wedge \beta$);
5. $\mathrm{DO}^- \nvdash \alpha, \neg\alpha \Rightarrow\ $;
6. $\mathrm{DO}^- \vdash\ \Rightarrow \alpha \vee \neg\alpha$ *iff the inference rule* $\dfrac{\alpha, \Gamma \Rightarrow \delta}{\Gamma \Rightarrow \delta \vee \neg\alpha}$ *holds in* DO^-.

From Proposition 5.6(1) and (4), we obtain the following theorem.

Theorem 5.7 *If* $\mathrm{DO}^{-e} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathrm{DO}^{-c} \vdash \Gamma \Rightarrow \Delta$, *where* Δ *contains at most one wff.*

Using Proposition 5.5, we can easily prove the dual of Theorem 4.17.

Theorem 5.8 $\mathrm{DO}^{-c} = \mathrm{LJ_P}^{-d}$ *; that is,* $\mathrm{DO}^{-c} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{LJ_P}^{-d} \vdash \Gamma \Rightarrow \Delta$, *where* Δ *contains at most one wff.*

Moreover, we can show that $\mathrm{DO}^{-c} = \mathrm{LK_P}$. For this, we need a preliminary work.

Proposition 5.9 *The following hold.*

1. *Let* Δ *consist of at most one wff. Then,* $\mathrm{DO}^- \vdash \bigwedge\!\!\!\bigwedge \Gamma \Rightarrow \Delta$ *if* $\mathrm{DO}^- \vdash \Gamma \Rightarrow \Delta$.
2. DO^- +'$\alpha, \beta \Rightarrow \alpha \wedge \beta$' $\vdash \Gamma \Rightarrow \bigwedge\!\!\!\bigwedge \Gamma$, *where* Γ *is not empty.*
3. $\mathrm{DO}^{-c} \vdash \neg\alpha \vee \beta, (\delta \vee \alpha) \wedge (\lambda \vee \neg\beta) \Rightarrow \delta \vee \lambda$.

We now define '$\alpha \to \beta$' by '$\neg\alpha \vee \beta$' in DO^{-c}. Then we can prove the equivalence of DO^{-c} and $\mathrm{LK_P}$.

Theorem 5.10 $\mathrm{DO}^{-c} \vdash \Gamma \Rightarrow \bigvee\!\!\!\bigvee \Delta$ *if and only if* $\mathrm{LK_P} \vdash \Gamma \Rightarrow \Delta$.

Proof ($\Rightarrow$) Trivial.

($\Leftarrow$) We need to show that all the inference rules of $\mathrm{LK_P}$ hold in DO^{-c}. We here consider the rules $\neg$L, $\neg$R, $\to$L, and $\to$R of $\mathrm{LK_P}$. We set $\delta = \bigvee\!\!\!\bigvee \Delta$ and $\lambda = \bigvee\!\!\!\bigvee \Lambda$.

$$(\neg\mathrm{L}) \quad \frac{\Gamma \Rightarrow \delta \vee \alpha}{\neg\alpha, \Gamma \Rightarrow \delta}$$

Using Proposition 5.6(4), we have

$$\dfrac{\dfrac{\Gamma \Rightarrow \delta \vee \alpha \qquad \dfrac{\delta, \neg\alpha \Rightarrow \delta \wedge \neg\alpha \qquad \dfrac{\alpha, \neg\alpha \Rightarrow}{\alpha, \neg\alpha \Rightarrow \delta \wedge \neg\alpha}}{\delta \vee \alpha, \neg\alpha \Rightarrow \delta \wedge \neg\alpha}}{\neg\alpha, \Gamma \Rightarrow \delta \wedge \neg\alpha} \qquad \dfrac{\delta \Rightarrow \delta}{\delta \wedge \neg\alpha \Rightarrow \delta}}{\neg\alpha, \Gamma \Rightarrow \delta}$$

$$(\neg\mathrm{R})\quad \frac{\alpha, \Gamma \Rightarrow \delta}{\Gamma \Rightarrow \delta \vee \neg\alpha}$$

This is from Proposition 5.6(1), (4), and (6).

$$(\rightarrow \mathrm{L})\quad \frac{\Gamma \Rightarrow \delta \vee \alpha \qquad \beta, \Pi \Rightarrow \lambda}{\neg\alpha \vee \beta, \Gamma, \Pi \Rightarrow \delta \vee \lambda}$$

From the upper sequent $\Gamma \Rightarrow \delta \vee \alpha$, we have $\Gamma, \Pi \Rightarrow \delta \vee \alpha$ and then $\bigwedge\!\!\bigwedge \Gamma \wedge \bigwedge\!\!\bigwedge \Pi \Rightarrow \delta \vee \alpha$, by Proposition 5.9(1). From $\beta, \Pi \Rightarrow \lambda$, we have $\Pi \Rightarrow \lambda \vee \neg\beta$, by $(\neg\mathrm{R})$ above, from which we obtain $\Gamma, \Pi \Rightarrow \lambda \vee \neg\beta$ and then $\bigwedge\!\!\bigwedge \Gamma \wedge \bigwedge\!\!\bigwedge \Pi \Rightarrow \lambda \vee \neg\beta$, by Proposition 5.9(1), again. Then, using Proposition 5.9(2) and (3), we have this:

$$\dfrac{\Gamma, \Pi \Rightarrow \bigwedge\!\!\bigwedge \Gamma \wedge \bigwedge\!\!\bigwedge \Pi \qquad \dfrac{\dfrac{\bigwedge\!\!\bigwedge \Gamma \wedge \bigwedge\!\!\bigwedge \Pi \Rightarrow \delta \vee \alpha \qquad \bigwedge\!\!\bigwedge \Gamma \wedge \bigwedge\!\!\bigwedge \Pi \Rightarrow \lambda \vee \neg\beta}{\bigwedge\!\!\bigwedge \Gamma \wedge \bigwedge\!\!\bigwedge \Pi \Rightarrow (\delta \vee \alpha) \wedge (\lambda \vee \neg\beta)} \qquad \neg\alpha \vee \beta, (\delta \vee \alpha) \wedge (\lambda \vee \neg\beta) \Rightarrow \delta \vee \lambda}{\bigwedge\!\!\bigwedge \Gamma \wedge \bigwedge\!\!\bigwedge \Pi, \neg\alpha \vee \beta \Rightarrow \delta \vee \lambda}}{\neg\alpha \vee \beta, \Gamma, \Pi \Rightarrow \delta \vee \lambda}$$

$$(\rightarrow \mathrm{R})\quad \frac{\alpha, \Gamma \Rightarrow \delta \vee \beta}{\Gamma \Rightarrow \delta \vee (\neg\alpha \vee \beta)}$$

We can obtain this rule by $(\neg\mathrm{R})$ above. □

5.2 DO and DOM We now define the dual system of LO, which we will call the *Dual Orthologic* (DO, for short).

Definition 5.11 Let DO be the system obtained from LO as follows:

1. Axioms and structural inference rules are the same as those of LO.
2. Logical rules

$$\neg\mathrm{L}\quad \frac{\Gamma \Rightarrow \alpha}{\neg\alpha \Rightarrow \neg\Gamma} \qquad\qquad \neg\mathrm{R}\quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}$$

$$\wedge\mathrm{L}\quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \quad \frac{\beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \qquad \wedge\mathrm{R}\quad \frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$$

(Γ contains at most one wff.)

$$\vee\mathrm{L}\quad \frac{\alpha, \Gamma \Rightarrow \Delta \qquad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta} \qquad \vee\mathrm{R}\quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \quad \frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

The following proposition is almost trivial.

Proposition 5.12 $\mathrm{DO} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{DO} \vdash \Gamma \Rightarrow \bigvee\!\!\bigvee \Delta$.

Then we have the dual of Theorem 4.16.

Theorem 5.13 $\mathrm{DO} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{DO}^{-e} \vdash \Gamma \Rightarrow \bigvee\!\!\bigvee \Delta$.

Finally, we consider the orthomodular logic, or the quantum logic. Nishimura added the following rule of inference to his system GO to obtain GOM:

$$\text{0-modular}\quad \frac{\neg\beta \Rightarrow \neg\alpha \qquad \neg\alpha, \beta \Rightarrow}{\neg\alpha \Rightarrow \neg\beta}$$

We use the following rule OM, which is equivalent to his 0-modular rule in GO or in LO:

$$\text{OM} \quad \frac{\alpha \Rightarrow \beta \qquad \neg\alpha, \beta \Rightarrow}{\beta \Rightarrow \alpha}$$

We then define the dual of the rule OM, which we will call "Dual-OM":

$$\text{Dual-OM} \quad \frac{\Rightarrow \neg\alpha, \beta \qquad \beta \Rightarrow \alpha}{\alpha \Rightarrow \beta}$$

Definition 5.14

1. LOM (Orthomodular Logic) is the system LO + OM.
2. DOM (Dual Orthomodular Logic) is the system DO + Dual-OM.

By Proposition 4.5, it is clear that LOM = GOM. Also, by the translation mapping *, it is easy to see that DOM is the dual of LOM. Thus we have the following.

Theorem 5.15 *The following hold:*

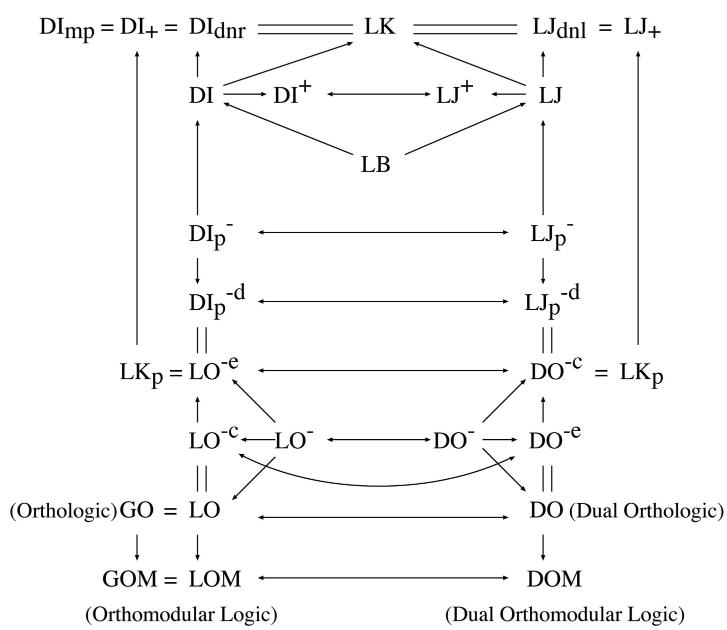
1. LOM = GOM*;*
2. LOM $\vdash \Gamma \Rightarrow \Delta$ *iff* DOM $\vdash (\Gamma \Rightarrow \Delta)^*$.

6 Concluding Remarks

DI^+ and LJ^+ can be regarded as isomorphic systems by the translation mapping (isomorphism) *. With this mapping, DI and LJ are related dually, though not completely. When we consider the relation between DI and LO, the two systems $DI_P{}^-$ and LO^- become very important and some sequents like '$\Rightarrow \alpha, \neg\alpha$', '$\alpha \wedge \neg\alpha \Rightarrow$', and '$\alpha \vee \beta \Rightarrow \alpha, \beta$' play important roles for connecting DI with LO. Especially, the last one can be regarded as the key sequent for the distributive laws.

This paper contains only syntactical considerations of various logics in the form of sequent calculus and we also need to consider their model theoretic properties and relations. We point out one thing here. As the translation mapping * shows, the logical connective $\rightarrow$ of LJ corresponds to the pseudo-difference operator $\dot{-}$ of DI^+, which is not in DI. This means that the "dual" algebra of a complete Heyting algebra, that is, the complete Brouwerian algebra is not a proper model for DI. For the Brouwerian algebras, see McKinsey and Tarski [8] and Goodman [6]. Rauszer [11] and Goré [7] might also be useful.

We can finally indicate the relationship among the major systems considered in this paper, diagrammatically. In the diagram below, 'A $\rightarrow$ B' indicates that the system B is an extension of A; 'A = B' indicates that A is equivalent to B ; and 'A $\longleftrightarrow$ B' indicates that A and B are the duals according to the translation mapping *. The system LB in the diagram is the sequent calculus obtained from LK by restricting sequents so that both the antecedent and the succedent of a sequent are singular. The cut elimination theorem holds for LB, which is shown in Aoyama [1]. Similar diagrams of logical systems have been presented in [17], [10], Sambin et al. [13], Chiara and Giuntini [2], and so on. They show interesting relations among various logical systems and seem to suggest very promising ideas of research both syntactical and semantical.



The Relationship among the Major Systems in this Paper

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Faculty of Humanities
Tokaigakuen University
Nakahira 2-901
Tempaku ward, Nagoya City 468-8514
JAPAN
aoyama@tokaigakuen-c.ac.jp