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A Generalization to the q-Convex Case of a Theorem of Fornæss and Narasimhan

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1. Introduction

Fornæss and Narasimhan proved (in [8, Thm. 5.3.1]) that, for any complex space X, the identity $\mathrm{WPSH}(X) = \mathrm{PSH}(X)$ holds, where $\mathrm{WPSH}(X)$ denotes the weakly plurisubharmonic functions on X and $\mathrm{PSH}(X)$ denotes, as usual, the plurisubharmonic functions on X.

When X has no singularities, this identity is clear. For the singular case, however, the inclusion $\mathrm{WPSH}(X) \subseteq \mathrm{PSH}(X)$ is no longer trivial; one must find locally a plurisubharmonic extension to the ambient space of an embedding of X.

In this paper we give another proof for this identity (Theorem 3.3). It is shorter and easier and has the advantage that it can be generalized to q-plurisubharmonic functions (Theorem 4.16). However it has the disadvantage that it works only for continuous functions. The q-plurisubharmonic functions were introduced by Hunt and Murray in [10] (see also [9]), but we will call here q-plurisubharmonic what they call $(q-1)$-plurisubharmonic.

We also obtain a generalization of a theorem of Siu [16]; namely, we show (Lemma 4.18) that every q-complete subspace with corners of a complex space X admits a neighborhood in X that is q-complete with corners. This will be needed in the proof of our main result.

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2. Preliminaries

Let X be a complex space (with singularities). We denote by $\mathrm{PSH}(X)$ the plurisubharmonic functions on X. We use $\mathrm{SPSH}(X)$ to denote the *strongly plurisubharmonic functions* on X, that is, those PSH functions for which we have: for every $\theta \in \mathcal{C}_0^\infty(X,\mathbb{R})$, there exists an $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta \in \mathrm{PSH}(X)$ for $0 \le \varepsilon \le \varepsilon_0$.

We will denote by $\mathrm{WPSH}(X)$ the class of *weakly plurisubharmonic functions* on X (as they are defined in [8]), that is, the class of upper semicontinuous functions $\varphi \colon X \to [-\infty, \infty)$ such that, for any holomorphic function $f\colon \Delta \to X$

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(where Δ denotes the unit disc in $\mathbb{C}$), the composition $\varphi \circ f$ is subharmonic on Δ. We use $\mathrm{SWPSH}(X)$ to denote the *strongly weakly plurisubharmonic functions* on X, that is, those $\mathrm{WPSH}(X)$ functions for which we have: for every $\theta \in \mathcal{C}_0^\infty(X, \mathbb{R})$, there exists an $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta$ is in $\mathrm{WPSH}(X)$ for $0 \leq \varepsilon \leq \varepsilon_0$.

In our alternative proof of Fornæss–Narasimhan's theorem, we will use an extension theorem of Richberg (see [15, Satz 3.3]).

THEOREM 2.1 (Richberg). *Let X be a complex space and Y a closed complex subspace of X. Then, for every function ψ on Y that is continuous* (*resp. smooth*) *and strongly plurisubharmonic, there exist a neighborhood V of Y and a function $\tilde{\psi}$ on V that is continuous* (*resp. smooth*), *strongly plurisubharmonic, and such that $\tilde{\psi}|_Y = \psi$.*

We also shall need a theorem by Colţoiu in [3].

THEOREM 2.2 (Colţoiu). *Let X be a complex space that admits a strongly plurisubharmonic exhaustion function $\varphi\colon X \to [-\infty, \infty)$. Then X is* 1*-convex.*

REMARK 2.3. If φ in Theorem 2.2 is supposed to be real-valued, as remarked in [3], then it follows that the exceptional set of X (i.e., the maximal compact analytic subset) is empty, hence X is Stein. This had been proved before by Fornæss and Narasimhan in [8, Thm. 6.1].

3. Another Proof of Fornæss–Narasimhan's Theorem

We first prove a lemma that shows the interplay between SWPSH and SPSH functions on a complex space under certain conditions.

LEMMA 3.1. *Let Ω be an open subset of a reduced Stein space X with* $\dim X < +\infty$ *and such that Ω admits a SWPSH exhaustion function $\varphi\colon \Omega \to \mathbb{R}$. Then Ω is Stein.*

Proof. Without loss of generality, we may assume that $\varphi > 0$. The proof is by induction on $n = \dim X$.

If $n = 0$ then X has only isolated points and is therefore a manifold, so there is nothing to prove.

Suppose now that the lemma is true for all complex spaces Y with $\dim Y \leq n-1$, and let $\dim X = n$. Consider $Y = \mathrm{Sing}(X)$, the singular locus of X. We have $\dim Y \leq n-1$ and, since $\varphi|_{Y\cap\Omega} \in \mathrm{SWPSH}(Y \cap \Omega)$ is an exhaustion function for $Y \cap \Omega$, by the induction hypothesis it follows that $Y \cap \Omega$ is Stein. So $Y \cap \Omega$ admits a smooth SPSH exhaustion function, which we shall denote by ψ_1.

Now Theorem 2.1 yields a SPSH and smooth extension of ψ_1 to an open neighborhood V of $Y \cap \Omega$ in Ω, denoted by $\tilde{\psi}\colon V \to \mathbb{R}$. By shrinking V, if necessary, we can suppose that $\tilde{\psi}$ is defined in a neighborhood of $\bar{V}$ (the closure being in Ω) and that $\{x \in \bar{V} \mid \tilde{\psi}(x) \leq c\}$ is compact in $\bar{V}$ for all real numbers c.

However, since Y is a closed analytic subset of a Stein space X, there exist $f_1, \dots, f_m \in \mathcal{O}(X)$ such that $Y = \{x \in X \mid f_1(x) = \cdots = f_m(x) = 0\}$. If we define $p := \log(|f_1|^2 + \cdots + |f_m|^2)$ then $Y = \{x \in X \mid p(x) = -\infty\}$.

Let now $\chi : (0, \infty) \to \mathbb{R}$ be a smooth, convex, rapidly increasing function (to be made precise later), and define

$$\psi = \begin{cases} \max(\tilde{\psi}, \chi \circ \varphi + p) & \text{on } V, \\ \chi \circ \varphi + p & \text{on } \Omega \backslash V. \end{cases}$$

We choose χ such that:

(1) $\chi \circ \varphi + p > \tilde{\psi}$ on ∂V (the border being considered in Ω); and
(2) ψ is an exhaustion function of Ω.

These two conditions can be achieved for a suitable choice of χ, for example, in the following way.

Consider a sufficiently small open neighborhood W of $Y \cap \Omega$ in Ω such that $\bar{W} \subset V$ and such that $\tilde{\psi} > \chi \circ \varphi + p$ on $\bar{W}$. Let $(c_n)_n$ be a strictly increasing sequence of nonnegative numbers with $c_0 = 0$ and $\lim_{n\to\infty} c_n = +\infty$, and consider the relatively compact sets given by

$$A_i := \{x \in \Omega \mid c_i \leq \varphi(x) < c_{i+1}\}, \quad i \in \mathbb{N}.$$

All we need in order to satisfy our conditions (1) and (2) is the existence of a convex, smooth, and strictly increasing function $\chi : (0, \infty) \to \mathbb{R}$ that satisfies

$$\chi|_{[c_i, c_{i+1})} > \max(M_i, c_{i+1} + L_i),$$

where the positive constants M_i and L_i are chosen so that $|p| < L_i$ on $A_i \backslash W$ and $M_i \geq \tilde{\psi} - p$ on $A_i \cap \partial V$. The existence of such a χ is a well-known fact.

Now, to finish the proof of Lemma 3.1 we observe that, by the definition of ψ and our choice of χ, obviously $\psi \in \mathrm{PSH}(\Omega)$. If now $\tau > 0$ is a smooth strongly plurisubharmonic function on X, then $\psi + \tau|_\Omega \in \mathrm{SPSH}(\Omega)$ and $\psi + \tau|_\Omega$ is exhaustive. By Theorem 2.2, Ω is Stein and the proof of our Lemma 3.1 is complete. $\square$

The next result needed is due to Siu [16].

Theorem 3.2. *Let Y be a closed Stein subspace in a complex space X. Then Y has a Stein open neighborhood in X.*

Now we are ready to give our proof of Fornæss-Narasimhan's theorem for the case of continuous functions.

Theorem 3.3. *On any reduced complex space X, any continuous* $\mathrm{WPSH}(X)$ *function is a* $\mathrm{PSH}(X)$ *function.*

Proof. Because the problem is local, we may assume that X is a closed analytic subset in some Stein open subset U of $\mathbb{C}^n$.

Let $\varphi \in \mathrm{WPSH}(X)$ be continuous. Consider $\tilde{X} := X \times \mathbb{C}$, which is Stein, and

$$\Omega := \{(z, w) \in \tilde{X} \mid \varphi(z) + \log|w| < 0\}.$$

We notice that Ω is itself Stein. Indeed, to see this, choose $g > 0$ a smooth, SPSH exhaustion function for $X \times \mathbb{C}$ and define

$$h(z, w) = g(z, w) - \frac{1}{\varphi(z) + \log|w|},$$

which is in SWPSH(Ω) and exhausts Ω (here we need the continuity of φ). By Lemma 3.1, Ω is Stein. We have $\Omega \subset X \times \mathbb{C} \subset U \times \mathbb{C} \subset \mathbb{C}^{n+1}$. Consider now an open set W in $\mathbb{C}^{n+1}$ with the property that $W \cap (X \times \mathbb{C}) = \Omega$. By [16, Thm. 3.2] applied to $\Omega \subset W$, it follows that there exists an open Stein set V in $\mathbb{C}^{n+1}$ with $V \cap (X \times \mathbb{C}) = \Omega$. Since V is Stein, we have that $-\log \delta_w$ is plurisubharmonic on V, where δ_w denotes the boundary distance of V in the w-direction (or the Hartogs radius of V with respect to w). To define δ_w, fix a point $(z^0, w^0) \in V$ and look at all polydiscs of the form $|z_i - z_i^0| < r_i$, $i \in \{1, \ldots, n\}$, $|w - w^0| < r_{n+1}$, that are subsets of V. Then $\delta_w(z^0, w^0)$ is the supremum over all such r_{n+1}. Identifying X with $X \times \{0\}$, it follows at once from the definition of $\Omega = V \cap (X \times \mathbb{C})$ that $-\log \delta_w|_X = \varphi$ and so we have the required plurisubharmonic extension of φ. $\square$

4. A Generalization to the q-Convex Case

4.1. General Setup

In generalizing Fornæss-Narasimhan's theorem to the q-plurisubharmonic case (but for continuous functions only) we will follow the general ideas of the proof in Section 3. But first of all we will give the precise definitions of *q-plurisubharmonic* (in notation, q-PSH) and *weakly q-plurisubharmonic* (q-WPSH) functions on complex spaces. We recall the definitions for open sets in $\mathbb{C}^n$.

Definition 4.1 (see e.g. [9]). An upper semicontinuous function $\varphi\colon D \to [-\infty, \infty)$, where $D \subset \mathbb{C}^n$ is an open subset, is called *subpluriharmonic* if, for every relatively compact subset $G \subset\subset D$ and for every pluriharmonic function u defined on a neighborhood of $\bar{G}$, the inequality $\varphi|_{\partial G} \le u|_{\partial G}$ implies $\varphi \le u$ on $\bar{G}$.

Remark 4.2. One may verify that a function $\varphi \in \mathcal{C}^2(U, \mathbb{R})$, where $U \subset \mathbb{C}^n$ is an open subset, is subpluriharmonic if and only if its Levi form has at least one nonnegative (≥ 0) eigenvalue at every point of U.

Definition 4.3 [10]. A function defined on $D \subset \mathbb{C}^n$ and with values in $[-\infty, \infty)$ is called *q-plurisubharmonic* ($1 \le q \le n$) *in D* if it is upper semicontinuous and if it is subpluriharmonic on the intersection of every q-dimensional complex plane with D.

Remark 4.4. The notion n-plurisubharmonic means subpluriharmonic, and 1-plurisubharmonic means plurisubharmonic.

Now we define the q-plurisubharmonic functions on an arbitrary complex space.

Definition 4.5. Let X be a complex space and let $\varphi\colon X \to [-\infty, \infty)$ be an upper semicontinuous function on X. Then φ is called *q-plurisubharmonic on*

X if for every point $x \in X$ there exists a local embedding $i: U \hookrightarrow \tilde{U} \subset \mathbb{C}^n$, where U is a neighborhood of x, $\tilde{U}$ an open subset of $\mathbb{C}^n$, and there exists a q-plurisubharmonic function $\tilde{\varphi}$ on $\tilde{U}$ such that $\tilde{\varphi} \circ i = \varphi$.

REMARK 4.6. Even if φ in Definition 4.5 happens to be continuous, we do not require $\tilde{\varphi}$ to be continuous; it is always only assumed to be upper semicontinuous.

We also define the weakly q-plurisubharmonic functions on complex spaces as follows.

DEFINITION 4.7. Let X be a complex space. An upper semicontinuous function $\varphi: X \to [-\infty, \infty)$ is called *weakly q-plurisubharmonic on X* if for every holomorphic function $f: G \to X$, where G is open in $\mathbb{C}^q$, the function $\varphi \circ f$ is subpluriharmonic on G.

REMARK 4.8. 1. If a function is weakly q-plurisubharmonic, then it also is weakly q'-plurisubharmonic for every $q' \geq q$.

2. A real-valued $\mathcal{C}^2$-function defined on an open set D in $\mathbb{C}^n$ is weakly q-plurisubharmonic ($1 \leq q \leq n$) if and only if the Levi form of φ has at least $n - q + 1$ nonnegative eigenvalues at every point of D. Note that each q-convex function, in the sense of Andreotti and Grauert [1], is weakly q-plurisubharmonic.

3. It is known that, on a complex manifold, the two classes q-WPSH and q-PSH coincide (see a remark in [11] about a preprint of Fujita). In the manifold case, the nontrivial inclusion is q-PSH $\subseteq$ q-WPSH. This inclusion generalizes at once to the singular case. However, in the singular case the other inclusion, q-WPSH $\subseteq$ q-PSH, becomes nontrivial. This is because we must now find locally a q-plurisubharmonic extension of the respective function to the ambient space of an embedding.

4. For D open in $\mathbb{C}^n$, weakly q-plurisubharmonic functions on D are what Fujita [9] called "pseudoconvex functions of order $n - q$".

We may define the q-SPSH and q-SWPSH functions on a complex space X in a similar way to that in Section 2.

We denote by $F_q(X)$ the set of the q-convex functions with corners on X, as they were introduced by Diederich and Fornaess [6; 7]. Theorem 2.1, which was needed in Section 3, must in the q-convex case be replaced by the following.

THEOREM 4.9 [4]. *Let X be a complex space, $A \subset X$ a closed analytic subset, $f \in F_q(A)$, and $\eta > 0$ a continuous function on A. Then there exists an open neighborhood V of A in X and $\tilde{f} \in F_q(V)$ such that $|\tilde{f} - f| < \eta$ on A.*

We will also need the following approximation result due to Bungart [2].

THEOREM 4.10 (Bungart). *Let X be a complex manifold and $\varphi: X \to \mathbb{R}$ a continuous q-SPSH(X) function. Then, for any continuous function $\eta: X \to (0, \infty)$, there exists a function $\tilde{\varphi} \in F_q(X)$ such that $|\tilde{\varphi} - \varphi| < \eta$ on X.*

REMARK 4.11. In fact, Bungart proved this result only when X is an open subset of some Euclidian space $\mathbb{C}^n$. But as Matsumoto [11] remarked, this result still holds when X is a complex manifold. For the sake of completeness we give here a proof for the manifold case, using Bungart's theorem.

Proof of Theorem 4.10. Fix three locally finite open coverings $(U_i)_{i\in\mathbb{N}}$, $(V_i)_{i\in\mathbb{N}}$, $(W_i)_{i\in\mathbb{N}}$ of X such that $U_i \subset\subset V_i \subset\subset W_i \subset\subset X$ for all $i \in \mathbb{N}$ and such that each W_i is the domain of a biholomorphic map $i\colon W_i \to \tilde{W}_i$, where $\tilde{W}_i$ is an open set in $\mathbb{C}^{n_i}$.

For each index $i \in \mathbb{N}$, consider a function $\theta_i \in \mathcal{C}_0^\infty(X, \mathbb{R})$ such that $\theta_i \equiv -1$ on ∂V_i, $\theta_i \equiv 1$ on $\bar{U}_i$, and $\theta_i \equiv 0$ on $X \setminus W_i$. Let $\varepsilon_i > 0$ be small enough so that $2\varepsilon_i\theta_i \le \eta$ and $\varphi + \varepsilon_i\theta_i$ is still q-SPSH.

Since $\bar{V}_i \subset W_i \simeq \tilde{W}_i$, we can apply Bungart's theorem to obtain, for all $i \in \mathbb{N}$, a function $\varphi_i \in F_q(W_i)$ with the property that

$$|\varphi(x) + \varepsilon_i\theta_i(x) - \varphi_i(x)| < \min\left(\varepsilon_i, \frac{\eta(x)}{2}\right)$$

on a neighborhood of $\bar{V}_i$.

It follows that we have $\varphi_i < \varphi$ on ∂V_i and $\varphi_i > \varphi$ on $\bar{U}_i$. Hence we may define $\tilde{\varphi}\colon X \to \mathbb{R}$ by $\tilde{\varphi}(x) := \max\{\varphi_i(x) \mid x \in V_i\}$. Clearly $\tilde{\varphi} \in F_q(X)$, $\varphi \le \tilde{\varphi}$, and $\tilde{\varphi} < \varphi + \eta$ as desired. □

We shall also use the following result due to Fujita [9, Thm. 1].

THEOREM 4.12 (Fujita). *Let D be an open subset of $\mathbb{C}^n$ that is q-complete with corners, let $w \in \mathbb{C}^n$ ($\|w\| = 1$), and denote by δ_w the boundary distance function of D along the w-direction. Then $-\log\delta_w$ is weakly q-plurisubharmonic on D and thus also q-plurisubharmonic.*

REMARK 4.13. In fact, Fujita proves this result for the more general case of "pseudoconvex domains of order $(n - q)$".

Finally, we also will use a theorem of Peternell ([12, Lemma 5]; see also [5]). For this we need the following.

DEFINITION 4.14. Let X be a manifold. A function $v\colon X \to [-\infty, \infty)$ is called *almost plurisubharmonic* if it can be written locally as a sum of a plurisubharmonic and a smooth function. If X is a complex space, we require that v can be locally extended as an almost plurisubharmonic function in the ambient space of an embedding.

THEOREM 4.15 (Peternell). *If Y is a closed analytic subset in a complex space X, then there exists an almost plurisubharmonic function v on X such that $v \in \mathcal{C}^\infty(X \setminus Y)$ and $Y = \{x \in X \mid v(x) = -\infty\}$.*

4.2. The Equivalence of q-WPSH and q-PSH Functions

We can now state our main result as follows.

THEOREM 4.16. *Every continuous q-WPSH function on a reduced complex space X is a q-PSH function on X.*

In order to prove this theorem, we first show the following two lemmas.

LEMMA 4.17. *Let X be a reduced complex space of finite dimension for which there exists a continuous exhaustion function $\varphi\colon X \to \mathbb{R}$ that is in q-SWPSH(X). Then there exists a q-convex function with corners $\psi\colon X \to \mathbb{R}$, exhausting X.*

Proof. We may assume that $\varphi > 0$. In the regular case (i.e., if X is a complex manifold) then this lemma is a direct consequence of Bungart's approximation theorem, because in the manifold case the inclusion q-SWPSH $\subseteq q$-SPSH is trivial. In the singular case, the proof is by induction on $n = \dim(X)$.

The case $n = 0$ is obvious. Now suppose that Lemma 4.17 holds for all complex spaces Y with $\dim Y \le n-1$, and let $\dim X = n$.

Consider $Y = \mathrm{Sing}(X)$, the singular locus of X. Because $\dim Y \le n-1$ and $\varphi|_Y$ satisfies the conditions of our lemma, we conclude that there exists an exhaustion function $\psi_1\colon Y \to \mathbb{R}$ that is q-convex with corners. By Theorem 4.9, we can find a neighborhood V of Y in X and a $\tilde{\psi}_1 \in F_q(V)$ such that $|\tilde{\psi}_1 - \psi_1| < 1$ on Y.

By shrinking V if necessary, we can suppose that $\tilde{\psi}_1$ is defined on a neighborhood of $\bar{V}$ and that $\{x \in \bar{V} \mid \tilde{\psi}_1(x) < c\}$ is relatively compact in $\bar{V}$ for all real numbers c. By Peternell's theorem, there exists an almost plurisubharmonic function $\theta\colon X \to [-\infty, \infty)$ such that $\theta|_{\mathrm{Reg}(X)}$ is smooth and such that $Y = \{x \in X \mid \theta(x) = -\infty\}$.

Now let $\chi : [0, \infty) \to \mathbb{R}_+$ be a continuous, convex, increasing function that is linear on segments. This means that there is a division $0 = a_0 < a_1 < \cdots < a_n < \cdots$ of $[0, \infty)$ such that, on $[a_i, a_{i+1}]$, we have $\chi(t) = A_i t + B_i$ with $A_i > 0$, and the convexity of χ gives $A_{i+1} \ge A_i$.

If χ increases rapidly at infinity then $(\chi\circ\varphi+\theta)|_{\mathrm{Reg}(X)}$ is in q-SWPSH(Reg(X)). This can be seen as follows. Take a locally finite open covering $(U_j)_{j\in\mathbb{N}}$, $U_j \subset\subset X$, of X such that, for each j on a neighborhood of $\bar{U}_j$, one has $\theta = \theta_{1,j} + \theta_{2,j}$ with $\theta_{1,j}$ smooth and $\theta_{2,j}$ plurisubharmonic. Then, if the constants $A_i > 0$ in the definition of χ are chosen large enough, $\chi\circ\varphi + \theta_{1,j}$ is q-SWPSH on U_j. We can thus find χ as before so that $(\chi\circ\varphi+\theta)|_{\mathrm{Reg}(X)}$ is in q-SWPSH(Reg(X)) $\subset q$-SPSH(Reg(X)). Also, if χ increases rapidly then we may assume that $(\chi\circ\varphi+\theta)|_{\partial V} > \tilde{\psi}_1|_{\partial V}$ and that $(\chi\circ\varphi+\theta)|_{X\setminus V}$ exhausts $X\setminus V$.

By Bungart's approximation theorem, there is a function $u\colon \mathrm{Reg}(X) \to \mathbb{R}$ that is q-convex with corners and such that:

(1) $|u - (\chi\circ\varphi+\theta)| < 1$ on Reg(X);
(2) $u|_{\partial V} > \tilde{\psi}_1|_{\partial V}$.

We define now $\psi\colon X \to \mathbb{R}$ as follows:

$$\psi = \begin{cases} \max(\tilde{\psi}_1, u) & \text{on } V\setminus Y, \\ \tilde{\psi}_1 & \text{on } Y, \\ u & \text{on } X\setminus V. \end{cases}$$

Then clearly ψ is an exhaustion function on X and ψ is q-convex with corners. Hence our lemma is proved. □

The second needed statement is the following generalization of Siu's theorem (previously formulated as Theorem 3.2).

LEMMA 4.18. *Let S be a closed analytic subset of a complex space X, and assume that S is q-complete with corners. Then there exists an open neighborhood V of S in X such that V is q-complete with corners.*

Proof. Since $S \subset X$ is a closed complex subspace, by Peternell's theorem there exists an almost plurisubharmonic function λ on X such that $S = \{x \in X \mid \lambda(x) = -\infty\}$ and such that $\lambda|_{X \setminus S} \in \mathcal{C}^\infty(X \setminus S)$.

Denote by $\psi\colon S \to \mathbb{R}$ a positive, q-convex exhaustion function with corners. Applying Theorem 4.9, we deduce that there exists a q-convex function with corners, $\tilde{\psi}$, in a neighborhood U of S such that $|\tilde{\psi} - \psi| < 1$ on S. We can assume that $\tilde{\psi} > 0$. We may suppose, by eventually shrinking U, that $\tilde{\psi}$ is defined on a neighborhood of $\bar{U}$ and that $\tilde{\psi}$ exhausts $\bar{U}$.

Consider $\chi : [0, \infty) \to \mathbb{R}$, a continuous, convex, increasing function that is linear on segments and such that:

(1) if $V = \{x \in U \mid \chi \circ \tilde{\psi}(x) + \lambda(x) < 0\}$, then $\partial V \cap \partial U = \emptyset$; and
(2) the function $\varphi := \max(-1/(\chi \circ \tilde{\psi} + \lambda), \tilde{\psi})$ defined on V is q-convex with corners.

The choice of χ satisfying (2) is possible as in Lemma 4.17. We also can realize condition (1) by choosing a sequence of real numbers $(\lambda_n)_n \searrow -\infty$ such that $\{x \in U \mid \tilde{\psi}(x) < n,\ \lambda(x) < \lambda_n\}$ is relatively compact in U and requiring that $\chi : [0, \infty) \to \mathbb{R}$ additionally satisfy $\chi|_{[n-1,n)} \geq -\lambda_n$ for all $n \in \mathbb{N}$.

It then follows that the set $V = \{x \in U \mid \chi \circ \tilde{\psi}(x) + \lambda(x) < 0\}$ is an open q-complete with corners neighborhood of S, where φ is the exhaustion function; hence, Lemma 4.18 is proved. □

We are now in a position to prove Theorem 4.16.

Proof of Theorem 4.16. Because the problem is local, we can assume (without loss of generality) that X is a closed analytic subset in a Stein open set $U \subset \mathbb{C}^n$. Let $\varphi \in q$-WPSH(X) be continuous.

We have $X \times \mathbb{C} \subset U \times \mathbb{C} \subset \mathbb{C}^{n+1}$ and consider $\Omega \subset X \times \mathbb{C}$, the open set given by

$$\Omega = \{(z, w) \in X \times \mathbb{C} \mid |w| < e^{-\varphi(z)}\}.$$

On Ω there exists a continuous q-SWPSH exhaustion function. Indeed, denote by $s\colon X \times \mathbb{C} \to \mathbb{R}$ a smooth, SPSH$(X \times \mathbb{C})$, positive exhaustion function and consider

$$s(z, w) - \frac{1}{\varphi(z) + \log|w|} : \Omega \to \mathbb{R}.$$

This function has the desired properties, so that for Ω we can apply Lemma 4.17 and thus obtain a q-convex with corners exhaustion function $\psi\colon \Omega \to \mathbb{R}$. But this means that Ω is q-complete with corners.

Consider now an open set W in $\mathbb{C}^{n+1}$ with the property that $W \cap (X \times \mathbb{C}) = \Omega$. Then Lemma 4.18 can be applied for the situation $\Omega \subset W$. We conclude with the existence of an open set $\tilde{\Omega} \subset \mathbb{C}^{n+1}$ that is q-complete with corners and for which $\tilde{\Omega} \cap (X \times \mathbb{C}) = \Omega$ holds.

Now it is enough to consider δ_w, the distance to the boundary of $\tilde{\Omega}$ along the w-direction. By Theorem 4.12, $-\log \delta_w$ is a q-PSH$(\tilde{\Omega})$ function (not necessarily continuous). By the definition of Ω, it follows that $-\log \delta_w|_X = \varphi$ and so we have the desired conclusion that φ is a q-PSH(X) function. $\square$

REMARK 4.19. In the manifold case, the standard proof for the inclusion PSH $\subseteq$ WPSH can not be used to prove that q-PSH $\subseteq$ q-WPSH for $q > 1$ because the class of q-plurisubharmonic functions is not additive for $q > 1$.

However, using the methods just described, one can show for a manifold that the inclusion q-PSH $\subseteq$ q-WPSH holds for continuous functions. Then we can also get rid of the continuity condition by using an approximation result of Slodkowski [17].

More precisely, to prove q-PSH$(M) \subseteq q$-WPSH(M) for continuous functions when M is a manifold, let $\varphi \in q$-PSH(M). Because the problem is local, we can suppose (without loss of generality) that $M = U$ is an open Stein set in $\mathbb{C}^n$. As in the proof of Theorem 4.16, we introduce the set

$$\Omega = \{(z, w) \in U \times \mathbb{C} \mid |w| < e^{-\varphi(z)}\}$$

and observe that now the function

$$s(z, w) - \frac{1}{\varphi(z) + \log|w|} : \Omega \to \mathbb{R}$$

is continuous, exhaustive, and q-SPSH. Applying Theorem 4.10, it follows that Ω is q-complete with corners. Using Theorem 4.12, as before it follows that $-\log \delta_w$ is q-WPSH on $\tilde{\Omega}$. Its restriction to U, which coincides with φ, is therefore also q-WPSH, as desired.

Now, if φ is no longer continuous then we apply a result of Slodkowski [17, Rem. 2.10]. Namely, every q-PSH function φ on an open set $U \subseteq \mathbb{C}^n$ can be approximated (on a compact set K) by a pointwise convergent and nonincreasing sequence (φ_n) of continuous q-PSH functions (defined on a neighborhood of K).

Now, since the (φ_n) are continuous and q-PSH functions, they are also q-WPSH functions. But it is known (see e.g. [9]) that the pointwise limit function of a nonincreasing sequence of q-WPSH functions is itself q-WPSH.

Note that the reverse inclusion, q-WPSH $\subseteq$ q-PSH, is trivial in the manifold case. We thus have the equality q-WPSH$(M) = q$-PSH(M) on each manifold M.

References

[1] A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes,* Bull. Soc. Math. France 90 (1962), 193–259.

[2] L. Bungart, *Piecewise smooth approximations to q-plurisubharmonic functions,* Pacific J. Math. 142 (1990), 227–244.

[3] M. Colţoiu, *A note on Levi's problem with discontinuous functions,* Enseign. Math. (2) 31 (1985), 299–304.
[4] ———, *n-Concavity of n-dimensional complex spaces,* Math. Z. 210 (1992), 203–206.
[5] J. P. Demailly, *Cohomology of q-convex spaces in top degrees,* Math. Z. 204 (1990), 283–295.
[6] K. Diederich and J. E. Fornæss, *Smoothing q-convex functions and vanishing theorems,* Invent. Math. 82 (1985), 291–305.
[7] ———, *Smoothing q-convex functions in the singular case,* Math. Ann. 273 (1986), 665–671.
[8] J. E. Fornæss and R. Narasimhan, *The Levi problem on complex spaces with singularities,* Math. Ann. 248 (1980), 47–72.
[9] O. Fujita, *Domaines pseudoconvexes d'ordre général et fonctions pseudoconvexes d'ordre général,* J. Math. Kyoto Univ. 30 (1990), 637–649.
[10] L. R. Hunt and J. J. Murray, *q-Plurisubharmonic functions and a generalized Dirichlet problem,* Michigan Math. J. 25 (1978), 299–316.
[11] K. Matsumoto, *Boundary distance functions and q-convexity of pseudoconvex domains of general order in Kähler manifolds,* J. Math. Soc. Japan 48 (1996), 85–107.
[12] M. Peternell, *Continuous q-convex exhaustion functions,* Invent. Math. 85 (1986), 249–262.
[13] A. Popa, *Sur un théorème de Fornæss et Narasimhan,* C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), 11–14.
[14] A. Popa-Fischer, *Generalized Kähler metrics on complex spaces and a supplement to a theorem of Fornæss and Narasimhan,* Ph.D. dissertation, Bergische Universität Wuppertal, 2000.
[15] R. Richberg, *Stetige streng pseudokonvexe Funktionen,* Math. Ann. 175 (1968), 257–286.
[16] Y.-T. Siu, *Every Stein subvariety admits a Stein neighborhood,* Invent. Math. 38 (1976/77), 89–100.
[17] Z. Slodkowski, *The Bremmermann–Dirichlet problem for q-plurisubharmonic functions,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11 (1984), 303–326.

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