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Leśniewski's Strategy and Modal Logic

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Abstract Leśniewski used formal systems and artificial languages to capture and explore concepts expressed in ordinary language. This strategy, which is the appropriate strategy for the philosophical logician, is illustrated by developing a system of modal logic to investigate the concept of analyticity. The ordinary concept of analyticity applies only to sentences; it is de dicto. This is a shortcoming of the ordinary concept, which is overcome by extending the concept to constitute the corresponding de re concept. A semantic account and a deductive system are developed for a first-order language with identity to capture this concept. The system is shown to be sound and complete with respect to the semantic account.

1 The strategy In [4] Leśniewski attacks the very idea of a set, and writes off set theory as an illicit enterprise. He cites remarks about sets made by Cantor, Frege, Hausdorff, Sierpinski, Fraenkel, Zermelo, and Russell, and finds fault with all of them. There are two principal faults that Leśniewski fastens on: (1) The existence of the null set is simply postulated. The null set is an invention, a fiction of mathematicians. Sets are fictitious entities. (2) Attempts to construe a set as the extension of an idea are unintelligible. Talk of sets doesn't even make good sense.

Being fictitious is different from being incoherent, which is the status of objects which can only be "described" in an unintelligible fashion. But both of Leśniewski's criticisms exemplify a common theme: He believes there is a certain strategy that is appropriate in developing logical and mathematical theories, which the advocates (the perpetrators) of set theory have not pursued.

It is not clear to me what Leśniewski accepted, or would have accepted, as a principle of significance or intelligiblity. But he was surely wrong in thinking that talk about sets lacks significance. My own view is that an expression is significant if it belongs to a linguistic or conceptual system whose expressions are linked by various semantic relations, and if the system can be used to describe experience or to describe things linked in identifiable ways to objects we experience. The expressions of set theory certainly qualify; Leśniewski was misguided in rejecting set theory. His own foundation for mathematics is not satisfactory; it is as good as that presented in *Principia Mathematica* (minus its nonlogical

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axioms). And Leśniewski's system, as he presents it, is not characterized by the confusion that we sometimes find in *Principia Mathematica*. But this is not good enough.

However, I want to endorse Leśniewski's strategy for developing logical systems. Leśniewski developed artificial languages (and formal systems) with the explicit goal of capturing features of natural languages. He wanted to incorporate selected features in artificial languages, and to explore the consequences of these features for them. He was trying to come up with a perspicuous language for describing the world. Leśniewski used artificial languages to explore concepts expressed or embodied in ordinary language. Doing this can help us get a better understanding of these concepts; it can reveal their strengths and weaknesses. And artificial languages provide the resources for repairing the shortcomings of ordinary concepts.

The founders of set theory may have thought they were investigating ordinary concepts (or ordinary mathematical concepts), and analyzing the natures of objects falling under these concepts. But it seems more accurate to deny that our concept of a set is an ordinary concept. Set theory postulates the existence of sets, characterizes them, and develops the consequences of this postulation and characterization. Sets are what set theory says they are—we have no independent access to the set-theoretic universe, which would enable us to judge a certain formulation to be inaccurate.

In contrast, Leśniewski constructed the artificial languages' protothetic and ontology. These incorporate what he took to be essential features of natural languages, and they constitute the logical skeletons of languages suited for talking about things in the world. (Leśniewski's languages need to be supplemented with nonlogical expressions before they can be used to make useful nonlogical statements.) In his system of mereology, Leśniewski explored the ordinary concept of a collective class (to use Sobocinski's terminology). Leśniewski initially hoped to use mereology to provide an acceptable substitute for set theory, but his investigations revealed that mereological concepts are not suited for that task. Subsequent theorists have tried to supplement mereology with additional concepts or postulates, but this is not widely regarded as a promising approach.

I think that the artificial languages that Leśniewski produced in carrying out his strategy are something of a disappointment. Protothetic is unnecessarily complicated, and ontology is inferior to first-order languages in capturing the important features of natural languages. First-order languages are not entirely adequate. (In [1] and [3] I have supplemented a first-order language in a way which I think makes it more adequate.) But in a first-order language, the category of singular terms is a fundamental syntactic category. In contrast, the corresponding fundamental category of ontology is that of common nouns; singular terms are regarded as common nouns that label a single object. This misrepresents the syntax of natural languages.

I am not applauding the results that Leśniewski achieved by employing his strategy. I am applauding the strategy itself. I don't think that this strategy is appropriate for mathematics; mathematicians may need to invent new concepts and postulate new objects to achieve suitable mathematical ends. There is no reason to hold it against sets that the concept of a set is not one of our ordinary concepts. But it is appropriate for a philosophical logician, as opposed to a mathematical logician, to construct artificial languages that incorporate selected features of natural ones, and to use artificial languages to investigate concepts expressed or embodied in natural languages.

Natural languages are highly complex, and have a multiplicity of features. In an artificial language we can separate these, and explore them one (or two, etc.) at a time. For example, natural languages have plural forms, while most artificial logical languages do not. This simplifies the artificial languages, and makes them appropriate for exploring singular but not plural constructions. Figuring out how to express a given concept in a relatively simple artificial language—which may have to be enriched for this purpose—can provide a much deeper understanding of the concept than is achieved by other means.

I believe that Leśniewski devised the appropriate strategy for philosophical logicians. But arguments pro and con this claim are likely to be inconclusive; the best argument for the claim is actually to employ the strategy, and get someplace with it. Unfortunately, Leśniewski's own achievements do not make a convincing case for his strategy. In the remainder of this paper I will use his strategy to deal with modal logic, as a way of illustrating the fruitfulness of the strategy.

2 Modal concepts In this century, modal logic really "took off" after Kripke's development of set-theoretic semantics for modal languages. The key elements in these semantic accounts have come to be called possible worlds, which are used to determine when a sentence is necessary or possible (are used to explain what it is for a sentence to be necessary or possible). While modal logic with set-theoretic semantics is interesting, I do not find it enlightening. I do not feel that I now know what necessity and possibility really are. Instead, I am perplexed about the right, or best, way to relate my ordinary concepts of necessity and possibility to these formal constructions.

It does not seem reasonable to postulate the existence of entities in the context of modal logic. The situation is not like that of set theory, where sets help to understand and explain numbers, order, and measurement. And it is not like postulating entities in scientific theories, where nonobservable items help explain observable ones. In the modal sphere, we have ordinary concepts of possibility and necessity, but there are no readily identifiable modal phenomena. It is unreasonable to postulate entities to explain our modal concepts; we need to understand these concepts, not reduce them to items we invent rather than discover.

To use modal logic to help understand our ordinary modal concepts it is not necessary that we come up with definitions for these concepts. It is sufficient if we can use modal logic to determine how the concepts "behave"—how they are related to each other and to other concepts. To accomplish this, it is helpful to reflect on ordinary concepts of necessity and possibility. We do not have just one concept of necessity, or one concept of possibility, we have several. I will call a concept of possibility and the corresponding concept of necessity a *modal concept pair*. In ordinary English, we actually use the words 'possible', 'possibility', etc. to express different concepts of possibility, but we do not always use 'necessary', 'necessity', etc. for the corresponding concepts of "necessity". I will begin by considering what I think are the principal concepts of possibility. (Each principal concept is the paradigm case of a related family of concepts.) I recognize four principal concepts of possibility; two are epistemic, and two I will call metaphysical.

Epistemic possibility comes in absolute and relative versions. A sentence is absolutely epistemically possible if its truth is not ruled out by its meaning. For this concept, it is appropriate that being ruled out be understood epistemically. It is not clear to me whether an epistemic conception must be understood deductively; deductive conceptions would involve being able to deduce a contradiction from incompatible sentences (for example, being able to deduce this by specified means, or by any acceptable means). If a truth-conditional conception could count as epistemic, it would still seem that we must be able, in principle, to determine that incompatible sentences cannot be true together. Some philosophers would use the expression 'logical possibility' for absolute epistemic possibility, but I prefer to use 'logically possible' for sentences whose truth is not ruled out by their logical form, and I understand logical form as in [3]. My concept of logical possibility belongs to the family whose paradigm member is absolute epistemic possibility. The concept of necessity that corresponds to absolute epistemic possibility is analytic truth. (And logical truth corresponds to logical possibility.)

Relative epistemic possibility is relative to the knowledge of a person (or community) at a given time. A sentence is relatively epistemically possible if its truth is not ruled out by current knowledge; a sentence is relatively epistemically necessary if it follows from current knowledge. Different conceptions of following from correspond to different conceptions of incompatibility, yielding different versions of relative epistemic necessity.

The metaphysical concept pairs also admit of an absolute and a relative version. Relative metaphysical possibility is relative to the way the world is at present. A sentence is possible in this sense if its truth is not ruled out by present and past states of the world. We ordinarily conceive the world to be such that the present and past are fixed, or determined, and that some things in the future are determined, but not everything. We think of the future as containing some alternative possibilities. Different alternatives cannot all be realized. Some one of them will, yet it is not now determined which will be. But even if everything in the future were determined, we could still employ this concept, though it would not be very interesting, for possibility and necessity would coincide. In considering what sentences are and are not ruled out by present and past states of the world, it is no longer appropriate to use an epistemic conception of incompatibility. I am not sure what is the appropriate way to understand this incompatibility; I will content myself with saying that present and past states *preclude* the sentences they rule out.

To understand absolute metaphysical possibility and necessity, we must realize that we ordinarily conceive the world to contain individuals of various kinds, and that we conceive of them as having natures which determine what they are like and, in some respects, how they behave. A sentence is absolutely metaphysically possible if its truth is not precluded by the natures of the relevant individuals. This sense of possibility is involved in such sentences as the following:

For most fish, it is not possible to live on land.
A person can have a heart transplant and lead an active life.

A sentence is necessary in the corresponding sense if its truth is required by the natures of the individuals involved.

3 Capturing the concepts One way to explore concepts is by developing a deductive system. In an artificial or a natural language one can incorporate expressions for the relevant concepts, and then present rules and axioms which indicate how these concepts are related to one another. This was Leśniewski's procedure, and it was the practice of those working in modal logic before the advent of set-theoretic semantics for modal logic. This technique is still useful, but when used in isolation it does not always yield the desired insight. It is difficult to determine whether a deductive system captures all the important features of a concept or set of concepts; C. I. Lewis developed a number of systems of modal logic, but could not determine which, if any, was the right one. This is partly due to his not reflecting sufficiently on the different modal concept pairs, and deciding which pair he wanted to capture. It is also due to relying exclusively on the formulation and (deductive) development of deductive systems.

When we supplement a deductive system with an account of the truth conditions of the sentences in an artificial language we very much increase our understanding of the language and the concepts it expresses. The failure to achieve a good "fit" between truth conditions and deductive system can reveal that a concept has not been captured, and it may also show what is required to capture it. The standard semantics for systems of modal logic have not been developed with ordinary modal concepts in mind, and they do not provide much in the way of conceptual clarification. But if we begin with these concepts, and develop truth conditions for sentences which contain expressions of these concepts, we can increase our understanding of our modal concepts.

In giving an account of the truth conditions of sentences in an artificial language it is convenient to follow the lead of Leśniewski's student Tarski, and make use of set-theoretic techniques. This does not show that the concepts we are trying to capture involve set-theoretic notions. Set theory provides the resources for constructing a mathematical model (in the model airplane sense of 'model') that illustrates the truth conditions of sentences. The set-theoretic truth conditions are a "scaled down" version of the real thing.

In [2] I gave an account of the truth conditions of a modal language that exemplifies Leśniewski's strategy. However, not many people seem to have understood what I was doing, or why I would do it that way. The "idea" of my account was not to define necessity, or possibility, it was instead to get clear about how these "work". Given a modal concept pair, we can distinguish two "varieties" of truth, and two of falsity. A sentence can be necessarily true (I use T to indicate necessary truth) or contingently true—true-but-not-necessary (t). A false sentence can be impossible (F) or false-but-not-impossible (f). We gain an understanding of the way our modal concepts work when we devise a means to assign the values T, t, F, f to sentences in a language, and our distribution of values is intuitively seen to be correct. (Having the right intuition simply shows that we have grasped the appropriate concepts: that we know how to use certain English—or French, or Italian, etc.—expressions.)

Different modal concept pairs work differently, but the different pairs share some common features; a "minimal" deductive system-semantic account will capture these common features. If L_0 is a language with (unspecified) atomic sentences and these logical symbols: $\sim$, $\vee$, &, $\supset$, $\Box$, $\Diamond$, the following matrix shows the (intuitively correct) behavior of the four modal concept pairs:

A	B	$\sim A$	$(A \vee B)$	$(A \,\&\, B)$	$(A \supset B)$
T	T	F	T	T	T
T	t	F	T	t	t
T	f	F	T	f	f
T	F	F	T	F	F
t	T	f	T	t	T
t	t	f	T,t	t	T,t
t	f	f	T,t	f,F	f
t	F	f	t	F	f
f	T	t	T	f	T
f	t	t	T,t	f,F	T,t
f	f	t	f	f,F	T,t
f	F	t	f	F	t
F	T	T	T	F	T
F	t	T	t	F	T
F	f	T	f	F	T
F	F	T	F	F	T

For some assignments of values to the components of a sentence two values are possible for the complex; the connectives are not completely functional for the four values. A completely safe modal matrix is the following:

A	$\Diamond A$	$\Box A$
T	T,t	T,t
t	T,t	f,F
f	T,t	f,F
F	f,F	f,F

Not all assignments of values consistent with the matrices are intuitively satisfactory. If A and B are distinct atomic sentences, then this assignment should be allowed:

A	B	$(A \vee B)$
t	t	t

But a sentence $(A \vee \sim A)$ should have value T—though the matrices would allow this sentence to have value t. A sentence that is necessarily true is one that "has to" be true; it cannot help being true. Any sentence that the matrices will not allow to be false is one that is necessarily true. But once we restrict our attention to valuations that assign T to all such sentences, we find additional sentences that never come out false. These sentences are also necessarily true.

A systematic account of valuations that do the right thing is below:

A *0th-level T_0-valuation* is a function $\mathcal{V}$ that assigns one of T, t, F, f to each sentence of L_0, in a manner consistent with the matrices above.

Let $\mathcal{V}$ be an nth-level T_0-valuation. $\mathcal{V}$ is an *n + 1st-level T_0-valuation* if $\mathcal{V}$ assigns T to every sentence A which is true (T or t) for every nth-level T_0-valuation.

$\mathcal{V}$ is a *T_0-valuation of L_0* if $\mathcal{V}$ is an nth-level T_0-valuation for every $n \geq 0$.

The system T_0 is a natural deduction system for L_0 that employs tree proofs. It has a sound and complete set of rules for the nonmodal connectives, and these rules for $\Box$:

$\Box$ Elimination $\dfrac{\Box A}{A}$

$\Box$ Introduction $\dfrac{A}{\Box A}$ A is the conclusion of a (sub)proof with no uncancelled hypotheses.

(T) $\dfrac{\Box(A \supset B) \quad \Box A}{\Box B}$

The diamond is defined in terms of the box, so it does not get rules of its own. In [2] I showed that T_0 is sound and complete for T_0-valuations.

With respect to the epistemic modal concept pairs the semantic account above represents a conjecture. Epistemic criteria were not used in determining when an nth-level valuation is also an n + 1st-level valuation; but T_0 is deductively complete, and deduction is an epistemic procedure. The combined deductive system-semantic account is genuinely a minimal modal logic.

If we modify the modal matrix to yield the following, we get semantic accounts that fit the systems S4 and S5:

S4

A	$\Diamond A$	$\Box A$
T	T	T
t	T,t	f,F
f	T,t	f,F
F	F	F

S5

A	$\Diamond A$	$\Box A$
T	T	T
t	T	F
f	T	F
F	F	F

(The appropriate matrices are used to define $S4_0$-valuations and $S5_0$-valuations.) With respect to the pair absolute epistemic possibility/analyticity, I think the S5 matrix is correct. The deductive system $S5_0$ is obtained from T_0 by adding this rule:

$$\textbf{(S5)} \quad \frac{\Diamond A}{\Box \Diamond A}$$

4 Extending a concept L_0 is a very simple language; developing semantic accounts and deductive systems for L_0 gives only modest insights into the various modal concept pairs. With respect to the pair semantic possibility/analyticity (absolute epistemic possibility is semantic possibility), the interesting cases of analyticity depend on nonlogical expressions occurring in sentences. Our logical approach does not give us access to these, but the logical approach does provide a framework within which we can begin to explore the modal concept pairs. This approach must be supplemented before it will yield a more complete understanding.

Some readers may think that arguments by Quine and others have shown that the logical account of analyticity is the best we can do. Those arguments only show that analyticity belongs to a class of semantic concepts that are interdefinable, but that cannot be defined in terms of concepts (or by means of techniques) that (latter day) empiricists regard as privileged. While the arguments are correct, they do not discredit the semantic concepts. We have no reason to adopt a principle of significance that requires expressions to be definable in terms of some privileged class of concepts (expressions) or experiences, and we have no reason for thinking that there is not a lot to be learned about analyticity.

Even though a modal concept pair is imperfectly understood on the basis of logic alone, a successful treatment of a quantificational language will yield considerably more understanding than does a treatment of L_0. This claim is true in general, but it may not seem true in the case of semantic possibility/analyticity. The great difficulties in developing semantic accounts for modal logic arise in determining what it means for an open formula to have modal values with respect to individuals as values of its free variables, and how these values should be awarded. (These constitute the problem of making sense of *de re* modality.) It is clear that for many of our ordinary concepts of possibility we do allow that a predicate (or a property) can be possible—or not—for an individual. But the pair semantic possibility/analyticity are not really ordinary concepts, they are philosophical ones. And as far as the standard concept of analyticity is concerned, only sentences (propositions) can be analytic. With respect to this concept of analyticity the question whether '$(A(x) \vee \sim A(x))$' is analytic for 2 as a value of x is senseless.

However, this restriction of the ordinary concept of analyticity is a shortcoming of the ordinary concept. With respect to relative epistemic possibility (that is, possibility "for all we know") it makes sense to speak of a property being possible, or not, for an individual. Every sentence which is absolutely epistemically necessary—which is analytic—is also always relatively epistemically necessary. The "minimum" amount of knowledge we can allow a person to have is knowledge of the language she speaks. For such a person (who knew as little as possible), the relative and absolute epistemic modal pairs would amount to nearly the same. If there are *de re* relative modalities, then we also need *de re* absolute modalities; the received concept of analyticity needs to be extended to constitute a suitable "backup" for relative epistemic necessity.

Before I consider how we should extend the received concept, I must acknowledge that there is more than one such concept. I am not going to attempt a survey of these concepts at this time, and give reasons for focusing on one rather than another. I'll just pick one. A rough idea of the concept I have in mind is given by this epistemic definition: An analytic sentence is one that can be determined to be true on the basis of understanding it. (Note that analytic sentences are not required to be self-evident. It does not have to be easy to determine that an analytic sentence is true.) This definition is pretty rough, but it will serve at present. If we are successful in using logical techniques to capture the concept, we will have made headway in moving beyond the rough definition.

Given our ordinary concept of analyticity which applies to sentences, there are different ways this concept may be extended to yield a concept suited to a first-order language (a concept which allows *de re* modalities). The different extended concepts are all legitimate concepts, and can be investigated by developing systems of modal logic. But the different extended concepts do not all have equally valid claims to be the "repaired" concept of analyticity. They do not all constitute "limiting cases" of relative epistemic necessity.

In [2] I formulated a semantic account-deductive system that was intended to express the appropriate extended concept of analyticity. However, the completeness proof in that paper contains an error (Lemma 3 is incorrect). The deductive system is not complete for the semantic account presented, and that semantic account is not satisfactory. In the following sections I will correct these failings.

5 De re analyticity The language L is a first-order language with identity containing the same operators and connectives as L_0, and these additional expressions:

Individual variables: $x, y, z, x_1, \ldots$
Individual constants: $a, b, c, a_1, \ldots$
For every $n > 0$, n-place predicates: $F^n, G^n, H^n, F_1^n, \ldots$.

The universal quantifier $(\forall\alpha)$ is primary, the existential quantifier is a defined symbol; vacuous quantification is not permitted in L.

In developing a logical system (deductive system and statement of truth conditions) for L, we are trying to capture/express a certain concept which applies to natural languages. However, the language L is unlike natural languages in several respects; because of this, modal concepts suited to L will differ from the corresponding concepts which apply to, say, English. The differences are especially prominent as concerns singular terms. In L it is given as a matter of convention that every individual constant names a real individual: For any constant α, it is analytic that α exists. It is also given as a matter of convention that the domain of interpretation is nonempty.

I will treat L as much as possible as if it were a natural language. Some individual constants will be thought of as meaningful and others will not be (the latter are *purely proper names*). For names with meanings it is given as a matter of convention that exactly one individual satisfies the meaning of a name. I will also allow some predicates to give access to a particular individual ('is presently a mayor of New York City'), and that for some predicates it is given as a matter of convention that they do or do not label real individuals. (For example, I think it is part of the meaning of 'dog' that dogs are real, and part of the meaning of 'unicorn' that unicorns are not.)

In presenting semantic accounts for quantificational languages I find it convenient to "pretend" that individuals in a domain can be treated like names. If A is a formula containing free occurrences of distinct individual variables $\alpha_1, \ldots, \alpha_n$, and $\rho_1, \ldots, \rho_n$ are individuals in our domain of interpretation, then $A(\alpha_1, \ldots, \alpha_n; \rho_1, \ldots, \rho_n)$ is the *pseudoformula* obtained from A by replacing the free occurrences of $\alpha_1, \ldots, \alpha_n$ (respectively) by $\rho_1, \ldots, \rho_n$. The value of $A(\alpha_1, \ldots, \alpha_n; \rho_1, \ldots, \rho_n)$ for a certain function is the same as the value of A for $\rho_1, \ldots, \rho_n$ as values of $\alpha_1, \ldots, \alpha_n$. (A pseudoformula with no free variables is a *pseudosentence*.)

With such a treatment, issues about *de re* modality reduce to problems concerning the assignment of modal values to pseudosentences. If A is a pseudowff containing free occurrences of the individual variable α and no other, and ρ is an individual in our domain, under what conditions should $A(\alpha;\rho)$ receive the value T (or F)? The pseudosentence $A(\alpha;\rho)$ should be counted as analytic just in case knowing/understanding the language L is sufficient to determine that the pseudosentence is true. Some philosophers and logicians hold that *de re* epistemic modalities depend on some form of acquaintance, but I do not. I think that the access we have to most things is indirect. It is often symbolic—through information "incorporated" in language and information expressed with language. Following Frege, I think that we know a given individual under one or more aspects. I will develop an account of truth conditions for L based on these ideas. Since this semantic account models epistemic ideas, it should not be taken to be ontologically valid.

I will construe an individual in a domain as a collection of aspects. Each aspect "gives" the whole individual, but a person can know less than all the aspects "constituting" an individual. A name or other expression will typically give access to only some aspects of an individual (it will give access to an individual through only some of its aspects). The formula A will not be analytic simply with respect to some individual as the value of the free variable α; it can be analytic with respect to this individual under one aspect, and fail to be analytic with respect to the individual under a different aspect. (It can be analytic of Smith under his aspect of mayor that he is a politician, but not under his aspect of owning the house at 174 Elm Street.) For the formula to be analytic with respect to an individual under an aspect there must be some expression which gives access to that aspect; it must be part of the meaning of some expression that there is an individual with that aspect.

In developing the truth conditions for L, I will consider three domains. $\mathcal{E}$ is a nonempty domain of aspects. $\mathfrak{D}_\mathrm{I}$ is the nonempty domain of individuals; the members of $\mathfrak{D}_\mathrm{I}$ are disjoint nonempty sets of elements of $\mathcal{E}$. In considering pseudoformulas and pseudosentences, it is not individuals in $\mathfrak{D}_\mathrm{I}$ that are treated like names, but elements of individuals in $\mathfrak{D}_\mathrm{I}$—aspects replace free variables.

$L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$ is the *pseudolanguage* obtained by expanding L with pseudosentences constructed with aspects of elements of $\mathfrak{D}_I$.

A *modal valuation* $\mathcal{V}$ of $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$ is a function which (1) determines a nonempty domain $\mathfrak{D}_{\mathcal{V}}$ whose elements are disjoint nonempty sets of aspects, where every element of $\mathfrak{D}_{\mathcal{V}}$ is a subset of an element of $\mathfrak{D}_I$; (2) assigns one of T, t, F, f to each pseudosentence of $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$; (3) divides the individual constants into two disjoint sets:

C_M (the meaningful constants),
C_{PN} (the constants with no linguistic meaning, the purely proper names)

and assigns nonempty sets of aspects to individual constants such that

(i) If $\alpha \in C_M$, then $\mathcal{V}(\alpha) \in \mathfrak{D}_{\mathcal{V}}$
(ii) If $\alpha, \beta \in C_M$, $\mathcal{V}(\alpha) \cap \mathcal{V}(\beta) \neq \varnothing$, then $\mathcal{V}(\alpha) = \mathcal{V}(\beta)$
(iii) If $\alpha \in C_{PN}$, then $\mathcal{V}(\alpha) \in \mathfrak{D}_I$.

The aspects which belong to elements of $\mathfrak{D}_{\mathcal{V}}$ are *accessible*: if ρ is an aspect of an individual in $\mathfrak{D}_I$, but not of an individual in $\mathfrak{D}_{\mathcal{V}}$, then ρ is *inaccessible*. The inaccessible aspects are not beyond human comprehension, but an understanding of L is not sufficient for determining that there is an individual with such an aspect. It takes extralinguistic information to know what is labelled by a purely proper name, but it is a matter of language if a purely proper name is purely proper.

Many modal valuations are intuitively unsatisfactory; we must narrow down the class of modal valuations to capture the desired extension of the concept of analyticity. But first we must consider the difference between accessible and inaccessible aspects. No pseudosentence containing an inaccessible aspect can be analytic (T) or contradictory (F); since we cannot know about those aspects on the basis of understanding the language, we also cannot know about those pseudosentences. We must add new matrices to those considered previously. The following matrices are for pseudosentences which contain an inaccessible aspect.

A	B	$\sim A$	$(A \vee B)$	$(A \,\&\, B)$	$(A \supset B)$
T	t	-	t	t	t
T	f	-	t	f	f
t	T	f	t	t	t
t	t	f	t	t	t
t	f	f	t	f	f
t	F	f	t	f	f
f	T	t	t	f	t
f	t	t	t	f	t
f	f	t	f	f	t
f	F	t	f	f	t
F	t	-	t	f	t
F	f	-	f	f	t

A	$\Box A$	$\Diamond A$
t	f	t
f	f	t

Let $(\forall\alpha)A$ be a pseudosentence of $L_{\langle\mathfrak{D}_I,\mathcal{E}\rangle}$. The *extended matrix conditions* for $\forall$ are as follows:

(1) $(\forall\alpha)A$ is true iff $A(\alpha;\rho)$ is true for every ρ which is an element of a member of $\mathfrak{D}_I$

(2) If $(\forall\alpha)A$ has the value T, then $A(\alpha;\rho)$ has T for every ρ which is an element of a member of $\mathfrak{D}_{\mathcal{V}}$

(3) $(\forall\alpha)A$ has the value F if $A(\alpha;\rho)$ has F for some aspect ρ.

The extended matrix conditions for $\exists$ can be derived from those for $\forall$.

For what follows, I will borrow Church's notation for substitution. The expression '$\text{Ṣ}^{\alpha}_{\beta}A|$' labels the result of replacing all *free* occurrences of α in A by β.

Let $\mathcal{V}$ be a modal valuation of $L_{\langle\mathfrak{D}_I,\mathcal{E}\rangle}$. $\mathcal{V}$ is a *0th-level S5A(nalytic)-valuation* iff

(1) $\mathcal{V}$ does not assign T or F to any pseudosentence which contains an inaccessible aspect

(2) $\mathcal{V}$ satisfies the matrices for connectives and the S5 matrix for $\Box$ (and $\Diamond$), except that the modified matrices characterize those pseudosentences which contain inaccessible aspects

(3) $\mathcal{V}$ satisfies the extended matrix conditions for $\forall$ (and $\exists$)

(4) Let A be a pseudo-wff which contains free occurrences of the individual variable γ and no others, let A contain no occurrence of $\Box$, and let ρ,σ be elements of a single member of $\mathfrak{D}_I$. Then $A(\gamma;\rho)$ is true for $\mathcal{V}$ iff $A(\gamma;\sigma)$ is true for $\mathcal{V}$

(5) Let A be as in (4), and let $\alpha \in C_{PN}$ and $\rho \in \mathcal{V}(\alpha)$. Then $\text{Ṣ}^{\gamma}_{\alpha}A|$ is true for $\mathcal{V}$ iff $A(\gamma;\rho)$ is true for $\mathcal{V}$

(6) Let A be a pseudo-wff containing free occurrences of the individual variable γ and no others, and let ρ,σ be elements of a single member of $\mathfrak{D}_{\mathcal{V}}$. Then $\mathcal{V}(A(\gamma;\rho)) = \mathcal{V}(A(\gamma;\sigma))$

(7) Let A be as in (6), and let $\alpha \in C_M$ and $\rho \in \mathcal{V}(\alpha)$. Then $\mathcal{V}(\text{Ṣ}^{\gamma}_{\alpha}A|) = \mathcal{V}(A(\gamma;\rho))$

(8) Let ρ,σ be aspects. $\rho = \sigma$ is true iff ρ,σ are elements of a single member of $\mathfrak{D}_I$. $\rho = \sigma$ has value T iff ρ,σ are elements of a single member of $\mathfrak{D}_{\mathcal{V}}$

(9) Let α be an individual constant and ρ an aspect. Then $\alpha = \rho$ is true iff there is an $X \in \mathfrak{D}_I$ such that $\mathcal{V}(\alpha) \subseteq X$ and $\rho \in X$. $\alpha = \rho$ has value T iff $\alpha \in C_M$ and $\rho \in \mathcal{V}(\alpha)$

(10) Let α,β be individual constants. $\alpha = \beta$ is true iff there is an $X \in \mathfrak{D}_I$ such that $\mathcal{V}(\alpha) \subseteq X$ and $\mathcal{V}(\beta) \subseteq X$. $\alpha = \alpha$ has value T. If α,β are distinct, then $\mathcal{V}(\alpha = \beta) = \text{T}$ iff $\alpha,\beta \in C_M$ and $\mathcal{V}(\alpha) = \mathcal{V}(\beta)$

(11) Let A be a pseudo-wff which contains free occurrences of the individual variable γ and no others, and let α,β be individual constants not occurring in A such that $\alpha,\beta \in C_{PN}$. Then $\mathcal{V}(\text{Ṣ}^{\gamma}_{\alpha}A|) = \text{T}$ iff $\mathcal{V}(\text{Ṣ}^{\gamma}_{\beta}A|) = \text{T}$.

The rationale for the various clauses should be obvious. Clause (11) is needed because any analytic sentence which contains a purely proper name cannot owe its analytic character to the meaning of that name; any other purely proper name should play the same role.

To give an inductive clause defining m + 1st-level S5A-valuations, we must consider a class of mth-level valuations. Let $\mathcal{V}$ be an mth-level S5A-valuation of $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$, $\mathfrak{D}_{I'}$ a domain of disjoint nonempty sets of elements of $\mathcal{E}$, and let $\mathcal{V}'$ be an mth-level S5A-valuation of $L_{\langle \mathfrak{D}_{I'}, \mathcal{E} \rangle}$ such that every element of $\mathfrak{D}_{\mathcal{V}}$ is a subset of an element of $\mathfrak{D}_{\mathcal{V}'}$, and for every pseudosentence A of $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$, $\mathcal{V}(A) = \mathrm{T}$ iff $\mathcal{V}'(A) = \mathrm{T}$. Then $\mathcal{V}'$ is a member of the *mth-level family of valuations determined by* $\mathcal{V}$.

Only the sentences of L and the pseudosentences of $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$ which contain accessible individuals can be understood on the basis of understanding L. As far as knowledge of the language is concerned, the inaccessible aspects might not be there. Different aspects might be there instead. In bringing in the mth-level family we consider the linguistically understood portion of $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$ to be a sublanguage of a (possibly) more inclusive linguistically understood pseudolanguage.

Let $\mathcal{V}$ be an mth-level S5A-valuation of $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$. $\mathcal{V}$ is an *m + 1st-level S5A-valuation* iff $\mathcal{V}$ assigns T to every pseudosentence A which (1) does not contain an inaccessible aspect, and (2) is true for every mth-level S5A-valuation in the mth-level family determined by $\mathcal{V}$.

$\mathcal{V}$ is an *S5A-valuation of* $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$ iff $\mathcal{V}$ is an mth-level S5A-valuation of $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$ for every $m \geq 0$.

6 Reasoning with the extended concept of analyticity In $L_{\langle \mathfrak{D}_I, \mathcal{E} \rangle}$ individual constants do not completely "reflect" the behavior of individuals or aspects of individuals; not even purely proper names do this. For example, for every individual constant α, the sentence $\alpha = \alpha$ is analytic (has value T); but if ρ is an inaccessible aspect then $\rho = \rho$ is not analytic—though (of course) it is true. And the sentence $(\forall x)(\exists y)\Box(x = y)$ might be true—its significance is that all aspects are accessible; but if α is a purely proper name we cannot infer $(\exists y)\Box(\alpha = y)$, for this is false.

For the deductive system S5A we will expand L with these *individual parameters*: 1, 2, 3, ... These occupy the same positions in formulas as individual constants. Their logical behavior is intended to reflect the behavior of aspects of individuals. We will call this expanded language L^+.

In presenting the deductive system I will use '$M(\alpha)$' to abbreviate '$(\exists y)\Box(\alpha = y)$.' If α is an individual constant, then $M(\alpha)$ is true (and has value T) iff $\alpha \in \mathrm{C_M}$. If $\alpha \in \mathrm{C_{PN}}$, then $M(\alpha)$ has value F. If α is an aspect, then $M(\alpha)$ is true (and has value T) iff α is accessible. If α is not accessible, then $M(\alpha)$ has value f.

In S5A the rules for connectives are the same as in a system of propositional logic. The rules $\Box$Elimination and (T) are taken from the propositional system $\mathrm{S5_0}$. The other rules are below.

(S5) $$\frac{\Diamond A \qquad M(m_1)\ldots M(m_r)}{\Box\Diamond A}$$ $m_1, \ldots, m_r$ are the individual parameters occurring in A.

$\Box$ Introduction $$\frac{A \qquad M(m_1)\ldots M(m_r)}{\Box A}$$ A is the conclusion of a proof with no uncancelled hypotheses. $m_1, \ldots, m_r$ are the individual parameters occurring in A.

∀ Elimination $\dfrac{(\forall\alpha)A}{\underset{\cdot}{\mathrm{S}}{}^{\alpha}_{m}A|}$ m is an individual parameter.

$\dfrac{(\forall\alpha)A}{\underset{\cdot}{\mathrm{S}}{}^{\alpha}_{\beta}A|}$ β is an individual constant. No free occurrence of α in A is within the scope of $\square$.

∀ Introduction $\dfrac{\underset{\cdot}{\mathrm{S}}{}^{\alpha}_{m}A|}{(\forall\alpha)A}$ m is an individual parameter that does not occur in A or in any uncancelled hypothesis.

= Introduction We can always introduce $\alpha = \alpha$ as a conclusion from no premises, where α is an individual constant or parameter.

= Elimination $\dfrac{\alpha = \beta \qquad A}{A'}$ A' is obtained from A by replacing occurrences of α by β, or by replacing occurrences of β by α, so long as no occurrence that is replaced is within the scope of $\square$. α,β are individual constants or parameters.

□ = Elimination $\dfrac{\square(\alpha = \beta) \qquad A}{A'}$ A' is obtained from A by replacing occurrences of α by β, or by replacing occurrences of β by α. α,β are individual constants or parameters.

M Introduction $\dfrac{\square\underset{\cdot}{\mathrm{S}}{}^{\alpha}_{m}A|}{M(m)}$ m is an individual parameter.

$\dfrac{\square(\alpha = \beta)}{M(\alpha)}$ α,β are distinct individual constants.

□M Introduction $\dfrac{M(\alpha)}{\square M(\alpha)}$ α is an individual constant or parameter.

□~M Introduction $\dfrac{\sim M(\alpha)}{\square\sim M(\alpha)}$ α is an individual constant.

~□~M Introduction $\dfrac{\sim M(m)}{\sim\square\sim M(m)}$ m is an individual parameter.

(Analytic) $\dfrac{\sim M(\beta) \qquad \sim M(\delta) \qquad \square\underset{\cdot}{\mathrm{S}}{}^{\alpha}_{\beta}A|}{\square\underset{\cdot}{\mathrm{S}}{}^{\alpha}_{\delta}A|}$ α is an individual variable occurring free in A. β,δ are individual constants not occurring in A.

The deductive system S5A is sound and complete with respect to the semantic account of the preceding section. I will outline the proofs of these results, stating the relevant lemmas and theorems, but I will not provide the proofs. These are straightforward; an interested reader can easily supply them.

For soundness, we first extend S5A to include pseudosentences of $L_{\langle \mathfrak{D}_{\mathrm{I}}, \mathcal{E}\rangle}$.

Aspects are treated like individual parameters, except that aspects are not generalized by $\forall$ Introduction.

A proof in S5A or $\mathrm{S5A}_{\langle \mathfrak{D}_{\mathrm{I}}, \mathcal{E} \rangle}$ is *standardized* if each occurrence of $\forall$ Introduction generalizes a different individual parameter, and that parameter occurs only in the subproof above the occurrence of $\forall$ Introduction that generalizes the parameter.

Lemma 1 *Let A be a pseudo-wff containing free occurrences of the individual variable α and no others, where no free occurrence of α is within the scope of $\Box$, $\mathcal{V}$ a 0th-level S5A-valuation, β an individual constant and ρ an aspect such that there is an $X \in \mathfrak{D}_{\mathrm{I}}$ for which $\rho \in X$ and $\mathcal{V}(\beta) \subseteq X$. Then $\underset{\cdot}{\mathrm{S}}^{\alpha}_{\beta} A|$ is true for $\mathcal{V}$ iff $A(\alpha;\rho)$ is true for $\mathcal{V}$.*

This is proved by induction on the length of A.

Lemma 2 *Let $\mathcal{V}$ be a 0th-level S5A-valuation, α, β aspects or individual constants such that $\alpha = \beta$ is true for $\mathcal{V}$, A a pseudosentence containing occurrences of α, and let A' be obtained from A by replacing one occurrence of α by β, where the occurrence that is replaced is not within the scope of $\Box$. Then A is true for $\mathcal{V}$ iff A' is true for $\mathcal{V}$.*

Lemma 3 *Let Γ be a proof in $S5A_{\langle \mathfrak{D}_{\mathrm{I}}, \mathcal{E} \rangle}$ from uncancelled hypotheses $A_1, \ldots, A_r$ to the conclusion B. Let each of m, n be an aspect or a parameter that is not generalized in Γ by $\forall$ Introduction. Let Γ' be obtained from Γ by replacing every occurrence of m by n. Then Γ' is a proof from $A'_1, \ldots, A'_r$ to the conclusion B', where $A'_1, \ldots, A'_r, B'$ are obtained from $A_1, \ldots, A_r, B$ by replacing every occurrence of m by n.*

Lemma 4 *Let Γ be a proof in S5A or $S5A_{\langle \mathfrak{D}_{\mathrm{I}}, \mathcal{E} \rangle}$. Then there is a standardized proof of the same result.*

For evaluating pseudo-sentences occurring in proofs we will employ valuations $\mathcal{V}$ and *assignments* $\mathfrak{A}$. An assignment $\mathfrak{A}$ is a function which assigns aspects to individual parameters. The value of a pseudosentence A for $\mathcal{V}$ and $\mathfrak{A}$ is the same as the value of A' for $\mathcal{V}$, where A' is obtained from A by replacing every parameter m by $\mathfrak{A}(m)$.

The *rank* of a proof is the number of occurrences of inference figures it contains.

Lemma 5 *Let Γ be a rank m proof in $S5A_{\langle \mathfrak{D}_{\mathrm{I}}, \mathcal{E} \rangle}$ from uncancelled hypotheses $A_1, \ldots, A_r$ to the conclusion B, $\mathcal{V}$ an mth-level S5A-valuation and $\mathfrak{A}$ an assignment of aspects to the parameters in Γ. Let each of $A_1, \ldots, A_r$ be true for $\mathcal{V}$ and $\mathfrak{A}$. Then B is true for $\mathcal{V}$ and $\mathfrak{A}$.*

This is proved by induction on m.

Theorem 1 *The deductive systems $S5A_{\langle \mathfrak{D}_{\mathrm{I}}, \mathcal{E} \rangle}$ are sound. The system S5A is sound.*

For the completeness proof, let $\mathcal{E} = \{1, 2, \ldots; d_1, d_2, \ldots\}$. L^{EVEN} is L with the even numerals added to serve as parameters.

Let X be a set of sentences of L^{EVEN} that is consistent with respect to S5A when the even numerals serve as parameters, and let X be extended to a maximal consistent set X^*.

Let the sentences of L^+ that have the form $(\exists\alpha)A$ be enumerated: $(\exists\alpha_1)A_1$, $(\exists\alpha_2)A_2, \ldots$. Let g be a 1-1 function from the positive integers to the odd numerals such that $g(i)$ does not occur in any of $(\exists\alpha_1)A_1, \ldots, (\exists\alpha_i)A_i$. Let W be the set of all sentences $((\exists\alpha_i)A_i \supset \text{Ṣ}^{\alpha_i}_{g(i)}A_i|)$.

Lemma 1 *The set $X^* \cup W$ is consistent.*

Let $X^* \cup W$ be extended to a maximal consistent set Y of sentences of L^+.

Lemma 2 *The set Y is instantially sufficient i.e., if $(\exists\alpha)A \in Y$, then for some parameter m, $\text{Ṣ}^{\alpha}_{m}A| \in Y$.*

For each parameter m let $I(m) = \{n: n \text{ is a parameter and } m = n \in Y\}$. Then $\mathfrak{D}_{\mathrm{I}} = \{I(m): m \text{ is a parameter}\}$.

Let $\mathcal{V}$ be the function such that

(1) If A is a sentence of L^+, then $\mathcal{V}(A) = \mathrm{T}$ iff $\Box A \in Y$
$\mathcal{V}(A) = \mathrm{t}$ iff $A \in Y$, $\Box A \notin Y$
$\mathcal{V}(A) = \mathrm{f}$ iff $A \notin Y$, $\Box{\sim}A \notin Y$
$\mathcal{V}(A) = \mathrm{F}$ iff $\Box{\sim}A \in Y$.

(2) For each parameter m such that $M(m) \in Y$, let $A(m) = \{n: n \text{ is a parameter and } \Box(m = n) \in Y\}$. Then $\mathfrak{D}_{\mathcal{V}} = \{A(m): m \text{ is a parameter}\}$.

(3) $\mathrm{C_M} = \{\alpha: \alpha \text{ is an individual constant and } M(\alpha) \in Y\}$, $\mathrm{C_{PN}} = \{\alpha: \alpha \text{ is an individual constant and } M(\alpha) \notin Y\}$.

(4) If $\alpha \in \mathrm{C_M}$, $\mathcal{V}(\alpha) = \{m: m \text{ is a parameter and } \Box(\alpha = m) \in Y\}$, if $\alpha \in \mathrm{C_{PN}}$, $\mathcal{V}(\alpha) = \{m: m \text{ is a parameter and } \alpha = m \in Y\}$.

Lemma 3 *The elements of both $\mathfrak{D}_{\mathrm{I}}$ and $\mathfrak{D}_{\mathcal{V}}$ are disjoint nonempty sets of individuals from $\mathcal{E}$.*

Lemma 4 *Each element of $\mathfrak{D}_{\mathcal{V}}$ is a subset of an element of $\mathfrak{D}_{\mathrm{I}}$.*

Lemma 5 *Let α be an individual constant. If $\alpha \in \mathrm{C_M}$, then $\mathcal{V}(\alpha) \in \mathfrak{D}_{\mathcal{V}}$. If $\alpha \in \mathrm{C_{PN}}$, then $\mathcal{V}(\alpha) \in \mathfrak{D}_{\mathrm{I}}$.*

Lemma 6 *$\mathcal{V}$ is a 0th-level S5A-valuation of $L_{\langle\mathfrak{D}_{\mathrm{I}},\mathcal{E}\rangle}$.*

Lemma 7 *Suppose $\mathcal{V}$ is an mth-level S5A-valuation of $L_{\langle\mathfrak{D}_{\mathrm{I}},\mathcal{E}\rangle}$. Then $\mathcal{V}$ is an $m + 1$st-level S5A-valuation of $L_{\langle\mathfrak{D}_{\mathrm{I}},\mathcal{E}\rangle}$.*

Lemma 8 *$\mathcal{V}$ is an S5A-valuation of $L_{\langle\mathfrak{D}_{\mathrm{I}},\mathcal{E}\rangle}$.*

Theorem 2 *The system S5A is complete for S5A-valuations.*

7 ***Conclusion*** I have explained what I take to be Leśniewski's strategy, which I think is his most important contribution to logic, since his logical systems possess only modest interest, and his painstaking treatment of syntax is more painful than enlightening. But the strategy he devised is *the* appropriate strategy for a philosophical logician to employ.

I have illustrated how Leśniewski's strategy can be applied in the area of modal logic. This strategy is not a method which solves problems mechanically; to employ the strategy successfully requires insight, imagination, and luck. However, this strategy is helpful in getting its followers to focus attention on the right issues. The strategy shows us what it takes to achieve understanding of one or another topic, and enables us to know when we have reached this goal.

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