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# HARMONIC MAPS FROM THE RIEMANN SPHERE INTO THE COMPLEX PROJECTIVE SPACE AND THE HARMONIC SEQUENCES

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**Abstract**

When harmonic maps from the Riemann sphere into the complex projective space are energy bounded, it contains a subsequence converging to a bubble tree map $f^I : T^I \to \mathbf{C}P^n$. We show that their $\partial$-transforms and $\bar{\partial}$-transforms are also energy bounded. Hence their subsequences converge to harmonic bubble tree maps $f_1^{I_1} : T^{I_1} \to \mathbf{C}P^n$ and $f_{-1}^{I_{-1}} : T^{I_{-1}} \to \mathbf{C}P^n$ respectively. In this paper, we show relations between $f^I$, $f_1^{I_1}$ and $f_{-1}^{I_{-1}}$.

## 1. Introduction

In [12], Sacks & Uhlenbeck have shown that any harmonic maps defined on a closed surface with bounded energy contains a subsequence weakly converging to a set of harmonic maps and that a bubbling phenomenon may occur in the convergence. Gromov ([6]) also noticed a bubbling phenomenon in the study of pseudo holomorphic maps.

In this paper, we concentrate on harmonic maps from the Riemann sphere $S^2$, $g_0$ into the complex projective space $\mathbf{C}P^n$, $g$. Here we identify $S^2$, $g_0$ with $\mathbf{C}P^1$, $g$ and consider it as the complex manifold. Combining the resuls by Eells & Wood in [4, §6] with Wolfson in [14], for each full harmonic map $f : S^2 \to \mathbf{C}P^n$, we get a harmonic sequence

$$seq(f,r) : 0 \overset{\bar{\partial}}{\leftarrow} f_0 \overset{\partial}{\to} f_1 \overset{\partial}{\to} \cdots \overset{\partial}{\to} f_r \overset{\partial}{\to} \cdots \overset{\partial}{\to} f_n \overset{\partial}{\to} 0$$

with $f_r = f$.

Let $\mathscr{H}arm(\mathbf{C}P^n)$ be the set of harmonic maps in a Banach manifold $W^{1,p}(S^2, \mathbf{C}P^n)$ for $p > 2$. Refining the "Sacks-Uhlenbeck" limit, Parker & Wolfson ([11]) give a definition of "converging to a harmonic bubble tree map". Though their definition in [11] is for pseudo-holomorphic maps, as mentioned in it, the definition is applicable for harmonic maps. In [11] and [10], they have

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shown that, in this sense, harmonic maps with bounded energy contain a sequence converging to a harmonic bubble tree map satisfying appropriate conditions. Our main result is the following. As for details of notations or terminologies, we will define in the following sections.

Main Theorem. *Let* $S^2$, $g_0$ *be the Riemann sphere and* $\mathbf{C}P^n$, $g$ *be the complex projective space. Take a sequence* $\{f^k\}_k$ *in* $\mathcal{H}arm(\mathbf{C}P^n)$ *which are energy bounded. Then both* $\{\partial f^k\}_k$ *and* $\{\bar{\partial} f^k\}_k$ *are also energy bounded. Passing through subsequences,* $\{f^k\}_k$, $\{\partial f^k\}_k$ *and* $\{\bar{\partial} f^k\}_k$ *converge to either trivial maps or harmonic bubble tree maps*

$$f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbf{C}P^n$$

$$f_1^{I_1} = \bigvee_{\ell' \in I_1} f_1^{(\ell')} : T^{I_1} \to \mathbf{C}P^n$$

$$f_{-1}^{I_{-1}} = \bigvee_{\ell'' \in I_{-1}} f_{-1}^{(\ell'')} : T^{I_{-1}} \to \mathbf{C}P^n$$

*respectively satisfying the followings*:

**(1)** *If* $\partial f^{(\ell)}$ *is non-trivial, it is equivalent to* $f_1^{(\ell')}$ *for some* $\ell' \in I_1$; $f_1^{(\ell')} = \partial f^{(\ell)} \circ \sigma_\ell$ *satisfying* $\sigma_\ell(B_{f_1^{(\ell')}}) \subset B_{f^{(\ell)}}$.

**(2)** *When* $f_1^{(\ell')}$ *is not equivalent to any* $\partial f^{(\ell)}$, $f_1^{(\ell')}$ *is a holomorphic map of the length no greater than* $n - r - 1$.

**(3)** *If* $\bar{\partial} f^{(\ell)}$ *is non-trivial, it is equivalent to* $f_{-1}^{(\ell'')}$ *for some* $\ell'' \in I_{-1}$; $f_{-1}^{(\ell'')} = \bar{\partial} f^{(\ell)} \circ \bar{\sigma}_\ell$ *with* $\bar{\sigma}_\ell(B_{f_{-1}^{(\ell'')}}) \subset B_{f^{(\ell)}}$.

**(4)** *When* $f_{-1}^{(\ell'')}$ *is not equivalent to any* $\bar{\partial} f^{(\ell)}$, $f_{-1}^{(\ell'')}$ *is an anti-holomorphic map of the length no greater than* $r - 1$.

*Here* $r + 1$ *is the* $\bar{\partial}$*-order of* $f$.

Here and throughout this paper, to simplify notation, we adopt the convention of immediately renaming subsequences and so a subsequence of $\{f^k\}$ is still denoted by the same way.

Contents are as follows. In §2, we begin to introduce harmonic maps defined on $S^2$, $g_0$ into $\mathbf{C}P^n$, $g$. Associated to each harmonic map, we consider its harmonic sequence. We refer related results. In §3, we define a harmonic bubble tree map introduced by Parker & Wolfson in [11]. Then we show Main Theorem. In §4, we consider when harmonic maps into either $\mathbf{C}P^1$ or $\mathbf{C}P^2$ are gluable. Lastly, in §5, we consider examples of gluable or non-gluable harmonic bubble tree maps and their harmonic sequences.

## 2. A harmonic map and a harmonic sequence

Let $\mathbf{C}^{n+1}$ be the complex $(n+1)$-dimensional space equipped with the standard Hermitian inner product defined by

$$X \cdot Y = \sum_j x_j \bar{y}_j \quad \textit{where } X = (x_j)_{0 \le j \le n},\ Y = (y_j)_{0 \le j \le n} \in \mathbf{C}^{n+1}.$$

Put $|X| = \sqrt{X \cdot X}$. We equip the Fubini-Study metric $g$ on $\mathbf{C}P^n$ of constant holomorphic sectional curvature 4. As for the geometry of $\mathbf{C}P^n$, refer [7, IX. 6. Example 6.3]. When $n = 1$, we get an isomorphism $S^2 \simeq \mathbf{C}P^1$ through a stereographic projection

$$S^2 - \{\infty\} \to \mathbf{C} \simeq U_0 = \{[z_0 : z_1] \in \mathbf{C}P^1 \,|\, z_0 \neq 0\} = \mathbf{C}P^1 - \{[0 : 1]\}$$

which takes the north pole to the origin, the south pole $\infty$ to infinity, and the equator to the unit circle. Here $[z_0 : z_1]$ is the homogeneous coordinate system of $\mathbf{C}P^1$. Let $S^2$, $g_0$ be the sphere with the Reimann metirc $g_0$ induced from $\mathbf{C}P^1$. As mentioned in §1, we also equip the complex structure on $S^2$ induced from $\mathbf{C}P^1$. On a coordinate neighbourhood $U_0$, the metric $g_0$ is customary represented by $ds_0^2 = \varphi\bar{\varphi} = \frac{dz d\bar{z}}{(1+|z|^2)^2}$ for $z = \frac{z_1}{z_0} \in \mathbf{C} \simeq U_0$. Here $\varphi$ is determined up to a complex factor of absolute value 1.

Throughout this paper, take and fix a real $p > 2$. As $1 > \frac{2}{p}$, we can get a Banach manifold $W^{1,p}(S^2, \mathbf{C}P^n)$ consisting of maps $f : S^2 \to \mathbf{C}P^n$ whose derivatives of order $\leq 1$ are $L_p$ integrable. A map $f \in W^{1,p}(S^2, \mathbf{C}P^n)$ is harmonic if it is a critical point of the energy functional $E : W^{1,p}(S^2, \mathbf{C}P^n) \to \mathbf{R}$ defined by

$$E(f) = \int_{S^2} |df|^2 \frac{\sqrt{-1}}{2} \varphi \wedge \bar{\varphi}$$

where $|df|^2$ is the Hilbert-Schmidt norm $\langle g_0, f^*g \rangle_{HS}$. Thus we consider the set $\mathscr{H}arm(\mathbf{C}P^n)$ of harmonic maps as a subspace of $W^{1,p}(S^2, \mathbf{C}P^n)$. Because of the regularity and the Sobolev embedding theorem $C^0 \supset W^{1,p}$, $\mathscr{H}arm(\mathbf{C}P^n)$ is contained in the set $C^s(S^2, \mathbf{C}P^n)$ of all $C^s$ maps for any $s \geq 0$. Since $f$ is defined between Kähler manifolds, any holomorphic or anti-holomorphic map is harmonic. Refer [9] and also [3, (8.15) Corollary]. Denote by $\mathscr{H}ol(\mathbf{C}P^n)$ the subspace of $\mathscr{H}arm(\mathbf{C}P^n)$ consisting of holomorphic maps.

Now we introduce a $\partial$ transform and a $\bar{\partial}$ transform in [1] which is the same correspondence given in [4, §3]. For a smooth map $f : S^2 \to \mathbf{C}P^n$, let $\pi_f : V(f) \to S^2$ be the tautological complex line bundle whose fiber at $z \in S^2$ is $f(z)$. For a $\mathbf{C}$-line $X$ in $\mathbf{C}^{n+1}$, denote by $X^\perp$ the orthogonal complement of $X$ in $\mathbf{C}^{n+1}$. Define a smooth map $f^\perp : S^2 \to G(n, n+1)$ by $f^\perp(z) = f(z)^\perp$. Here $G(n, n+1)$ is the complex Grassmann manifold consisting of $n$-dimensional subspaces in $\mathbf{C}^{n+1}$. We equip the standard Riemann metric $g_n$ and the complex structure on it. Refer [7, IX, Example 6.4]. $f^\perp$ also defines the tautological bundle $V(f^\perp) \to S^2$. By [1, §2], both $V(f)$ and $V(f^\perp)$ are holomorphic bundles over $S^2$.

Take a unitary frame $Z_0, Z_1, \ldots, Z_n$ of $\mathbf{C}^{n+1}$ so that $Z_0$ defines $f$. Then put

$$dZ_0 = \omega_0 Z_0 + \sum_{r \geq 1} \omega_r Z_r, \quad f^*\omega_r = a_r\varphi + b_r\bar{\varphi}$$

and define maps

$$\partial : V(f) \to V(f^{\perp}) \otimes T^{(1,0)}, \quad \partial(\xi^0 Z_0) = \left(\xi^0 \sum_r a_r Z_r\right) \otimes \varphi,$$

$$\bar{\partial} : V(f) \to V(f^{\perp}) \otimes T^{(0,1)}, \quad \bar{\partial}(\xi^0 Z_0) = \left(\xi^0 \sum_r b_r Z_r\right) \otimes \bar{\varphi}.$$

Here $T^{(1,0)}$ (resp. $T^{(0,1)}$) is the cotangent bundle on $S^2$ of type $(1,0)$ (resp. $(0,1)$). We get the followings.

THEOREM 1 ([1], §2). *If $f \in \mathscr{H}arm(\mathbf{C}P^n)$, $\partial$ is a holomorphic bundle map and $\bar{\partial}$ is an anti-holomorphic bundle map.*

Denote by $[V(f)]$ the projectivization of $V(f)$. Though $\varphi$ is determined only up to a complex factor of absolute value 1, we get the fundamental colliniation of $f$

$$[V(f)] \ni [f(z)] \to [\partial f(z)] \in [V(f^{\perp})]$$

if $\partial f(z) \neq 0$. As mentioned in [1, §2], when $f$ is harmonic, by Theorem 1, we can get a well-defined non-trivial map $\partial f : S^2 \to \mathbf{C}P^n$ as far as $f$ is not anti-holomorphic. We call it the $\partial$ transform of $f$. When $f$ is anti-holomorphic, we define the $\partial$ transform of $f$ as a zero map. Similarly we also get the fundamental colliniation

$$[V(f)] \ni [f(z)] \to [\bar{\partial} f(z)] \in [V(f^{\perp})]$$

if $\bar{\partial} f(z) \neq 0$. If $f$ is not holomorphic, this defines a non-trivial map $\bar{\partial} f : S^2 \to \mathbf{C}P^n$ which we call the $\bar{\partial}$ transform of $f$. When $f$ is holomorphic, the $\bar{\partial}$ transform of $f$ is defined as a zero map.

THEOREM 2 ([1], Theorem 2.2). *Take $f \in \mathscr{H}arm(\mathbf{C}P^n)$. Then we get the followings.*
(1) *$f^{\perp} : S^2 \to G(n, n+1)$ is harmonic.*
(2) *Both the $\partial$ transform of $f$ and its $\bar{\partial}$ transform are harmonic.*
(3) *If $\partial f$ is non-trivial, $\bar{\partial}\partial f = f$.*
(4) *If $\bar{\partial} f$ is non-trivial, $\partial\bar{\partial} f = f$.*

We say that $f \in \mathscr{H}arm(\mathbf{C}P^n)$ is full if its image lies in no proper projective subspace of $\mathbf{C}P^n$. Associated to a full map $f \in \mathscr{H}ol(\mathbf{C}P^n)$, take a lift $Z : S^2 \supset U \to \mathbf{C}^{n+1} - \{0\}$ over a chart $U$. Classically we get the Frenet frame $\{Z_r\}_{r \geq 0}$ of $f$ which is obtained by the Gram-Schmidt's orthogonalization of $\{\frac{\partial^r}{\partial z^r} Z\}_r$ except at finite point of $S^2$. Since the zeros of

$$Z \wedge \frac{\partial}{\partial z} Z \wedge \cdots \wedge \frac{\partial^n}{\partial z^n} Z$$

are finite and are removable, this frame can be uniquely extended over $S^2$. Refer [15, §4]. We get

$$dZ_r = -\bar{a}_{r-1}\bar{\varphi}Z_{r-1} + \omega_r Z_r + a_r\varphi Z_{r+1}$$

for $0 \le r \le n$ where $a_{-1} = a_n = 0$. For $0 \le r \le n$, let $f_r : S^2 \to \mathbf{C}P^n$ be the non-trivial map defined by $Z_r$. By definition, $f_{r+1}$ is the $\partial$ transform of $f_r$ and $f_{r-1}$ is the $\bar{\partial}$ transform of $f_r$. Hence, by Theorem 2, $f_r$ is harmonic for any $r$. We call the sequence of harmonic maps

$$seq(f,0) : 0 \xleftarrow{\bar{\partial}} f_0 = f \xrightarrow{\partial_0} f_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{r-1}} f_r \xrightarrow{\partial_r} \cdots \xrightarrow{\partial_{n-1}} f_n \xrightarrow{\partial_n} 0$$

a harmonic sequence of $f$ with the length $n$.

When $f \in \mathscr{H}ol(\mathbf{C}P^n)$ is not full, we can choose an isometry $T^A : \mathbf{C}P^n \to \mathbf{C}P^n$ induced from a unitary transformation $A : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ so that

$$f = T^A \circ \iota \circ f^A : S^2 \xrightarrow{f^A} \mathbf{C}P^{n_0} \overset{\iota}{\subset} \mathbf{C}P^n \xrightarrow{T^A} \mathbf{C}P^n$$

by a full $f^A \in \mathscr{H}ol(\mathbf{C}P^{n_0})$ and the inclusion $\iota$. We define a harmonic sequence of $f$ of the length $n_0$

$$seq(f,0) : 0 \xleftarrow{\bar{\partial}} f_0 = f \xrightarrow{\partial_0} f_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{r-1}} f_r \xrightarrow{\partial_r} \cdots \xrightarrow{\partial_{n_0-1}} f_{n_0} \xrightarrow{\partial_{n_0}} 0$$

by compositions $f_r = T^A \circ \iota \circ f_r^A$;

$$seq(f^A,0) : 0 \xleftarrow{\bar{\partial}} f_0^A = f^A \xrightarrow{\partial_0} f_1^A \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{r-1}} f_r^A \xrightarrow{\partial_r} \cdots \xrightarrow{\partial_{n_0-1}} f_{n_0}^A \xrightarrow{\partial_{n_0}} 0.$$

Here $seq(f,0)$ is defined independently on the choice of a unitary matrix $A$. Following to [4, Definition 5.1], we define the $\partial$-order of $f \in \mathscr{H}arm(\mathbf{C}P^n)$ by

$$\max_U \max_{z \in U} \dim span\{\partial^\alpha Z_U(z) \mid 0 \le \alpha\}$$

and also the $\bar{\partial}$-order of $f$ by

$$\max_U \max_{z \in U} \dim span\{\bar{\partial}^\beta Z_U(z) \mid 0 \le \beta\}.$$

Here $Z_U : S^2 \supset U \to \mathbf{C}^{n+1}$ is a lift of $f$ over a chart $U$ and $span\{\mathbf{v}^\alpha\}_\alpha$ is the subspace of $\mathbf{C}^{n+1}$ spanned by vectors $\{\mathbf{v}^\alpha\}_\alpha$. These orders are determined independently on the choice of a lift $Z_U$. By [14, Theorem 3.1 & Theorem 3.4], we get the following.

THEOREM 3. *For any non-trivial $f \in \mathscr{H}arm(\mathbf{C}P^n)$, we get $f_0 \in \mathscr{H}ol(\mathbf{C}P^n)$ so that the harmonic sequence of $f_0$ contains $f$;*

$$seq(f,r) : 0 \xleftarrow{\bar{\partial}} f_0 \xrightarrow{\partial_0} f_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{r-1}} f_r = f \xrightarrow{\partial_r} \cdots \xrightarrow{\partial_r} f_{n_0} \xrightarrow{\partial_{n_0}} 0$$

*where $1 \le n_0 \le n$, $r+1$ is the $\bar{\partial}$-order of $f$ and $n_0 - r + 1$ is its $\partial$-order.*

We also call $seq(f,r)$ the harmonic sequence of $f$ with the length $n_0$. Obviously $f \in \mathscr{H}arm(\mathbf{C}P^n)$ is full exactly when the length of $seq(f,r)$ is $n$. Let $\mathscr{H}arm^*(\mathbf{C}P^n)$ be the subspace of $\mathscr{H}arm(\mathbf{C}P^n)$ consisting of full maps. Correspondingly $\mathscr{H}ol^*(\mathbf{C}P^n)$ is denoted for the space of all full maps in $\mathscr{H}ol(\mathbf{C}P^n)$.

Essentially, by [4, Theorem 6.9], we get the following. Theorem 3 gives the correspondence of the following theorem.

THEOREM 4. *There is a bijective correspondence between* $f \in \mathscr{H}arm^*(\mathbf{C}P^n)$ *and pairs* $(f_0, r)$ *where* $f_0 \in \mathscr{H}ol^*(\mathbf{C}P^n)$ *and* $r$ *is an integer with* $0 \le r \le n$.

For a smooth map $f \in W^{1,p}(S^2, \mathbf{C}P^n)$, we denote by $c_1(f)$ the first Chern number of the tautological bundle $V(f) \to S^2$. By [14], we get the followings.

LEMMA 2.1 [14, §2 & §3]. *For* $f \in \mathscr{H}ol^*(\mathbf{C}P^n)$, *choose the Frenet frame* $\{Z_r\}_r$ *of* $f$ *and put*

$$dZ_r = -\bar{a}_{r-1}\bar{\varphi}Z_{r-1} + \omega_r Z_r + a_r\varphi Z_{r+1}$$

*for* $0 \le r \le n$ *where* $a_{-1} = a_n = 0$. *Then each* $f_r$ *defined by* $Z_r$ *holds*

$$E(f_r) = \int (|a_{r-1}|^2 + |a_r|^2)\frac{\sqrt{-1}}{2}\varphi \wedge \bar{\varphi},$$

$$c_1(f_r) = \frac{1}{\pi} \cdot \int (|a_{r-1}|^2 - |a_r|^2)\frac{\sqrt{-1}}{2}\varphi \wedge \bar{\varphi}.$$

Denote by $R_\partial(f)$ the ramification index of $\partial : V(f) \to V(\partial f) \otimes T^{(1,0)}$ which is the number of zeros of $\partial$ counted according to multiplicity. Similarly $R_{\bar\partial}(f)$ is the ramification index of $\bar\partial : V(f) \to V(\bar\partial f) \otimes T^{(0,1)}$. As for the following lemma, we refer [5] and also [14, §3].

LEMMA 2.2. *For* $f \in \mathscr{H}arm(\mathbf{C}P^n)$, *if* $\partial f$ *is non trivial, we get*

$$c_1(\partial f) = c_1(f) + R_\partial(f) + 2.$$

*When* $\bar\partial f$ *is non-trivial, we also get* $c_1(\bar\partial f) = c_1(f) - R_{\bar\partial}(f) - 2$.

By Lemma 2.1 and 2.2, we get the following inequalities.

LEMMA 2.3 [14, Theorem 3.1]. *For* $f \in \mathscr{H}ol^*(\mathbf{C}P^n)$, *choose the Frenet frame* $\{Z_r\}_r$ *as in Lemma* 2.1. *Then we get the followings for any* $r$.

(1) $\sum_{r+1 \le q \le n} \sum_{r \le s \le q-1} R_\partial(f_s) < \frac{1}{\pi}E(f_r) + (n+1) \cdot |c_1(f_r)|$.

(2) $\sum_{0 \le q \le r-1} \sum_{q \le s \le r-1} R_\partial(f_s) < \frac{1}{\pi}E(f_r) + (n+1) \cdot |c_1(f_r)|$.

## 3. Harmonic bubble tree maps

It is well-known that $\mathscr{H}arm(\mathbf{C}P^n)$ may be non-compact with respect to $W^{1,2}$-topology. To consider this bubbling phenomenon, we refer Parker & Wolfson ([11]) and Parker ([10]).

Let $TS^2 \to S^2$ be the complex tangent bundle over the complex manifold $S^2$, $g_0$. Compactifying each vertical fiber, we get a bundle $\Sigma(S^2) \to S^2$ with fibers

$S_z = S^2$ where we identify $z$ of $S_z$ with the south pole $\infty$ of $S^2$ and equip the complex structure on $\Sigma(S^2)$. By the induction on $k \geq 1$, we define a bundle

$$\Sigma^k(S^2) := \Sigma(\Sigma^{k-1}(S^2)) \to \Sigma^{k-1}(S^2).$$

A bubble domain at level $k$ is a fiber $S_z^k = S^2$ of $\Sigma^k(S^2) \to \Sigma^{k-1}(S^2)$ and a bubble domain tower is a union $T^I = \bigvee_{\ell \in I} S^{(\ell)}$ of the base space $S^{(0)}$ of $\Sigma(S^2) \to S^2$ and finite number of bubble domains $S^{(\ell)}$ $(\ell \in I, \ell \geq 1)$ with

$$\pi_\ell : \Sigma S^{(\ell')} \supset S^{(\ell)} = S_{z_\ell}^{k_\ell} = \pi_\ell^{-1}(z_\ell) \to z_\ell \in S^{(\ell')}.$$

We denote by $\infty_\ell$ the south pole of $S^{(\ell)}$. Motivated by Parker [10], if a map

$$f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I = \bigvee_{\ell \in I} S^{(\ell)} \to \mathbf{C}P^n$$

consisits of non-trivial maps $f^{(\ell)}$ satisfying $f^{(\ell)}(\infty_\ell) = f^{(\ell')}(z_\ell)$ when $\pi_\ell^{-1}(z_\ell) = S^{(\ell)}$, we call $f^I$ a bubble tree map, $f^{(0)}$ a base map, $f^{(\ell)}$ a bubble map for $\ell \in I - \{0\}$ and $z_\ell \in S^{(\ell')}$ a bubble point of $f^{(\ell')}$. Denote by $B_{f^{(\ell)}}$ the set of bubble points of $f^{(\ell)}$.

We call $f^I$ a harmonic bubble tree map if $f^{(\ell)}$ is a harmonic map for each $\ell \in I$. Similarly we call $f^I$ a holomorphic (resp. an anti-holomorphic) bubble tree map if $f^{(\ell)}$ is holomorphic (resp. anti-holomorphic) for any $\ell \in I$.

We say that a sequence $\{f^k\}_{k \geq 1}$ in $\mathscr{H}arm(\mathbf{C}P^n)$ converges to a harmonic bubble tree map $f^I : T^I \to \mathbf{C}P^n$ if each $f^k$ defines a bubble tree map $f^{k,I} = \bigvee_{\ell \in I} f^{k,\ell} : T^I \to \mathbf{C}P^n$ by the iterated renormalization procedure and if $\{f^{k,I}\}_k$ converges to $f^I$ uniformly in $C^0 \cap W^{1,2}$ and uniformly in $C^r$ $(r \geq 1)$ on any compact set of $T^I - \bigcup_\ell(\{\infty_\ell\} \cup B_{f^{(\ell)}})$. Here $f^{k,\ell} = f^k \circ \sigma_{k,\ell}$ by a fractional linear transformation $\sigma_{k,\ell}$ of $S^{(\ell)} = S^2$ fixing the south pole on a compact set of $S^{(\ell)} - \{\infty_\ell\} \cup B_{f^{(\ell)}}$ for $k$ large enough. For details, refer [11, §4]. By [10, Theorem 2.2 & Corollary 2.3], we get the following.

Theorem 5. *Let $\{f^k\}_k$ be a sequence in $\mathscr{H}arm(\mathbf{C}P^n)$ with $\sup_k E(f^k) < \infty$. Then a subsequence converges to a harmonic bubble tree map $f^I = \bigvee_\ell f^{(\ell)} : T^I \to \mathbf{C}P^n$ satisfying*

$$\lim_k E(f^k) = \sum_\ell E(f^{(\ell)}) \quad \text{and} \quad \alpha = \lim_k c_1(f^k) = \sum_\ell c_1(f^{(\ell)}).$$

For $\mathbf{C}P^n$, $g$, we get a constant $B_0$ so that any $f \in \mathscr{H}arm(\mathbf{C}P^n)$ with $E(f) < 2B_0$ is trivial (refer [12]). We choose $B_0$ as a scaling constant. Put $H^+ = \{z \mid |z| \geq 1\} \subset \mathbf{C}$. By the choice of the translation and the rescaling in the renormalization, if a sequence of harmonic maps converges to a harmonic bubble tree map $f^I = \bigvee_\ell f^{(\ell)} : T^I \to \mathbf{C}P^n$, each bubble map $f^{(\ell)}$ is parametrized satisfying

$$\text{(BC)} \qquad \int_{H^+} |df^{(\ell)}|^2 \frac{\sqrt{-1}}{2}\, dz d\bar{z} = B_0$$

and $B_{f^{(\ell)}}$ is contained in the northern hemisphere of $f^{(\ell)}$ when $\ell \neq 0$.

In the case of $\mathbf{C}P^n$, the map

$$c_1 : \pi_2(\mathbf{C}P^n) \simeq H_2(\mathbf{C}P^n; \mathbf{Z}) \to \mathbf{Z}$$

defined by $c_1([f]) = c_1(f)$ is an isomorphism. Let $\mathscr{H}arm_\alpha(\mathbf{C}P^n)$ be the subspace of $\mathscr{H}arm(\mathbf{C}P^n)$ consisting of $f$ with $c_1(f) = \alpha$. For each $\alpha \in \mathbf{Z}$, $\mathscr{H}arm_\alpha(\mathbf{C}P^n)$ is non-empty. By Theorem 5, if $\{f^k\}_k$ in $\mathscr{H}arm(\mathbf{C}P^n)$ converges to a harmonic bubble tree map, $f^k \in \mathscr{H}arm_\alpha(\mathbf{C}P^n)$ for any $k$ large enough.

Lemma 3.1. *Let $\{f^k\}_k$ be a sequence in $\mathscr{H}arm_\alpha(\mathbf{C}P^n)$ with $E(f^k) \le E$ for any $k$. Then we get*

$$E(\partial f^k) + E(\bar{\partial} f^k) \le 4E + 2\pi\{2 + (n+3) \cdot |\alpha|\}$$

*for any $k$.*

*Proof.* By Lemma 2.2, $c_1(\partial f^k) = c_1(f^k) + R_\partial(f^k) + 2$ and, by Lemma 2.3,

$$R_\partial(f^k) < \frac{1}{\pi} E(f^k) + (n+1)|c_1(f^k)|.$$

Hence

$$|c_1(\partial f^k)| \le |c_1(f^k)|(n+2) + 2 + \frac{E}{\pi}.$$

As $c_1(f^k) = \alpha$, by Lemma 2.1, we get

$$E(\partial f^k) = E(f^k) - \pi c_1(f^k) - \pi c_1(\partial f^k) \le 2E + \pi\{2 + (n+3)|\alpha|\}.$$

As for $E(\bar{\partial} f^k)$, we can show similarly. □

We say that $f_0$ is equivalent to $f_1$ in $W^{1,p}(S^2, \mathbf{C}P^n)$ if $f_1 = f_0 \circ \sigma$ by some linear fractional transformation $\sigma : S^2 \to S^2$ fixing the south pole. We also say that $\tilde{f}^I = \bigvee_{\ell \in I} \tilde{f}^{(\ell)} : \tilde{T}^I \to \mathbf{C}P^n$ is equivalent to $f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbf{C}P^n$ if $\tilde{f}^{(\ell)}$ is equivalent to $f^{(\ell)}$;

$$\tilde{f}^{(\ell)} = f^{(\ell)} \circ \sigma_\ell : \tilde{S}^{(\ell)} = \tilde{S}^{(\ell')}_{\tilde{z}_\ell} \xrightarrow{\sigma_\ell} S^{(\ell)} = S^{(\ell')}_{z_\ell} \xrightarrow{f^{(\ell)}} \mathbf{C}P^n$$

with $\sigma_\ell(\tilde{z}_\ell) = z_\ell$ for each $\ell \in I$. Here $\tilde{T}^I = \bigvee_{\ell \in I} \tilde{S}^{(\ell)}$ and $\sigma_0$ is necessary the identity.

Now we begin to show Main Theorem. As $E(f^k) \le E$ for any $k$, a subsequence of $\{f^k\}_k$ converges and so we can assume that $f^k \in \mathscr{H}arm_\alpha(\mathbf{C}P^n)$ for any $k$. Hence, by Lemma 3.1, we get $E(\partial f^k) \le E_1$ for any $k$. Therefore, passing through subsequences, both $\{f^k\}_k$ and $\{\partial f^k\}_k$ converge to $f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbf{C}P^n$ and $f_1^{I_1} = \bigvee_{\ell' \in I_1} f_1^{(\ell')} : T^{I_1} \to \mathbf{C}P^n$ respectively. More precisely, consider the renormalization $f^{k,I} = \bigvee_\ell f^{k,\ell} : T^I = \bigvee_{\ell \in I} S^{(\ell)} \to \mathbf{C}P^n$ of $f^k$ converging to $f^I = \bigvee_\ell f^{(\ell)}$. Put $f_1^k = \partial f^k$ and consider again its renormalization

$$f_1^{k,I_1} = \bigvee_{\ell' \in I_1} f_1^{k,\ell} : T^{I_1} = \bigvee_{\ell' \in I_1} S^{(\ell')} \to \mathbf{C}P^n$$

whose subsequence converges to $f_1^{I_1} = \bigvee_{\ell' \in I_1} f_1^{(\ell')}$. If

$$f_1^{k,\ell'} = \partial f^k \circ \sigma_1^{k,\ell'} = \partial f^{k,\ell} \circ \tilde{\sigma}_1^{k,\ell'}$$

on a geodesic disc $D'$ in $S^{(\ell')} - B_{f_1^{(\ell')}} \cup \{\infty_\ell\}$ for any $k$ large enough, $\{f_1^{k,\ell'}\}_k$ converges to $f_1^{(\ell')}$ which is either equal to non-trivial $\partial f^{(0)}$ or equivalent to non-trivial $\partial f^{(\ell)}$ for some $\ell \in I - \{0\}$. On the other hand, if $\partial f^{(\ell)}$ is non-trivial, we can get $f_1^{(\ell')}$ equivalent to $\partial f^{(\ell)}$. As the convergence of $\{f^{k,\ell}\}_k$ is with respect to $C^s$-norm for any $s \geq 0$, if $f_1^{(\ell')} = \partial f^{(\ell)} \circ \sigma_\ell$, we get

$$\sigma_\ell(B_{f_1^{(\ell')}}) \subset B_{f^{(\ell)}}.$$

Now suppose that $\{f_1^{k,\ell'}\}_k$ converges to $f_1^{(\ell')}$ which is not equivalent to any $\partial f^{(\ell)}$ for $\ell \in I$. As $\sigma_1^{k,\ell'} : S^2 \to S^2$ is a holomorphic map given by

$$z = \sigma_1^{k,\ell'}(w) = \alpha_1^{k,\ell'} w + \beta_1^{k,\ell'},$$

$\overline{\partial_w z}$ is a non-zero constant $\alpha_1^{k,\ell'}$. Hence

$$\bar{\partial} f_1^{k,\ell'} = \bar{\partial}((\partial f^k) \circ \sigma_1^{k,\ell'}) = (\bar{\partial}\partial f^k) \circ \sigma_1^{k,\ell'} = f^k \circ \sigma_1^{k,\ell'}$$

on $D'$ for any $k$ large enough where the constant $\alpha_1^{k,\ell'}$ vanishes because of the homogeneous coordinate. A subsequence of $\{\bar{\partial} f_1^{k,\ell'}\}_k$ converges to zero on $D'$. By the uniqueness continuation theorem ([13]), this means the holomorphicity of $f_1^{(\ell')}$. The length of $f_1^{(\ell')}$ is obvious.

By replacing $\partial$ transform with $\bar{\partial}$ transform, we can show the corresponding assertion. This completes the proof of Main Theorem.

## 4. $\mathscr{H}arm_\alpha(\mathbf{C}P^1)$ and $\mathscr{H}arm_\alpha(\mathbf{C}P^2)$

We say that a harmonic bubble tree map $f^I : T^I \to \mathbf{C}P^n$ is gluable if a sequence of harmonic maps converges to a harmonic bubble tree map $\tilde{f}^I : \tilde{T}^I \to \mathbf{C}P^n$ equivalent to $f^I : T^I \to \mathbf{C}P^n$. Firstly we consider the case when $n = 1$. Note that any map in $\mathscr{H}arm(\mathbf{C}P^1)$ is either holomorphic or anti-holomorphic.

Lemma 4.1. *Let $f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbf{C}P^1$ be a holomorphic bubble tree map. Then $\partial f^I = \bigvee_{\ell \in I} \partial f^{(\ell)}$ is a well-defined anti-holomorphic bubble tree map defined on $T^I$. If $f^I$ is gluable, so is $\partial f^I$.*

*Proof.* Let $f = [p_0 : p_1] \in \mathscr{H}ol(\mathbf{C}P^1)$ be non-trivial where $p_0$ and $p_1$ have no common zero. Then, by calculations,

$$\partial f = [(p_1 p_0' - p_1' p_0)\bar{p}_1 : -(p_1 p_0' - p_1' p_0)\bar{p}_0].$$

If $p_1 p_0' - p_1' p_0 = 0$ on a domain, $p_0 \equiv K \cdot p_1$ and so we deduce a contradiction. Hence $\partial f = [\bar{p}_1 : -\bar{p}_0]$.

Now take a holomorphic bubble tree map $f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbf{C}P^1$. As shown above, when $f^{(\ell)}(\infty) = f^{(\ell')}(z_\ell)$, $\partial f^{(\ell)}(\infty) = \partial f^{(\ell')}(z_\ell)$. This shows the first assertion.

When a sequence $\{f^k\}_k$ in $\mathscr{H}ol(\mathbf{C}P^1)$ converges to $f^I$, by calculations, $\{\partial f^k\}_k$ converges to $\partial f^I$. $\square$

Now we consider the case when $n = 2$. We start to refer results of existence theorems. Denote by $\mathscr{H}ol_{\alpha,r}(\mathbf{C}P^2)$ the subspace of $\mathscr{H}ol(\mathbf{C}P^2)$ consisting of $f$ with $c_1(f) = \alpha$ and $R_\partial(f) = r$. We also put $\mathscr{H}ol^*_{\alpha,r}(\mathbf{C}P^2) = \mathscr{H}ol_{\alpha,r}(\mathbf{C}P^2) \cap \mathscr{H}ol^*(\mathbf{C}P^2)$. Obviously $\mathscr{H}ol^*_{\alpha,r}(\mathbf{C}P^2) = \mathscr{H}ol_{\alpha,r}(\mathbf{C}P^2)$ if $2\alpha + r + 2 < 0$.

We also consider the subspace $\mathscr{H}arm_{\alpha,E}(\mathbf{C}P^2)$ of $\mathscr{H}arm_\alpha(\mathbf{C}P^2)$ consisting of $f$ with $E(f) = \pi E$. Note that any map in $\mathscr{H}arm_{\alpha,E}(\mathbf{C}P^2)$ is full when $E \neq 0$, $|\alpha|$. We get the followings.

Theorem 6 ([2], Lemma 1.3 & Theorem 1.4). *For* $0 \le r \le -\alpha - 2$, $\mathscr{H}ol_{\alpha,r}(\mathbf{C}P^2)$ *is a smooth connected complex submanifold of* $\mathscr{H}ol(\mathbf{C}P^2)$ *of complex dimension* $2 - 3\alpha - r$. *Moreover there is a homeomorphism*

$$\mathscr{H}ol_{\alpha,r}(\mathbf{C}P^2) \ni f \to \partial f \in \mathscr{H}arm_{\alpha+2+r,\,-(3\alpha+r+2)}(\mathbf{C}P^2).$$

*Remark* 4.1. *By* [8, Proposition 2.7], $\mathscr{H}ol_{\alpha,r}(\mathbf{C}P^2)$ *is non-empty exactly when* $0 \le r \le -\frac{3}{2}\alpha - 3$.

As for the gluing, we get the following.

Proposition 4.2. *Let* $f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbf{C}P^2$ *be a harmonic bubble tree map with* $f^{(\ell)} \in \mathscr{H}arm_{\alpha_\ell, E_\ell}(\mathbf{C}P^2)$ *and* $E_\ell \neq |\alpha_\ell|$ *for any* $\ell \in I$. *If* $f^I$ *is gluable, both* $\partial f^I = \bigvee_{\ell \in I} \partial f^{(\ell)} : T^I \to \mathbf{C}P^2$ *and* $\bar\partial f^I = \bigvee_{\ell \in I} \bar\partial f^{(\ell)} : T^I \to \mathbf{C}P^2$ *are well-defined gluable bubble tree maps.*

*Proof.* If necessary, replace $f^I$ by an equivalent harmonic bubble tree map (which we denote by the same way) and take a sequence $\{f^k\}_k$ in $\mathscr{H}arm(\mathbf{C}P^2)$ converging to $f^I$. Without loss of generality, we can assume that $f^k \in \mathscr{H}arm_{\alpha,E}(\mathbf{C}P^2)$ with $E \neq |\alpha|$ for any $k$. We get a harmonic sequence

$$seq(f^k, 1) : 0 \xleftarrow{\bar\partial} \bar\partial f^k \xrightarrow{\partial} f^k \xrightarrow{\partial} \partial f^k \xrightarrow{\partial} 0.$$

Passing through a subsequence, $\{\partial f^k\}_k$ converges to $f_1^{I_1} = \bigvee_{\ell \in I_1} f_1^{(\ell)} : T^{I_1} \to \mathbf{C}P^2$. When $f_1^{(\ell')}$ is not equivalent to any $\partial f^{(\ell)}$, by Main Theorem, $f_1^{(\ell')}$ is a holomorphic map whose $\partial$ transform is trivial. Then $f_1^{(\ell')}$ becomes tirival and we deduce a contradiction. By the assumption, $\partial f^{(\ell)}$ is non-trivial. Hence $I_1 = I$ and each $f_1^{(\ell)}$ is equivalent to $\partial f^{(\ell)}$. This implies that $\partial f^I = \bigvee_{\ell \in I} \partial f^{(\ell)}$ is a well-defined anti-holomorphic bubble tree map. Moreover, by Main Theorem again, $\partial f^I$ is defined on $T^I$. Similarly we can show the corresponding result for $\bar\partial f^I$. $\square$

PROPOSITION 4.3. *For* $0 \le r \le -\alpha - 2$, *let* $\{f^k\}_k$ *be a sequence in* $\mathscr{H}ol_{\alpha,r}(\mathbf{C}P^2)$ *converging to* $f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbf{C}P^2$ *with* $f^{(\ell)} \in \mathscr{H}ol^*_{\alpha_\ell, r_\ell}(\mathbf{C}P^2)$ *for any* $\ell$. *Suppose that* $\{\partial f^k\}_k$ *converges to a harmonic bubble tree map* $f_1^{I_1} : T^{I_1} \to \mathbf{C}P^2$. *Then* $f_1^{I_1}$ *is equivalent to a well-defined harmonic bubble tree map* $\partial f^I = \bigvee_{\ell \in I} \partial f^{(\ell)} : T^I \to \mathbf{C}P^2$ *exactly when*

$$\sum_{\ell} R_\partial(f^{(\ell)}) = r - 2 \times (|I| - 1).$$

*Here* $|I|$ *is denoted for the number of elements of* $I$.

*Proof.* Since $E(\partial f^k) = (-3\alpha - 2 - r)\pi > 0$, by Main Theorem, a subsequence of $\{\partial f^k\}_k$ converges to $f_1^{I_1} : T^{I_1} \to \mathbf{C}P^2$.

Firstly we note that $\partial f^{(\ell)}$ is non-trivial. As $E(f^k) = \sum_\ell E(f^{(\ell)})$, by Lemma 2.1 and Lemma 2.2, we get

$$E(\partial f^k) - \sum_\ell E(\partial f^{(\ell)}) = \pi \left\{ \sum_\ell R_\partial(f^{(\ell)}) - R_\partial(f^k) + 2 \cdot |I| - 2 \right\} \ge 0.$$

Here this is equal to zero exacly when $\{\partial f^k\}_k$ converges to a bubble tree map equivalent to a well-defined bubble tree map $\partial f^I$. □

## 5. Example

In this section, we show examples to consider relations between a harmonic bubble tree map $f^I$ and its $\partial$ transform. We consider the case when $n = 2$.

For any $f \in \mathscr{H}ol(\mathbf{C}P^2)$, put $f = [p_0 : p_1 : p_2]$ where $[p_0 : p_1 : p_2]$ are homogeneous coordinates of $\mathbf{C}P^2$. Put $h_f = [h_0 : h_1 : h_2]$ where

$$(h_0, h_1, h_2) = (p_1' p_2 - p_1 p_2', -p_0' p_2 + p_0 p_2', p_0' p_1 - p_0 p_1').$$

When $p_0$, $p_1$, $p_2$ have no common zeros, $R_\partial(f)$ is the number of common zeros of three holomorphic maps $h_0$, $h_1$, $h_2$ as far as $2 \cdot \max_j \deg p_j - 2 = \max_j \deg h_j$. For details, refer [2, §2].

From now on, we denote by $T^I = S^{(0)} \vee S^{(1)}$ the bubble domain tower defined by the base space $S^{(0)} = S^2$ and a bubble domain $S^{(1)} = \pi_1^{-1}(0) \subset \Sigma S^{(0)}$. Denote by

$$\mathscr{H}ol_{-2,0}(\mathbf{C}P^2) * \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$$

the set of holomorphic bubble tree maps $f^I = f^{(0)} \vee f^{(1)} : T^I \to \mathbf{C}P^2$ with $f^{(\ell)} \in \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$ for $\ell = 0, 1$. Since $\mathscr{H}ol^*_{-2,0}(\mathbf{C}P^2) = \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$, by Theorem 6, $\mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$ is a complex manifold of the complex dimension 8.

*Example* 5.1. Take $f^I \in \mathscr{H}ol_{-2,0}(\mathbf{C}P^2) * \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$. By Proposition 4.3, if a sequence of $f_k \in \mathscr{H}ol_{-4,2}(\mathbf{C}P^2)$ converges to $f^I$, a subsequence of $\{\partial f_k\}_k$

converges to a harmonic bubble tree map equivalent to $\partial f^I := \partial f^{(0)} \vee \partial f^{(1)} : T^I \to \mathbf{C}P^2$.

In this case, we also get $E(\partial^2 f_k) = 4\pi$ and $E(\partial^2 f^{(\ell)}) = 2\pi$ for $\ell = 0, 1$. Hence, by Main Theorem, passing throught a subsequence, $\{\partial^2 f_k\}_k$ converges to a harmonic bubble tree map equivalent to $\partial^2 f^I := \partial^2 f^{(0)} \vee \partial^2 f^{(1)} : T^I \to \mathbf{C}P^2$.

*Example* 5.2. Let $f^I = f^{(0)} \vee f^{(1)} : T^I \to \mathbf{C}P^2$ be the holomorphic bubble tree map defined by

$$f^{(0)}(z) = [1 : z : z^2], \quad f^{(1)}(z) = [z^2 : z : 1].$$

As $R_\partial(f^{(\ell)}) = 0$ for $\ell = 0, 1$, $f^I \in \mathscr{H}ol_{-2,0}(\mathbf{C}P^2) * \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$.

A sequence of harmonic maps

$$f_R(z) = \left[1 : z + \frac{1}{Rz} : z^2 + \frac{1}{R^2 z^2}\right]$$

converges to a holomorphic bubble tree map equivalent to $f^I : T^I \to \mathbf{C}P^2$. By calculations, we also get $R_\partial(f_R) = 2$. Hence, by Proposition 4.3, $\{\partial f_R\}_R$ converges to a harmonic bubble tree map equivalent to a well-defined harmonic bubble tree map $\partial f^I : T^I \to \mathbf{C}P^2$. Moreover

$$\partial^2 f_R(z) = \left[\bar{z}^2 + \frac{1}{R^2 \bar{z}^2} + \frac{4}{R} : -2\bar{z} - \frac{2}{R\bar{z}} : 1\right]$$

converge to an anti-holomorphic bubble tree map equivalent to a well-defined $\partial^2 f^I : T^I \to \mathbf{C}P^n$. In fact, by using "Mathematica Ver.6.0", we can calculate

$$\partial f^{(0)}(z) = [-\bar{z} - 2z\bar{z}^2 : 1 - z^2\bar{z}^2 : 2z + z^2\bar{z}],$$
$$\partial f^{(1)}(z) = [2z + z^2\bar{z} : 1 - z^2\bar{z}^2 : -\bar{z} - 2z\bar{z}^2]$$

and

$$\partial^2 f^{(0)}(z) = [\bar{z}^2 : -2\bar{z} : 1], \quad \partial^2 f^{(1)}(z) = [1 : -2\bar{z} : \bar{z}^2].$$

Hence both $\partial f^I = \partial f^{(0)} \vee \partial f^{(1)} : T^I \to \mathbf{C}P^2$ and $\partial^2 f^I = \partial^2 f^{(0)} \vee \partial^2 f^{(1)} : T^I \to \mathbf{C}P^2$ are well-defined.

*Example* 5.3. We consider a bubble tree map $f^I = f^{(0)} \vee f^{(1)} : T^I \to \mathbf{C}P^2$ defined by

$$f^{(0)}(z) = [1 : z^2 : z], \quad f^{(1)}(z) = [z^2 : z : 1]$$

which is contained in $\mathscr{H}ol_{-2,0}(\mathbf{C}P^2) * \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$. For $R > 1$ large enough, we define holomorphic maps $f_R \in W^{1,p}(S^2, \mathbf{C}P^2)$ by

$$f_R(z) = \left[1 : z^2 + \frac{1}{Rz} : z + \frac{1}{R^2 z^2}\right]$$

which converge to a holomorphic bubble tree map equivalent to $f^I : T^I \to \mathbf{C}P^2$ if $R \to \infty$. Here $R_\partial(f_R) = 0$ and so $\{\partial f_R\}_R$ does not converge to a harmonic map equivalent to $\partial f^I$. In fact, we calculate $\partial f^{(\ell)}$ to get

$$\partial f^{(0)}(z) = [-\bar{z} - 2z\bar{z}^2 : 2z + z^2\bar{z} : 1 - z^2\bar{z}^2],$$
$$\partial f^{(1)}(z) = [2z + z^2\bar{z} : 1 - z^2\bar{z}^2 : -\bar{z} - 2z\bar{z}^2]$$

where

$$\partial f^{(0)}(0) = [0 : 0 : 1], \quad \partial f^{(1)}(\infty) = [0 : 1 : 0].$$

Hence these cannot define a bubble tree map on $T^I$. In fact, when $R \to +\infty$, $\partial f_R$ converge to a harmonic bubble tree map

$$f_1^{I_1} = \partial f^{(0)} \vee f_1^{(01)} \vee f_1^{(1)} : T^{I_1} = S^{(0)} \vee S^{(01)} \vee S^{(1)} \to \mathbf{C}P^2$$

where $f_1^{(1)}$ is equivalent to $\partial f^{(1)}$ and the map $f_1^{(01)} : S^{(01)} \to \mathbf{C}P^2$ is equivalent to $\tilde{f}_1^{(01)}$;

$$\tilde{f}_1^{(01)}(z) = [0 : 1 : -z^2].$$

Since the center of mass of $f_1^{(01)}$ is the north pole, we can define $T^{I_1}$ by

$$S^{(01)} = S_0^{(0)} \subset \Sigma S^{(0)}, \quad S^{(1)} = S_0^{(01)} \subset \Sigma S^{(01)}$$

and choose $f_1^{(01)}$ with $f_1^{(01)}(0) = \tilde{f}_1^{(01)}(0)$. We have

$$E(\partial f_R) = E(\partial f^{(0)}) + E(\partial f^{(1)}) + E(f_1^{(01)}) = 10\pi.$$

When $R \to +\infty$, $\partial^2 f_R$ given by

$$\begin{aligned}
\partial^2 f_R(z) &= [1 - 2R^2\bar{z}^3 + 4R\bar{z}^3 + R^3\bar{z}^6 : -2R\bar{z} + R^3\bar{z}^4 : R^2\bar{z}^2 - 2R^3\bar{z}^5] \\
&= \left[\bar{z}^3 + \frac{4}{R^2} - \frac{2}{R} + \frac{1}{R^3\bar{z}^3} : \bar{z} - \frac{2}{R^2\bar{z}^2} : -2\bar{z}^2 + \frac{1}{R\bar{z}}\right] \\
&= \left[\frac{1}{R} + \frac{4}{R^3\bar{z}^3} - \frac{2}{R^2\bar{z}^3} + \frac{1}{R^4\bar{z}^6} : \frac{1}{R\bar{z}^2} - \frac{2}{R^3\bar{z}^5} : -\frac{2}{R\bar{z}} + \frac{1}{R^2\bar{z}^4}\right]
\end{aligned}$$

also converges to an anti-holomorphic bubble tree map

$$f_2^{I_1} = \partial^2 f^{(0)} \vee f_2^{(01)} \vee f_2^{(1)} : T^{I_1} \to \mathbf{C}P^2$$

where $f_2^{(1)}$ and $f_2^{(01)}$ are equivalent to $\partial^2 f^{(1)}$ and $\partial \tilde{f}_1^{(01)}$ respectively;

$$\partial^2 f^{(0)}(z) = [\bar{z}^2 : 1 : -2\bar{z}],$$
$$\partial \tilde{f}_1^{(01)}(z) = [0 : \bar{z}^2 : 1],$$
$$\partial^2 f^{(1)}(z) = [1 : -2\bar{z} : \bar{z}^2].$$

These satisfy $E(\partial^2 f_R) = E(\partial^2 f^{(0)}) + E(\partial f_1^{(01)}) + E(\partial^2 f^{(1)}) = 6\pi.$

*Example* 5.4. Let $f^I = f^{(0)} \vee f^{(1)} \in \mathscr{H}ol_{-2,0}(\mathbf{C}P^2) * \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$ be defined by

$$f^{(0)}(z) = [p_0 : p_1 : p_2] = [1 : z^2 : 1 + z] \quad \text{and}$$
$$f^{(1)}(z) = [q_0 : q_1 : q_2] = [z^2 : 1 : z + z^2].$$

We can get $f_R \in \mathscr{H}ol^*_{-4}(\mathbf{C}P^2)$ defined by

$$f_R(z) = \left[1 : \frac{1}{R^2 z^2} + z^2 : 1 + z + \frac{1}{Rz}\right]$$

converging to a harmonic bubble tree map equivalent to $f^I$ when $R \to +\infty$. Since $R_\partial(f_R) = 2$, by Proposition 4.3, both $\{\partial f_R\}_R$ and $\{\partial^2 f_R\}_R$ converge to harmonic bubble tree maps equivalent to well-defined bubble tree maps $\partial f^I : T^I \to \mathbf{C}P^2$ and $\partial^2 f^I : T^I \to \mathbf{C}P^2$ respectively.

For $p(z) = a_0 + a_1 z + a_2 z^2$ and $q(z) = b_0 + b_1 z + b_2 z^2$, put $|p - q| = \sum_k |a_k - b_k|$.

Lemma 5.1. *Let $f^I = f^{(0)} \vee f^{(1)} : T^I \to \mathbf{C}P^2$ be a holomorphic bubble tree map in Example* 5.4 *with*

$$f^{(0)} = [p_0 : p_1 : p_2], \quad f^{(1)} = [q_0 : q_1 : q_2].$$

*Then, for any $\varepsilon > 0$ small enough, we can choose a holomorphic bubble tree map $\tilde{f}^I = \tilde{f}^{(0)} \vee \tilde{f}^{(1)} : T^I \to \mathbf{C}P^2$ in $\mathscr{H}ol_{-2,0}(\mathbf{C}P^2) * \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$ with*

$$\tilde{f}^{(0)} = [\tilde{p}_0 : \tilde{p}_1 : \tilde{p}_2], \quad \tilde{f}^{(1)} = [\tilde{q}_0 : \tilde{q}_1 : \tilde{q}_2]$$

*so that $\sum_\ell (|\tilde{p}_\ell - p_\ell| + |\tilde{q}_\ell - q_\ell|) < \varepsilon$ and that $\partial \tilde{f}^I = \partial \tilde{f}^{(0)} \vee \partial \tilde{f}^{(1)} : T^I \to \mathbf{C}P^2$ is well-defined but non-gluable.*

*Proof.* When the degrees of polynomials $p$ and $q$ are no greater than 2, we can choose $\varepsilon_0 > 0$ so that $p$ and $q$ have no common zeros as far as $|p - p_0| + |q - q_0| < \varepsilon_0$. Here $p_0(z) = 1$ and $q_0(z) = z^2$. Hence we can choose $\varepsilon > 0$ so that $\tilde{f}^{(\ell)} \in \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$ for $\ell = 0, 1$ if

$$\sum_{0 \le j \le 2} (|\tilde{p}_j - p_j| + |\tilde{q}_j - q_j|) < \varepsilon.$$

Put

$$\tilde{p}_\ell(z) = \alpha_{\ell 0} + \alpha_{\ell 1} z + \alpha_{\ell 2} z^2, \quad \tilde{q}_\ell(z) = \beta_{\ell 0} + \beta_{\ell 1} z + \beta_{\ell 2} z^2$$

with $\alpha_{00} = 1$ and $\beta_{02} = 1$. Since $\alpha_{00} = \beta_{02} = 1$, $\tilde{f}^{(0)}(0) = \tilde{f}^{(1)}(\infty)$ exactly when $\alpha_{\ell 0} = \beta_{\ell 2}$ for $\ell = 0, 1, 2$. Moreover the complex conjugates of $h_{\tilde{f}^{(\ell)}}$ is equal to $\partial^2 \tilde{f}^{(\ell)}$. Since $\tilde{f}^{(\ell)}$ is full, $\partial^2 \tilde{f}^{(\ell)}$ is non-trivial with $c_1(\partial^2 \tilde{f}^{(\ell)}) = 2$. Moreover $\partial^2 \tilde{f}^{(0)}(0) = \partial^2 \tilde{f}^{(1)}(+\infty)$ exactly when $\alpha_{\ell k}$ and $\beta_{\ell k}$ additionally satisfy

$$\beta_{11}(\alpha_{01}\alpha_{20} - \alpha_{21}) = \beta_{21}(\alpha_{01}\alpha_{10} - \alpha_{11}) - \beta_{01}(\alpha_{10}\alpha_{21} - \alpha_{11}\alpha_{20}).$$

If necessary, we rechoose $\varepsilon > 0$ so small that

$$\alpha_{20} > \frac{1}{2} \quad \textit{and} \quad -\frac{1}{2} > \alpha_{01}\alpha_{20} - \alpha_{21}.$$

Denote by $\tilde{U}_\varepsilon$ the set of all $(\tilde{f}^{(0)}, \tilde{f}^{(1)})$ whose polynomials have coefficients $\alpha_{\ell k}$, $\beta_{\ell k}$ satisfying above conditions. By definition, both $\tilde{f}^I = \tilde{f}^{(0)} \vee \tilde{f}^{(1)}$ and $\partial^2\tilde{f}^I = \partial^2\tilde{f}^{(0)} \vee \partial^2\tilde{f}^{(1)}$ are well-defined bubble tree maps defined on $T^I$. Moreover, in such a case, we can calculate to show that $\partial\tilde{f}^I = \partial\tilde{f}^{(0)} \vee \partial\tilde{f}^{(1)}$ are well-defined harmonic bubble tree map defined on $T^I$.

As the complex dimension of $\tilde{U}_\varepsilon$ is equal to 13 and that of $\mathscr{H}ol_{-4,2}(\mathbf{C}P^2)$ is 12 by Theorem 6, there is $\tilde{f}^I \in \tilde{U}_\varepsilon$ so that $\partial\tilde{f}^I$ is well-defined but not gluable. □

*Example* 5.5. We consider a holomorphic bubble tree map which contains a non-full map. Let $f^I = f^{(0)} \vee f^{(1)} : T^I \to \mathbf{C}P^2$ be the holomorphic bubble tree map defined by

$$f^{(0)}(z) = [1 : z : 0], \quad f^{(1)}(z) = [z : 1 : 1].$$

Then $f_R \in \mathscr{H}ol_{-2,0}(\mathbf{C}P^2)$ defined by

$$f_R(z) = \left[1 : z + \frac{1}{R^2 z} : \frac{1}{R^2 z}\right]$$

converge to $f^I$ when $R \to +\infty$. We get $E(f_R) = E(f^{(0)}) + E(f^{(1)}) = 2\pi$. By calculations, we get

$$\begin{aligned}\partial f_R(z) &= \left[-\bar{z} - \frac{1}{R^2\bar{z}} + \frac{2}{R^4 z^2\bar{z}} + \frac{\bar{z}}{R^2 z^2} : 1 - \frac{1}{R^2 z^2} + \frac{2}{R^4 z\bar{z}} : -\frac{1}{R^2 z^2} - \frac{2}{R^4 z\bar{z}} - \frac{2\bar{z}}{R^2 z}\right] \\ &= \left[-\frac{\bar{z}}{R^2} - \frac{1}{R^4\bar{z}} + \frac{2}{R^6 z^2\bar{z}} + \frac{\bar{z}}{R^4 z^2} : \frac{1}{R^2} - \frac{1}{R^4 z^2} + \frac{2}{R^6 z\bar{z}} : -\frac{1}{R^4 z^2} - \frac{2}{R^6 z\bar{z}} - \frac{2\bar{z}}{R^4 z}\right]\end{aligned}$$

which converge to

$$f_1^{I_1} = \partial f^{(0)} \vee f_1^{(01)} \vee f_1^{(1)} : T^{I_1} = S^{(0)} \vee S^{(01)} \vee S^{(1)} \to \mathbf{C}P^2$$

if $R \to +\infty$. Here $T^{I_1}$ is the same bubble domain tower in Example 5.3 and, $f_1^{(01)}$ and $f_1^{(1)}$ are equivalent to $\tilde{f}_1^{(01)}$ and $\partial f^{(1)}$ respectively;

$$\begin{aligned}\partial f^{(0)}(z) &= [-\bar{z} : 1 : 0], \\ \tilde{f}_1^{(01)}(z) &= [0 : 1 - z^2 : 1], \\ \partial f^{(1)}(z) &= [2 : -\bar{z} : -\bar{z}].\end{aligned}$$

In fact, making calculations, we can show that the center of mass of $\tilde{f}_1^{(01)}$ is the north pole. We also get

$$E(\partial f_R) = E(\partial f^{(0)}) + E(\partial f^{(1)}) + E(f_1^{(01)}) = \pi + \pi + 2\pi.$$

Moreover

$$\partial^2 f_R(z) = \left[-\frac{2}{R^2\bar{z}} : \frac{1}{R^2\bar{z}^2} : 1 - \frac{1}{R^2\bar{z}^2}\right]$$

converges to $f_2^{I_2} = \partial\tilde{f}_1^{(01)} : T^{I_2} = S^{(01)} \to \mathbf{C}P^2$;

$$\partial\tilde{f}_1^{(01)}(z) = [0 : 1 : -1 + \bar{z}^2].$$

In this case, $\partial\tilde{f}_1^{(01)}$ is the base map and $E(\partial^2 f_R) = E(\partial\tilde{f}_1^{(01)}) = 2\pi$.

## References


[1] S. S. Chern and J. G. Wolfson, Harmonic maps of the two-sphere into a complex Grassmann manifold II, Annals of Math. **125** (1987), 301–335.

[2] T. A. Crawford, The space of Harmonic maps from the 2-sphere to the complex projective plane, Canad. Math. Bull **40** (1997), 285–295.

[3] J. Eells and L. Lemaire, Selected topics in harmonic maps, CBMS Regional conference series in math. **50**, 1983.

[4] J. Eells and J. C. Wood, Harmonic maps from surfaces to complex projective spaces, Advances in Math. (1983), 217–263.

[5] P. Griffiths and J. Harris, Principles of algebraic geometry, J. Wiley and Sons, New York, 1978.

[6] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, Invent. math. **82** (1985), 307–347.

[7] S. Kobayashi and K. Nomizu, Foundations of differential geometry, **I**, II, A Wiley-Inter science publication, 1996.

[8] L. Lemaire and J. C. Wood, On the space of harmonic 2-spheres in $\mathbf{C}P^2$, Internat. J. Math. **7** (1996), 211–225.

[9] A. Lichnerowicz, Applications harmoniques et variétés Kählériennes, Symp. Math. III (Bologna, 1970), 341–402.

[10] T. H. Parker, Bubble tree convergence for harmonic maps, J. Differential Geom. **44** (1996), 595–633.

[11] T. H. Parker and J. G. Wolfson, Pseudoholomorphic maps and bubble trees, Jour. Geom. Anal. **3** (1993), 63–98.

[12] J. Sacks and K. Uhlenbeck, The existence of minimal immersions of two-spehers, Ann. of Math. (2) **113** (1981), 211–228.

[13] Y.-T. Siu, The complex-analicity of harmonic maps and the strong rigidity of compact Kähler manifolds, Ann. of Math. (2) **112** (1980), 73–111.

[14] J. G. Wolfson, Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifolds, J. Differential Geom. **27** (1988), 161–178.

[15] J. G. Wolfson, On minimal surfaces in a Kähler manifold of constant holomorphic sectional curvature, Trans. Amer. Math. Soc. **290** (1985), 627–646.



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