ON THE DEGREE OF FANO SCHEMES OF LINEAR SUBSPACES ON HYPERSURFACES

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Abstract

In this paper we propose and prove an explicit formula for computing the degree of Fano schemes of linear subspaces on general hypersurfaces. The method used here is based on the localization theorem and Bott's residue formula in equivariant intersection theory.

1. Introduction

Let $X \subset \mathbf{P}_{\mathbf{C}}^n$ be a general hypersurface of degree d. The *Fano scheme* $F_k(X)$ is defined to be the set of k-dimensional linear subspaces of $\mathbf{P}_{\mathbf{C}}^n$ which are contained in X. This is a subscheme of the Grassmannian $\mathbf{G}(k,n)$ of k-dimensional linear subspaces in $\mathbf{P}_{\mathbf{C}}^n$. For convenience, we set

$$\delta = (k+1)(n-k) - \binom{d+k}{d}.$$

Suppose that $d \neq 2$ (or $n \geq 2k+1$) and $\delta \geq 0$. Langer [10] showed that $F_k(X)$ is smooth of expected dimension δ. In this case, the degree of $F_k(X)$ is given by the following formula

$$(1) \qquad \deg(F_k(X)) = \int_{\mathbf{G}(k,n)} c_{\binom{d+k}{d}}(\mathrm{Sym}^d \, S^\vee) \cdot c_1(Q)^\delta,$$

where S and Q are respectively the tautological sub-bundle and quotient bundle on $\mathbf{G}(k,n)$, $\mathrm{Sym}^d \, S^\vee$ is the d-th symmetric power of the dual of S, and $c_i(E)$ is the i-th Chern class of the vector bundle E. Note that $\int_Y \alpha$ denotes the degree of the cycle class α on Y defined in [8, Definition 1.4]. Formula (1) can be found, for example, in [8, Example 14.7.13] or [11, Section 3.5]. Using Schubert calculus, Debarre and Manivel [5, Theorem 4.3] showed that the degree of $F_k(X)$

is equal to a certain coefficient of an explicit polynomial, given as the product of linear forms. In this paper we propose and prove an explicit formula for computing the degree of $F_k(X)$ via equivariant intersection theory.

We denote by $\mathscr{I}$ the set of $(k+1)$-subsets of $\{1, \ldots, n+1\}$, where a $(k+1)$-subset of $\{1, \ldots, n+1\}$ is a subset consisting of $k+1$ elements of $\{1, \ldots, n+1\}$. Consider the polynomial ring $\mathbf{C}[h_1, \ldots, h_{n+1}]$ in $n+1$ variables $h_1, \ldots, h_{n+1}$. For each $I \in \mathscr{I}$, we set

$$S_I = \prod_{v_i \in \mathbf{N}, \sum_{i \in I} v_i = d} \left(\sum_{i \in I} v_i h_i \right), \quad Q_I = \sum_{j \notin I} h_j,$$

and

$$T_I = \prod_{i \in I} \prod_{j \notin I} (h_i - h_j).$$

Then S_I, Q_I, and T_I are elements in $\mathbf{C}[h_1, \ldots, h_{n+1}]$.

Here is the main result of this paper:

THEOREM 1.1. *Let* $k, d, n \in \mathbf{N}$ *satisfy* $d \neq 2$ *(or* $n \geq 2k+1$*) and* $\delta \geq 0$*, and let* $X \subset \mathbf{P}^n_{\mathbf{C}}$ *be a general hypersurface of degree* d*. Then the degree of the Fano scheme* $F_k(X)$ *of* k*-dimensional linear subspaces on* X *is given by the following formula*:

$$\deg(F_k(X)) = (-1)^\delta \sum_{I \in \mathscr{I}} \frac{S_I Q_I^\delta}{T_I}.$$

The right-hand-side of the formula in Theorem 1.1 is a rational polynomial function, and the above theorem claims in other words that it is in fact a constant function, moreover it is an integer. Namely, for any numbers h_i such that $h_i \neq h_j$ for $i \neq j$, the right-hand-side of the formula becomes the same integer. Indeed, for concrete numbers (k, d, n) such that the above conditions are satisfied, it can be checked by computer algebra systems. See [3, Chapter 5] for the computation in SAGE [12] and SINGULAR [4].

If $k, d, n \in \mathbf{N}$ satisfy $d \neq 2$ (or $n \geq 2k+1$) and $\delta = 0$, then the Fano scheme $F_k(X)$ is zero-dimensional. In this case, the degree of $F_k(X)$ is equal to the number of k-dimensional linear subspaces on X.

In particular, if $k = 1$ and $\delta = 0$, then $d = 2n - 3$. In this case, we have

$$\mathscr{I} = \{\{i, j\} \mid 1 \leq i < j \leq n + 1\}.$$

For each $I = \{i, j\} \in \mathscr{I}$, we have

$$S_I = \prod_{a=0}^{2n-3} (ah_i + (2n - 3 - a)h_j)$$

and

$$T_I = \prod_{k \neq i,j} (h_i - h_k)(h_j - h_k).$$

Thus we have the following corollary.

COROLLARY 1.2. *Let $k, d, n \in \mathbf{N}$ satisfy $d \neq 2$ (or $n \geq 2k + 1$) and $\delta = 0$, and let $X \subset \mathbf{P}^n_{\mathbf{C}}$ be a general hypersurface of degree d. Then the number of k-dimensional linear subspaces on X is equal to*

$$\sum_{I \in \mathscr{I}} \frac{S_I}{T_I}.$$

In particular, the number of lines on a general hypersurface of degree $2n - 3$ in $\mathbf{P}^n_{\mathbf{C}}$ is equal to

$$\sum_{1 \leq i < j \leq n+1} \frac{\prod\limits_{a=0}^{2n-3} (ah_i + (2n - 3 - a)h_j)}{\prod\limits_{k \neq i,j} (h_i - h_k)(h_j - h_k)}.$$

Debarre and Manivel [5] showed that the degree of $F_k(X)$ is equal to the coefficient of the monomial $x_0^n x_1^{n-1} \cdots x_k^{n-k}$ in the product of the polynomial

$$\prod_{v_i \in \mathbf{N}, \sum_{i=0}^{k} v_i = d} \left(\sum_{i=0}^{k} v_i x_i \right) (x_0 + \cdots + x_k)^{\delta}$$

by the Vandermonde determinant

$$\prod_{0 \leq i < j \leq k} (x_i - x_j).$$

See [11, Theorem 3.5.18] for the special case of this formula for computing the number of linear subspaces on hypersurfaces.

In [1], Altman and Kleiman gave a formula for computing the degree of $F_1(X)$. Harris [9] gave a formula for computing the number of lines on a general hypersurface of degree $2n - 3$ in $\mathbf{P}^n$. In [13], van der Waerden showed that the number of lines on a general hypersurface of degree $d = 2n - 3$ in $\mathbf{P}^n_{\mathbf{C}}$ could also be computed as the coefficient of the monomial $x^n y^{n-1}$ in the polynomial

$$(x - y) \prod_{i=0}^{d} ((d - i)x + iy).$$

These results were obtained via Schubert calculus. However, in this paper, our approach is completely different. We use the localization theorem and Bott's residue formula in equivariant intersection theory.

The rest of the paper is organized as follows: Section 2 presents a quick review of equivariant intersection theory. Section 3 presents the proof of the main theorem.

2. Equivariant intersection theory

Edidin and Graham [6, 7] gave an algebraic construction to equivariant intersection theory. In this section, we review the basic notions and results of this theory. Let G be a linear algebraic group and let X be a scheme of finite type over $\mathbf{C}$ endowed with a G-action. For any non-negative integer i, we can find a representation V of G together with a dense open subset $U \subset V$ on which G acts freely and whose complement has codimension larger than $\dim X - i$ such that the principal bundle quotient $U \to U/G$ exists in the category of schemes (see [6, Lemma 9]). The diagonal action on $X \times U$ is then also free, which implies that under mild assumption, a principal bundle quotient $X \times U \to (X \times U)/G$ exists in the category of schemes (see [6, Proposition 23]). In what follows, we will tacitly assume that the scheme $(X \times U)/G$ exists and denote it by X_G.

2.1. Equivariant Chow groups.
We define the i-th G-equivariant Chow group of X to be

$$A_i^G(X) := A_{i+\dim U - \dim G}(X_G),$$

where A_* stands for the ordinary Chow group defined in [8]. By [6, Proposition 1], this is well-defined. The G-equivariant Chow group of X is defined to be

$$A_*^G(X) = \bigoplus_i A_i^G(X).$$

If X_G is smooth, then $A_*^G(X)$ inherits an intersection product from the ordinary Chow groups. This endows $A_*^G(X)$ with the structure of a graded ring, called the *G-equivariant Chow ring* of X. For example, if $G = T = (\mathbf{C}^*)^n$ is a split torus of dimension n, then the T-equivariant Chow ring of a point is isomorphic to a polynomial ring in n variables (see [6, Section 3.2]). Throughout this paper, we denote this ring by R_T.

2.2. Equivariant vector bundles and Chern classes.
A G-equivariant vector bundle is a vector bundle E on X such that the action of G on X lifts to an action of G on E which is linear on fibers. By [6, Lemma 1], E_G is a vector bundle over X_G. The *G-equivariant Chern classes* $c_i^G(E)$ are defined to be the Chern classes $c_i(E_G)$. If E has rank r, then the top Chern class $c_r^G(E)$ is called the *G-equivariant Euler class* of E and denoted $e^G(E)$.

Note that a G-equivariant vector bundle over a point is a representation of G (see also [6, Section 3.2]). Our primary interest is when $G = T = (\mathbf{C}^*)^n$ is a split torus of dimension n and $X = pt$ is a point. In this case, let $M(T)$ be the character group of the torus T. Suppose that $R_T = \mathbf{C}[h_1, \ldots, h_n]$. There is a group homomorphism $\psi : M(T) \to R_T$ given by $\rho_i \mapsto h_i$, where ρ_i is the character of T defined by $\rho_i(t_1, \ldots, t_n) = t_i$. This induces a ring isomorphism $\mathrm{Sym}(M(T)) \simeq R_T$. We call $\psi(\rho)$ the *weight* of ρ. In particular, h_i is the weight of ρ_i.

Example 2.1 [2, Example 9.1.1.1]. The diagonal action of $G = T = (\mathbf{C}^*)^n$ on $\mathbf{C}^n$ gives a T-equivariant vector bundle E over a point. The corresponding representation of T has characters ρ_i for $i = 1, \ldots, n$, and their weights are h_i. In this case, we have $E_T \cong \mathcal{O}(h_1) \oplus \cdots \oplus \mathcal{O}(h_n)$. This implies that $c_i^T(E) = c_i(E_T) = s_i(h_1, \ldots, h_n) \in R_T$, where s_i is the i-th elementary symmetric function. In particular, the T-equivariant Euler class of E is $e^T(E) = h_1 \cdots h_n \in R_T$.

Example 2.2. Consider the diagonal action of $T = (\mathbf{C}^*)^4$ on $\mathbf{C}^4$ given in coordinates by

$$(t_1, t_2, t_3, t_4) \cdot (x_1, x_2, x_3, x_4) = (t_1 x_1, t_2 x_2, t_3 x_3, t_4 x_4).$$

This induces an action of T on the Grassmannian $G(2, 4)$ with the isolated fixed points L_I corresponding to coordinate 2-planes in $\mathbf{C}^4$. Each L_I is indexed by the 2-subset I of the set $\{1, 2, 3, 4\}$ so that L_I is defined by the equations $x_j = 0$ for $j \notin I$. Let S be the tautological sub-bundle on $G(2, 4)$. At each L_I, the restriction of the action of T on the fiber $S|_{L_I}$ gives a representation of T with characters ρ_i for $i \in I$. This representation gives a T-equivariant vector bundle of rank 2 over a point. We also denote it by $S|_{L_I}$. If $I = \{i_1, i_2\}$, then we have $c_1^T(S|_{L_I}) = h_{i_1} + h_{i_2}$ and $c_2^T(S|_{L_I}) = h_{i_1} \cdot h_{i_2}$, where h_{i_1} and h_{i_2} are the weights of ρ_{i_1} and ρ_{i_2} respectively.

2.3. Localization and Bott's residue formula. Let X be a scheme endowed with an action of $T = (\mathbf{C}^*)^n$. We denote the fixed point locus by X^T. The localization theorem states that up to R_T-torsion, the T-equivariant Chow group of the fixed points locus X^T is isomorphic to that of X. Moreover, the localization isomorphism is given by the equivariant push-forward induced by the inclusion of X^T to X (see [7, Theorem 1]). For smooth varieties, the inverse to the equivariant push-forward can be written explicitly (see [7, Theorem 2]). Using these results, Edidin and Graham gave an algebraic proof of Bott's residue formula for Chern numbers of vector bundles on smooth complete varieties (see [7, Theorem 3]). Bott's residue formula shows that we can compute the degree of a zero-dimensional cycle class on a smooth complete variety X in terms of local contributions supported on the components of the fixed point locus of a torus action on X.

Let us describe Bott's residue formula in case where the fixed point locus X^T is finite. Let E be a T-equivariant vector bundle on X. For each point

$Z_j \in X^T$, assume that the action of T on $E|_{Z_j}$ has characters $\chi_j^1, \ldots, \chi_j^s$, where each χ_j^i is a linear combination of the basic characters $\rho_1, \ldots, \rho_n$ defined above. Write $\chi_j^i = a_j^i(\rho_1, \ldots, \rho_n)$ for this linear combination. Since ρ_i has weight h_i, the weight of χ_j^i is $a_j^i(\bar{h})$, where $a_j^i(\bar{h}) \in R_T$ denotes the linear combination of $h_1, \ldots, h_n$ obtained by replacing h_i by ρ_i in $a_j^i(\rho_1, \ldots, \rho_n)$. Thus the T-action on $E|_{Z_j}$ has weights $a_j^i(\bar{h})$. This implies that

$$c_k^T(E|_{Z_j}) = s_k(a_j^1(\bar{h}), \ldots, a_j^s(\bar{h})) \in R_T,$$

where s_k is the k-th elementary symmetric function. In addition, since Z_j is just a point, the normal bundle N_j of Z_j in X is the tangent space to X at Z_j. If the action of T on the tangent space to X at Z_j has weights $b_j^1(\bar{h}), \ldots, b_j^m(\bar{h})$, where $m = \dim X$, then

$$e^T(N_j) = \prod_{i=1}^{m} b_j^i(\bar{h}) \in R_T.$$

We arrive at the following formula

$$(2) \qquad \int_X P(c_k(E)) = \sum_j \frac{P(s_k(a_j^1(\bar{h}), \ldots, a_j^s(\bar{h})))}{\prod_{i=1}^{m} b_j^i(\bar{h})},$$

where $P(c_k(E))$ is a polynomial in the Chern classes of E.

Let us look closer at a simple example based on Bott's residue formula for the projective space $\mathbf{P}_{\mathbf{C}}^2$.

Example 2.3. Consider the natural action of $T = (\mathbf{C}^*)^3$ on $\mathbf{P}_{\mathbf{C}}^2$ given in coordinates by

$$(t_1, t_2, t_3) \cdot (x_1 : x_2 : x_3) = (t_1 x_1 : t_2 x_2 : t_3 x_3).$$

The fixed point locus is finite and consists of coordinate lines, say p_1, p_2, and p_3. Let c be the first Chern class of $E = \mathcal{O}_{\mathbf{P}_{\mathbf{C}}^2}(1)$, that is the class of a hyperplane. It is well-known that

$$\int_{\mathbf{P}_{\mathbf{C}}^2} c^2 = 1.$$

Applying Bott's residue formula (2) we get the identity

$$\frac{h_1^2}{(h_3 - h_1)(h_2 - h_1)} + \frac{h_2^2}{(h_3 - h_2)(h_1 - h_2)} + \frac{h_3^2}{(h_1 - h_3)(h_2 - h_3)} = 1,$$

which can be checked by hand.

3.   The proof of Theorem 1.1

Consider the diagonal action of $T = (\mathbf{C}^*)^{n+1}$ on $\mathbf{P}_{\mathbf{C}}^n$ given in coordinates by

$$(t_1, \ldots, t_{n+1}) \cdot (x_1 : \cdots : x_{n+1}) = (t_1 x_1 : \cdots : t_{n+1} x_{n+1}).$$

This induces an action of T on the Grassmannian $\mathbf{G}(k, n)$ with $\binom{n+1}{k+1}$ isolated fixed points L_I corresponding to $\binom{n+1}{k+1}$ coordinate k-planes in $\mathbf{P}_{\mathbf{C}}^n$.   Each fixed point L_I is indexed by a $(k+1)$-subset I of the set $\{1, \ldots, n+1\}$.   Let S and Q be the tautological sub-bundle and quotient bundle on $\mathbf{G}(k, n)$ respectively.   The key idea is that, at each L_I, the torus action on the fibers $S|_{L_I}$ and $Q|_{L_I}$ have characters ρ_i for $i \in I$ and ρ_j for $j \notin I$ respectively.   Since the tangent bundle on the Grassmannian is isomorphic to $S^\vee \otimes Q$, the characters of the torus action on the tangent space at L_I are

$$\{\rho_j - \rho_i \mid i \in I, j \notin I\}.$$

The normal bundle N_{L_I} of L_I in $\mathbf{G}(k, n)$ is just the tangent space of $\mathbf{G}(k, n)$ at L_I.   Hence

$$e^T(N_{L_I}) = \prod_{i \in I} \prod_{j \notin I} (h_j - h_i)$$

$$= (-1)^{(k+1)(n-k)} \prod_{i \in I} \prod_{j \notin I} (h_i - h_j),$$

where h_i is the weight of ρ_i defined above, and $R_T = \mathbf{C}[h_1, \ldots, h_{n+1}]$ is the T-equivariant Chow ring of a point.

At each L_I, we also need to compute $e^T(\operatorname{Sym}^d S^\vee|_{L_I})$ and $c_1^T(Q|_{L_I})$.   Since the characters of the torus action on $S^\vee|_{L_I}$ are $-\rho_i$ for $i \in I$, the torus action on $\operatorname{Sym}^d S^\vee|_{L_I}$ has characters

$$\left\{ \sum_{i \in I} v_i(-\rho_i) \mid v_i \in \mathbf{N}, \sum_{i \in I} v_i = d \right\}.$$

Hence

$$e^T(\operatorname{Sym}^d S^\vee|_{L_I}) = \prod_{v_i \in \mathbf{N}, \sum_{i \in I} v_i = d} \left(\sum_{i \in I} v_i(-h_i) \right)$$

$$= (-1)^{\binom{d+k}{d}} \prod_{v_i \in \mathbf{N}, \sum_{i \in I} v_i = d} \left(\sum_{i \in I} v_i h_i \right).$$

Since the characters of the torus action on $Q|_{L_I}$ are ρ_j for $j \notin I$, we have

$$c_1^T(Q|_{L_I}) = \sum_{j \notin I} h_j.$$

By (1) and (2), we obtain

$$\deg(F_k(X)) = \sum_{I \in \mathscr{I}} \frac{e^T(\operatorname{Sym}^d S^\vee|_{L_I})(c_1^T(Q|_{L_I}))^\delta}{e^T(N_{L_I})}$$

$$= (-1)^\delta \sum_{I \in \mathscr{I}} \frac{\left(\prod\limits_{v_i \in \mathbf{N}, \sum_{i \in I} v_i = d} \left(\sum\limits_{i \in I} v_i h_i\right)\right)\left(\sum\limits_{j \notin I} h_j\right)^\delta}{\prod\limits_{i \in I}\prod\limits_{j \notin I}(h_i - h_j)}.$$

We get the desired formula.

Another proof of Theorem 1.1 is as follows. Consider the action of $T = \mathbf{C}^*$ on $\mathbf{P}_{\mathbf{C}}^n$ given in coordinates by

$$t \cdot (x_1 : \cdots : x_{n+1}) = (t^{h_1} x_1 : \cdots : t^{h_{n+1}} x_{n+1}).$$

The induced action of T on $\mathbf{G}(k,n)$ also has $\binom{n+1}{k+1}$ isolated fixed points L_I as above. At each L_I, the torus action on the fibers $S|_{L_I}$ and $Q|_{L_I}$ have characters $h_i \rho$ for $i \in I$ and $h_j \rho$ for $j \notin I$ respectively, where ρ is the character of T defined by $\rho(t) = t$ for all $t \in T$. We denote the weight of ρ by h. In this case, the T-equivariant Chow ring of a point is $R_T = \mathbf{C}[h]$. With this set-up, we have

$$e^T(N_{L_I}) = \prod_{i \in I}\prod_{j \notin I}(h_j - h_i)h^{(k+1)(n-k)}$$

$$= (-1)^{(k+1)(n-k)} T_I h^{(k+1)(n-k)}.$$

Similarly, we also have

$$e^T(\operatorname{Sym}^d S^\vee|_{L_I}) = \prod_{v_i \in \mathbf{N}, \sum_{i \in I} v_i = d}\left(\sum_{i \in I} v_i(-h_i h)\right)$$

$$= (-1)^{\binom{d+k}{d}} S_I h^{\binom{d+k}{d}},$$

and

$$c_1^T(Q|_{L_I}) = \sum_{j \notin I} h_j h = Q_I h.$$

By (1) and (2), we obtain

$$\deg(F_k(X)) = (-1)^\delta \sum_{I \in \mathscr{I}} \frac{S_I Q_I^\delta h^{(k+1)(n-k)}}{T_I h^{(k+1)(n-k)}}.$$

Cancelling $h^{(k+1)(n-k)}$, we get the desired formula.

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