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EXTENSIONS OF THE EULER-SATAKE CHARACTERISTIC DETERMINE POINT SINGULARITIES OF ORIENTABLE 3-ORBIFOLDS

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Abstract

We compute the extensions of the Euler-Satake characteristic of a closed, effective, orientable 3-orbifold corresponding to free and free abelian groups in terms of the number and type of point singularities of the orbifold. Using these computations, we show that the free Euler-Satake characteristics determine the number and type of point singularities, and that it takes an infinite collection of free Euler-Satake characteristics to do so. Additionally, we show that the stringy orbifold Euler characteristic determines all of the free abelian Euler-Satake characteristics for an orbifold in this class.

1. Introduction

The *Euler-Satake characteristic* $\chi_{ES}(Q)$ of an orbifold Q, originally introduced in [13] where it is called the *Euler characteristic as a V-manifold* and independently in [18] as the *orbifold Euler characteristic*, is the first of many Euler characteristics defined for orbifolds. A rational number, it coincides with $\chi(M)/|G|$ in the case that Q is a *global quotient orbifold*, i.e. is presented as the quotient of a manifold M by a finite group G. In general, it is defined in terms of a simplicial decomposition analogous to the usual Euler characteristic of a topological space. Other Euler characteristics commonly considered for orbifolds include the usual Euler characteristic of the underlying space $\chi(\mathbf{X}_Q)$, as well as the *stringy orbifold Euler characteristic* $\chi_{orb}(Q)$ defined in [4] for global quotients and [12] for general orbifolds, see also [10].

In [3], it is demonstrated that the topological Euler characteristic of Q and the stringy orbifold Euler characteristic of Q are the first and second elements in a sequence of Euler characteristics in the case of global quotients. In [16, 17], these definitions are extended to show that for global quotients, an Euler characteristic can be associated to any group Γ, the Euler characteristics of [3] corresponding to free abelian groups. In [9], the definition of these Euler

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characteristics is generalized to arbitrary orbifolds where they are referred to as the Γ-*extensions of the Euler-Satake characteristic*.

This paper continues a program to understand the extent to which the extensions of the Euler-Satake characteristic determine the topology of the orbifold and its singular set. In [6], it is demonstrated that the collection of $\mathbf{Z}^\ell$-Euler-Satake characteristics completely determine the diffeomorphism type of a closed, effective, orientable 2-orbifold, and no finite collection of Γ-Euler-Satake characteristics determine this information. In [14], it is established that the collection of $\mathbf{Z}/2\mathbf{Z}$-, $\mathbf{Z}^\ell$-, and $\mathbf{F}_\ell$-Euler-Satake characteristics, where $\mathbf{F}_\ell$ denotes the free group with ℓ generators, determine the number and type of point singularities of a closed, effective 2-orbifold as well as the Euler characteristic of the underlying space, that infinitely many free and free abelian groups as well as $\mathbf{Z}/2\mathbf{Z}$ are required to do so, and that no other information can be determined from the Γ-Euler-Satake characteristics. Here, we compute the $\mathbf{Z}^\ell$- and $\mathbf{F}_\ell$-Euler-Satake characteristics of a closed, effective, orientable 3-orbifold. It is shown in Theorem 4.2 that the $\mathbf{F}_\ell$-Euler-Satake characteristics determine the number and type of point singularities of the orbifold, and by Corollary 2.2, no further information is determined by any Γ-Euler-Satake characteristics. In Proposition 4.3, we demonstrate that infinitely many $\mathbf{F}_\ell$-Euler-Satake characteristics are required determine this information. In this case, the $\mathbf{Z}^\ell$-Euler-Satake characteristics contain comparatively little information, and in fact are determined by the stringy orbifold Euler characteristic, see Corollary 4.1.

In Section 2, we recall the structure of the singular set of a closed, effective, orientable 3-orbifold and describe the Γ-extensions of the Euler-Satake characteristic in this case. In Section 3, we detail computations of $\chi_{\mathbf{F}_\ell}^{ES}(Q)$ and $\chi_{\mathbf{Z}^l}^{ES}(Q)$ for the orbifolds under consideration. In Section 4, we use these results to determine the degree to which the free and free abelain Euler-Satake characteristics determine the structure of the singular set of an orbifold in this class.

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2. Γ-sectors of effective, orientable 3-orbifolds

Let Q be a closed, effective, orientable 3-orbifold. Given a proper, étale Lie groupoid $\mathcal{G}$ presenting Q, each point x in the space of objects G_0 of $\mathcal{G}$ is contained in a neighborhood V such that the restricted groupoid $\mathcal{G}_{|V}$ is isomorphic to $G \ltimes \mathbf{R}^3$ where G is a finite subgroup of SO(3). Note that $G \ltimes \mathbf{R}^3$ denotes the translation groupoid of the G-space $\mathbf{R}^3$. The identification of $G \ltimes \mathbf{R}^3$ with $\mathcal{G}_{|V}$ induces a map $\pi : \mathbf{R}^3/G \to |\mathcal{G}|$ where $|\mathcal{G}|$ denotes the orbit space of $\mathcal{G}$, and then the triple $\{\mathbf{R}^3, G, \pi\}$ is an orbifold chart or local uniformizing system in the sense of [13, 18, 2].

The underlying space $\mathbf{X}_Q$ of Q is a closed, orientable 3-manifold, and the singular locus of Q consists of the disjoint union of a finite trivalent graph and a finite collection of circles; see [2, 5]. Each point on a circle or edge of the graph is covered by a chart of the form $\mathbf{Z}/k\mathbf{Z} \ltimes \mathbf{R}^3$ where $\mathbf{Z}/k\mathbf{Z}$ acts as rotations about an axis; we refer to k as the *order* of the edge. The vertices of the graph are covered by charts of the form $G \ltimes \mathbf{R}^3$ where G is the tetrahedral group T of order 12, the octahedral group O of order 24, the icosahedral group I of order 60, or the dihedral group D_{2n} of order $2n$. Up to conjugation in SO(3), the representation R_T of the tetrahedral group

$$T = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1\rangle$$

on $\mathbf{R}^3$ can be taken to be that induced by setting

$$R_T(a) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R_T(b) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The representation R_O of the octahedral group

$$O = \langle r, s \mid r^2 = s^4 = (rs)^3 = 1\rangle$$

is given by

$$R_O(r) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad R_O(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

and that of the icosahedral group

$$I = \langle p, q \mid p^2 = q^5 = (pq)^3 = 1\rangle$$

is induced by

$$R_I(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad R_I(q) = \begin{bmatrix} \phi/2 & \bar{\phi}/2 & 1/2 \\ \bar{\phi}/2 & 1/2 & -\phi/2 \\ -1/2 & \phi/2 & \bar{\phi}/2 \end{bmatrix},$$

where $\phi = (\sqrt{5}+1)/2$ and $\bar{\phi} = (\sqrt{5}-1)/2$. The representation of the dihedral group

$$D_{2n} = \langle \alpha, \beta \mid \alpha^n = \beta^2 = 1, \alpha\beta = \beta\alpha^{n-1}\rangle$$

is given by

$$R_{D_{2n}}(\alpha) = \begin{bmatrix} \cos 2\pi/n & -\sin 2\pi/n & 0 \\ \sin 2\pi/n & \cos 2\pi/n & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R_{D_{2n}}(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We refer to the vertices of the trivalent graph as *point singularities* and let $\mathscr{P}$ denote the collection such points. Each point singularity corresponds to the fixed point of a tetrahedral, octahedral, icosahedral, or dihedral group, the *type* of the point singularity. By a *dihedral point of order n*, we mean a dihedral point with isotropy group D_{2n}.

For a finitely generated discrete group Γ, the *orbifold of* Γ*-sectors* $\tilde{Q}_\Gamma$ *of* Q is most succinctly defined in terms of a proper, étale Lie groupoid $\mathscr{G}$ presenting Q. Given such a presentation, the space $\mathrm{HOM}(\Gamma, \mathscr{G})$ of groupoid homomorphisms inherits the structure of a smooth manifold with a left $\mathscr{G}$-action, and the orbifold $\tilde{Q}_\Gamma$ is presented by the groupoid $\mathscr{G} \ltimes \mathrm{HOM}(\Gamma, \mathscr{G})$. Note that $\tilde{Q}_\Gamma$ is not connected unless Q is a manifold, and the connected components need not have the same dimension. An orbifold chart of the form $G \ltimes \mathbf{R}^3$ with G finite induces charts for $\tilde{Q}_\Gamma$ parameterized by G-conjugacy classes of homomorphisms $\psi : \Gamma \to G$. The chart associated to a homomorphism ψ is of the form $C_G(\psi) \ltimes (\mathbf{R}^3)^{\langle \psi \rangle}$ where $(\mathbf{R}^3)^{\langle \psi \rangle}$ denotes the fixed-point set of the image of ψ and $C_G(\psi)$ denotes the centralizer of ψ in G. The Γ*-Euler-Satake characteristic* $\chi_\Gamma^{ES}(Q)$ *of* Q is then given by applying the Euler-Satake characteristic to the orbifold of Γ-sectors,

$$\chi_\Gamma^{ES}(Q) = \chi_{ES}(\tilde{Q}_\Gamma).$$

See [1, 11] for background on groupoid presentations of orbifolds, [7] for details on the construction of the orbifold of Γ-sectors, [8] for relationships with other constructions and presentations of orbifolds, and [9] for details on the Γ-Euler-Satake characteristics and their relationship to other orbifold Euler characteristics. Note that $\chi_{\mathbf{Z}^2}^{ES}(Q)$ coincides with the stringy orbifold Euler characteristic $\chi_{orb}(Q)$ of [4, 12].

Proposition 2.1. *Let Q be a closed, effective, orientable 3-orbifold and let Γ be a finitely generated discrete group. Let $\mathscr{P}$ denote the collection of point singularities of Q, G_p the isotropy group of a point $p \in Q$, and $\mathrm{HOM}(\Gamma, G_p)^d$ the set of homomorphisms $\psi \in \mathrm{HOM}(\Gamma, G_p)$ whose image has a fixed-point set of dimension d. Then*

$$\chi_\Gamma^{ES}(Q) = \sum_{p \in \mathscr{P}} \frac{|\mathrm{HOM}(\Gamma, G_p)^0|}{|G_p|}. \tag{2.1}$$

Proof. As Q is effective and orientable, it follows that the sectors of Q consist of the nontwisted sector corresponding to the trivial homomorphisms and diffeomorphic to Q as well as 0- and 1-dimensional sectors. By [13, Theorem 4], $\chi_{ES}(Q) = 0$, and all closed 1-dimensional orbifolds have zero Euler-Satake characteristic as well. As the Euler-Satake characteristic of a zero-dimensional orbifold $G \ltimes_{triv} \{point\}$ (where $\ltimes_{triv}$ denotes a trivial group action) is simply $1/|G|$, and as zero-dimensional sectors clearly correspond only to homomorphisms into the isotropy group of a point singularity, we have

$$\chi_\Gamma^{ES}(Q) = \sum_{p \in \mathscr{P}} \sum_{(\psi) \in \mathrm{HOM}(\Gamma, G_p)^0 / G_p} \frac{1}{|C_{G_p}(\psi)|},$$

where (ψ) denotes the G_p-conjugacy class of $\psi \in \mathrm{HOM}(\Gamma, G_p)^0$. Applying the fact that for each $\psi \in \mathrm{HOM}(\Gamma, G_p)$, $|G_p| = |(\psi)|\,|C_{G_p}(\psi)|$ completes the proof. □

The following is an immediate consequence.

COROLLARY 2.2. *Suppose Q and Q' are closed, effective, orientable 3-orbifolds that have the same number and type of point singularities. Then $\chi_\Gamma^{ES}(Q) = \chi_\Gamma^{ES}(Q')$ for every finitely generated discrete group Γ.*

Note that $\chi_{\mathbf{Z}}^{ES}(Q)$ coincides with $\chi(\mathbf{X}_Q)$, the Euler characteristic of the underlying space of Q; see [15]. As $\mathbf{X}_Q$ is in this case a closed 3-manifold, we have that $\chi_{\mathbf{Z}}^{ES}(Q) = 0$.

3. Free and free abelian Euler-Satake characteristics of effective, orientable 3-orbifolds

In this section, we present the following computation of extensions of the Euler-Satake characteristics associated to free and free abelian groups.

THEOREM 3.1. *Let Q be a closed, effective, orientable, 3-orbifold with t tetrahedral points, o octahedral points, i icosahedral points, d_{odd} dihedral points of odd orders n_j for $j = 1, \dots d_{odd}$, and d_{ev} dihedral points of even orders n_j for $j = d_{odd}+1, \dots, d_{odd}+d_{ev}$.*

I. *For each $\ell \geq 0$, the $\mathbf{Z}^\ell$-Euler-Satake characteristic of Q is given by*

$$\chi_{\mathbf{Z}^\ell}^{ES}(Q) = \frac{1}{12}(4^\ell - 3 \cdot 2^\ell + 2)(t + 2o + i + 3d_{ev}). \tag{3.1}$$

II. *For each $\ell \geq 0$, the $\mathbf{F}_\ell$-Euler-Satake characteristic of Q is given by*

$$\begin{aligned} \chi_{\mathbf{F}_\ell}^{ES}(Q) = {} & \frac{t}{2}(2 \cdot 12^{\ell-1} - 2 \cdot 3^{\ell-1} - 2^{\ell-1} + 1) \\ & + \frac{o}{2}(2 \cdot 24^{\ell-1} - 4^{\ell-1} - 3^{\ell-1} - 2^{\ell-1} + 1) \\ & + \frac{i}{2}(2 \cdot 60^{\ell-1} - 5^{\ell-1} - 3^{\ell-1} - 2^{\ell-1} + 1) \\ & + \frac{2^\ell - 1}{2} \sum_{j=1}^{d_{odd}+d_{ev}} (n_j^{\ell-1} - 1). \end{aligned} \tag{3.2}$$

Proof. To prove I., note that every nontrivial element of SO(3) acts on $\mathbf{R}^3$ as a rotation about a line. Recalling that $\mathrm{HOM}(\Gamma, G_p)^d$ denotes the homomorphisms $\psi \in \mathrm{HOM}(\Gamma, G_p)$ whose image has a fixed-point set of dimension $d, \psi \in \mathrm{HOM}(\Gamma, G_p)$ is an element of $\mathrm{HOM}(\Gamma, G_p)^1$ if and only if the image of ψ is isomorphic to $\mathbf{Z}/n\mathbf{Z}$ for some n. Hence, $\mathrm{HOM}(\mathbf{Z}^\ell, G_p)^0$ consists of those $\psi \in \mathrm{HOM}(\mathbf{Z}^\ell, G_p)$ whose image is not cyclic. Note that the image of $\psi \in \mathrm{HOM}(\mathbf{Z}^\ell, G_p)$ must be abelian, and the only abelian subgroups of SO(3) are cyclic or isomorphic to D_4. Therefore, if r denotes the number of distinct subgroups of G_p isomorphic to D_4, we have

$$|\mathrm{HOM}(\mathbf{Z}^\ell, G_p)^0| = r|\mathrm{HOM}(\mathbf{Z}^\ell, D_4)^0|. \tag{3.3}$$

Note that $\psi \in \mathrm{HOM}(\mathbf{Z}^\ell, D_4)$ fixes a point if and only if ψ is surjective, so that

$$|\mathrm{HOM}(\mathbf{Z}^\ell, D_4)^0| = [(4^\ell - 1) - 3(2^\ell - 1)] = 4^\ell - 3 \cdot 2^\ell + 2. \tag{3.4}$$

By inspection, the only subgroup of T isomorphic to D_4 is $\langle a, (ab)^2 b\rangle$. The four subgroups of O isomorphic to D_4 are $\langle r, (rs^2)^2\rangle$; $\langle s^2, rs^2r\rangle$; $\langle s^2, rs^2rs\rangle$; and $\langle rs^2r, rs^3rs^2\rangle$; while the five subgroups of I isomorphic to D_4 are the five conjugates of $C_I(p) = \langle p, pq^2pq^3pq^2\rangle$. If n is odd, there are no subgroups of D_{2n} isomorphic to D_4, while if n is even, the subgroups isomorphic to D_4 are the $n/2$ conjugates of $C_{D_{2n}}(\beta) = \langle \alpha^{n/2}, \beta\rangle$. Combining these observations and Equations 3.3 and 3.4 with Proposition 2.1 yields Equation 3.1, the formula for for $\chi_{\mathbf{Z}^\ell}^{ES}(Q)$.

We now consider the proof of II. Let a_n denote the number of distinct lines in $\mathbf{R}^3$ with G_p-isotropy group isomorphic to $\mathbf{Z}/n\mathbf{Z}$, and then $|\mathrm{HOM}(\mathbf{F}_\ell, G_p)^1| = \sum_{n=2}^{\infty} a_n |\mathrm{HOM}(\mathbf{F}_\ell, \mathbf{Z}/n\mathbf{Z})^1|$. It is easy to see by considering the image of a fixed set of generators of $\mathbf{F}_\ell$ that $|\mathrm{HOM}(\mathbf{F}_\ell, \mathbf{Z}/n\mathbf{Z})| = n^\ell$. All of the nontrivial homomorphisms in $\mathrm{HOM}(\mathbf{F}_\ell, \mathbf{Z}/n\mathbf{Z})$ fix lines so that $|\mathrm{HOM}(\mathbf{F}_\ell, \mathbf{Z}/n\mathbf{Z})^1| = n^\ell - 1$. As $\mathrm{HOM}(\mathbf{F}_\ell, G_p)$ consists of $\mathrm{HOM}(\mathbf{F}_\ell, G_p)^0$, $\mathrm{HOM}(\mathbf{F}_\ell, G_p)^1$, and the trivial homomorphism, we have

$$|\mathrm{HOM}(\mathbf{F}_\ell, G_p)^0| = |\mathrm{HOM}(\mathbf{F}_\ell, G_p)| - 1 - \sum_{n=2}^{\infty} a_n(n^\ell - 1). \tag{3.5}$$

It remains only to determine the values of a_n for the possible isotropy groups G_p.

With respect to the T-action on $\mathbf{R}^3$, there are $a_3 = 4$ distinct lines fixed by a subgroup isomorphic to $\mathbf{Z}/3\mathbf{Z}$ conjugate to $\langle b\rangle$ and $a_2 = 3$ lines fixed by a subgroup isomorphic to $\mathbf{Z}/2\mathbf{Z}$ conjugate to $\langle a\rangle$. Hence,

$$|\mathrm{HOM}(\mathbf{F}_\ell, T)^0| = 12^\ell - 1 - 4(3^\ell - 1) - 3(2^\ell - 1), \quad \text{and hence}$$

$$\frac{|\mathrm{HOM}(\mathbf{F}_\ell, T)^0|}{|T|} = \frac{1}{2}(2 \cdot 12^{\ell-1} - 2 \cdot 3^{\ell-1} - 2^{\ell-1} + 1).$$

With respect to the O-action on $\mathbf{R}^3$, there are $a_4 = 3$ lines fixed by a subgroup isomorphic to $\mathbf{Z}/4\mathbf{Z}$ conjugate to $\langle s\rangle$, $a_3 = 4$ lines fixed by a subgroup isomorphic to $\mathbf{Z}/3\mathbf{Z}$ conjugate to $\langle rs\rangle$, and $a_2 = 6$ lines fixed by a subgroup isomorphic to $\mathbf{Z}/2\mathbf{Z}$ conjugate to $\langle r\rangle$. Therefore,

$$|\mathrm{HOM}(\mathbf{F}_\ell, O)^0| = 24^\ell - 1 - 3(4^\ell - 1) - 4(3^\ell - 1) - 6(2^\ell - 1), \quad \text{and hence}$$

$$\frac{|\mathrm{HOM}(\mathbf{F}_\ell, O)^0|}{|O|} = \frac{1}{2}(2 \cdot 24^{\ell-1} - 4^{\ell-1} - 3^{\ell-1} - 2^{\ell-1} + 1).$$

With respect to the I-action on $\mathbf{R}^3$, there are $a_5 = 6$ lines fixed by a subgroup isomorphic to $\mathbf{Z}/5\mathbf{Z}$ conjugate to $\langle q\rangle$, $a_3 = 10$ lines fixed by a subgroup isomorphic to $\mathbf{Z}/3\mathbf{Z}$ conjugate to $\langle pq\rangle$, and $a_2 = 15$ lines fixed by a subgroup isomorphic to $\mathbf{Z}/2\mathbf{Z}$ conjugate to $\langle p\rangle$. Hence,

$$|\mathrm{HOM}(\mathbf{F}_\ell, I)^0| = 60^\ell - 1 - 6(5^\ell - 1) - 10(3^\ell - 1) - 15(2^\ell - 1), \quad \text{and hence}$$

$$\frac{|\mathrm{HOM}(\mathbf{F}_\ell, I)^0|}{|I|} = \frac{1}{2}(2 \cdot 60^{\ell-1} - 5^{\ell-1} - 3^{\ell-1} - 2^{\ell-1} + 1).$$

Finally, with respect to the D_{2n}-action on $\mathbf{R}^3$, there is $a_n = 1$ line fixed by $\langle\alpha\rangle \cong \mathbf{Z}/n\mathbf{Z}$ and $a_2 = n$ lines fixed by a subgroup isomorphic to $\mathbf{Z}/2\mathbf{Z}$ conjugate to $\langle\beta\rangle$, so that

$$|\mathrm{HOM}(\mathbf{F}_\ell, D_{2n})^0| = (2n)^\ell - 1 - (n^\ell - 1) - n(2^\ell - 1), \quad \text{and hence}$$

$$\frac{|\mathrm{HOM}(\mathbf{F}_\ell, D_{2n})^0|}{|D_{2n}|} = \frac{(2^\ell - 1)(n^{\ell-1} - 1)}{2}.$$

Combining these computations yields Equation 3.2, the formula for $\chi^{ES}_{\mathbf{F}_\ell}(Q)$. $\square$

4. Consequences of Theorem 3.1

Our first observation is that the higher free abelian extensions of the Euler-Satake characteristic all contain exactly the same information as the stringy orbifold Euler characteristic $\chi_{orb}(Q) = \chi^{ES}_{\mathbf{Z}^2}(Q)$.

Corollary 4.1. *Let Q be a closed, effective, orientable 3-orbifold. Every $\mathbf{Z}^\ell$-Euler-Satake characteristic of Q is determined by $\chi^{ES}_{\mathbf{Z}^2}(Q)$.*

Proof. In fact, for $\ell \geq 2$, Equation 3.1 implies that

$$\chi^{ES}_{\mathbf{Z}^\ell}(Q) = \frac{1}{6}(4^\ell - 3 \cdot 2^\ell + 2)\chi^{ES}_{\mathbf{Z}^2}(Q).$$

Note that $\chi_{ES}(Q) = 0$, and $\chi^{ES}_{\mathbf{Z}}(Q) = \chi(\mathbf{X}_Q) = 0$. $\square$

The $\mathbf{F}_\ell$-Euler-Satake characteristics, on the other hand, determine the point singularities of Q by the following.

THEOREM 4.2. *Let Q and Q' be closed, effective, orientable 3-orbifolds such that for some infinite collection $\mathcal{L}$ of positive integers ℓ, we have $\chi^{ES}_{\mathbf{F}_\ell}(Q) = \chi^{ES}_{\mathbf{F}_\ell}(Q')$ $\forall \ell \in \mathcal{L}$. Then Q and Q' have the same number of point singularities of each type.*

Proof. Let Q and Q' be closed, effective, orientable 3-orbifolds, and let $\mathcal{L}$ be an infinite set of nonnegative integers. Let t, o, i and d denote the number of tetrahedral, octahedral, icosahedral, and dihedral points of Q, respectively, and let d_r denote the number of dihedral points of order r for each $r \geq 2$. Similarly, let t', o', i', d', and d'_r denote the number of point singularities of Q' of each type. Let $t'' = t - t'$, $o'' = o - o'$, $i'' = i - i'$, $d'' = d - d'$, and $d''_r = d_r - d'_r$ for each r.

For each integer $r \geq 1$ let f_r denote the sequence $(r^{\ell-1})_{\ell \in \mathcal{L}}$, considered as an element of the linear space $\mathbf{R}^\infty$ of sequences of real numbers. Similarly, let

$$\chi = (\chi^{ES}_{\mathbf{F}_\ell}(Q))_{\ell \in \mathcal{L}} \quad \text{and} \quad \chi' = (\chi^{ES}_{\mathbf{F}_\ell}(Q'))_{\ell \in \mathcal{L}},$$

also considered as elements of $\mathbf{R}^\infty$. Then by Equation 3.2, we can express χ and χ' as linear combinations

$$\chi = \sum_{r=1}^{\infty} c_r f_r \quad \text{and} \quad \chi' = \sum_{r=1}^{\infty} c'_r f_r,$$

each with finitely many nonzero coefficients. In particular, setting $c''_r = c_r - c'_r$ for each r, we have

$$\begin{aligned} c''_1 &= (t'' + o'' + i'' + d'')/2, & c''_5 &= -(i'' + d''_5)/2, \\ c''_2 &= -(t'' + o'' + i'' + d''_2)/2 - d'', & c''_{12} &= t'' + d''_6 - d''_{12}/2, \\ c''_3 &= -(o'' + i'' + d''_3)/2 - t'', & c''_{24} &= o'' + d''_{12} - d''_{24}/2, \\ c''_4 &= -(o'' + d''_4)/2 + d''_2, & c''_{60} &= i'' + d''_{30} - d''_{60}/2. \end{aligned}$$

If $r \notin \{1, 2, 3, 4, 5, 12, 24, 60\}$, then $c''_r = d''_{r/2} - \dfrac{d''_r}{2}$ when r is even and $c''_r = -\dfrac{d''_r}{2}$ when r is odd. We have by hypotheses that $\sum_{r=1}^{\infty} c''_r f_r = \chi - \chi' = 0$, so that, as the f_r are linearly independent, $c''_r = 0$ for each r.

As Q and Q' are closed and hence have a finite number of point singularities, the d_r and d'_r are zero for sufficiently large r. Hence, it is easy to see that $d_r = d'_r$ for $r \neq 2, 3, 6, 12, 30$. Then as $c''_5 = 0$, it follows that $i'' = 0$ and hence as $c''_{60} = 0$ that $d''_{30} = 0$. The resulting equations $c''_r = 0$ for $r = 1, 2, 3, 4, 6, 12, 24$ along with

$$d'' = d''_2 + d''_3 + d''_6 + d''_{12}$$

yield a system of eight equations in the unknowns t'', o'', d'', d''_2, d''_3, d''_6, and d''_{12}. By a simple computation, the solutions of this system all satisfy $o'' = -d''_{12}$. Hence, the only nonnegative solution is the trivial solution, completing the proof. $\square$

PROPOSITION 4.3. *Let L be a positive integer. Then there are distinct closed, effective, orientable 3-orbifolds Q and Q' such that for each $\ell \leq L$,*

$$\chi^{ES}_{\mathbf{F}_\ell}(Q) = \chi^{ES}_{\mathbf{F}_\ell}(Q').$$

Proof. The proof of [6, Lemma 3.11] constructs for each $L \geq 2$ collections of integers $2 \leq n_1 \leq \cdots \leq n_k$ and $2 \leq m_1 \leq \cdots \leq m_k$, which can all be taken to be odd, such that for each $\ell \leq L$,

$$\sum_{j=1}^{k} n_j^{\ell-1} = \sum_{j=1}^{k} m_j^{\ell-1},$$

and such that $n_i \neq m_j$ for each i, j. Let Q be the orbifold with underlying space S^3 and singular set given by the connected trivalent graph with dihedral vertices $v_1, v_2, \ldots, v_{2k}$ of orders $n_1, n_1, n_2, n_2, \ldots, n_k, n_k$, one edge of order n_j connecting v_{2j-1} and v_{2j} for $1 \leq j \leq k$, and two edges of order 2 connecting v_{2j} and v_{2j+1} for each $1 \leq j < k$ as well as v_{2k} and v_1. Similarly, let Q' be the orbifold with underlying space S^3 and singular set the connected trivalent graph with dihedral vertices $w_1, w_2, \ldots, w_{2k}$ of orders $m_1, m_1, m_2, m_2, \ldots, m_k, m_k$, one edge of order m_j connecting w_{2j-1} and w_{2j} for $1 \leq j \leq k$, and two edges of order 2 connecting each w_{2j} and w_{2j+1} for $1 \leq j < k$ as well as w_{2k} and w_1. Then by Equation 3.2, for each $\ell \leq L$ we have

$$\begin{aligned}\chi^{ES}_{\mathbf{F}_\ell}(Q) &= (2^\ell - 1)\sum_{j=1}^{k}(n_j^{\ell-1} - 1) \\ &= (2^\ell - 1)\sum_{j=1}^{k}(m_j^{\ell-1} - 1) = \chi^{ES}_{\mathbf{F}_\ell}(Q'),\end{aligned}$$ □

In particular Proposition 4.3 implies that Theorem 4.2 cannot be improved upon by considering the Euler-Satake characteristics associated to any finite collection of free groups.

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