J. Math. Soc. Japan
Vol. 66, No. 1 (2014) pp. 205–221
doi: 10.2969/jmsj/06610205

Equivariant version of Rochlin-type congruences

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(Received Sep. 1, 2011)
(Revised Apr. 23, 2012)

Abstract. W. Zhang showed a higher dimensional version of Rochlin congruence for $8k+4$-dimensional manifolds. We give an equivariant version of Zhang's theorem for $8k+4$-dimensional compact Spin^c-G-manifolds with spin boundary, where we define equivariant indices with values in $R(G)/RSp(G)$. We also give a similar congruence relation for $8k$-dimensional compact Spin^c-G-manifolds with spin boundary, where we define equivariant indices with values in $R(G)/RO(G)$.

1. Introduction.

In this note we define equivariant versions of the mod 2 index for Dirac operators when a compact Lie group G acts on the base manifolds preserving the operators. We describe some fundamental properties of the equivariant indices including a generalization of the Rochlin congruences for $8k$ and $8k+4$ dimensional compact G-spin^c manifolds with spin boundary (Theorems 10 and 17). When the group action is trivial, W. Zhang gave two proofs for the inequivariant version of the congruences ([**11**], [**10**]). Our contribution is to generalize Zhang's theorem in equivariant settings. It would be possible to extend Zhang's arguments to equivariant cases. In this note, however, we give an alternative proof based on a fundamental excision property of indices. The equivariant indices we shall define take values in $R(G)/RSp(G)$ or $R(G)/RO(G)$, where $R(G), RO(G)$ and $RSp(G)$ are the Grothendieck groups of finite dimensional representation spaces over complex, real, and quaternionic numbers respectively. We define such indices for the four cases below.

1. When Z is a closed G-spin manifold Z with $\dim Z \equiv 2 \bmod 8$, we define $\mathrm{ind}_G^{R/RSp} Z \in R(G)/RSp(G)$.
2. When X is a compact G-spin^c manifold with $\dim X \equiv 4 \bmod 8$, and a reduction to a G-spin structure is given on the boundary ∂X, we define $\mathrm{ind}_G^{R/RSp} X \in R(G)/RSp(G)$.
3. When Z is a closed G-spin manifold Z with $\dim Z \equiv 6 \bmod 8$, we define $\mathrm{ind}_G^{R/RO} Z \in R(G)/RO(G)$.
4. When X is a compact G-spin^c manifold with $\dim X \equiv 0 \bmod 8$, and a reduction to a G-spin structure is given on the boundary ∂X, we define $\mathrm{ind}_G^{R/RO} X \in R(G)/RO(G)$.

To show some basic properties of the equivariant indices, we need a sum formula and a product formula of indices. To formulate these two formulas we use the indices of Dirac-type operators with the Atiyah-Patodi-Singer boundary condition.

2010 *Mathematics Subject Classification.* Primary 19K56; Secondary 57S15.
Key Words and Phrases. equivariant index, spin structure.

The definitions and properties of the $R(G)/RSp(G)$-valued indices and the $R(G)/RO(G)$-valued indices are quite parallel. In this note we formulate both, but give proofs only for the $R(G)/RSp(G)$-cases because the proofs are parallel as well.

A purpose of this note is to provide proofs for the two statements (Definition-Lemma 7 and Corollary 12) which are used in [**8**].

Remark 1. 1. When $G=1$, the $R(G)/RO(G)$-valued index is always 0. However it is not always equal to 0 for non-trivial group actions. An example is given for $G=\boldsymbol{Z}/3$ in Section 4.

2. For a real vector bundle E on a closed G-spin^c manifold Z with $\dim Z \equiv 2 \bmod 8$, we can generalize our definition to obtain $\mathrm{ind}_G^{R/RSp}(X,E) \in R(G)/RSp(G)$ using the Dirac operator with coefficient E. Similarly for a quaternionic vector bundle F on a closed G-spin^c manifold Z with $\dim Z \equiv 2 \bmod 8$, we have $\mathrm{ind}_G^{R/RO}(X,E) \in RO(G)/RSp(G)$. Similar parallel generalizations are possible for other cases.

2. $R(G)/RSp(G)$-valued equivariant indices.

2.1. Definition of $\mathrm{ind}_G^{R/RSp}$.

We will define two kinds of equivariant indices with values in $R(G)/RSp(G)$. We prepare some notations for the definitions.

2.1.1. G-spin^c structure and Dirac-type operators.

Let G be a compact Lie group. We write $\widetilde{GL_n^+}$ for the double covering of $GL^+(n,\boldsymbol{R})$ $(n \geq 2)$. In this note we use the following definition of spin and spin^c-structure.

Definition 2. 1. A G-spin^c-*structure* [resp. G-spin-*structure*] on a rank-n real vector bundle E over a topological space M is a pair of G-equivariant principal $\widetilde{GL_n^+}\times_{\pm1}U(1)$-bundle P^c [resp. principal $\widetilde{GL_n^+}$-bundle P] and an isomorphism between E and the $\boldsymbol{R}^n$-bundle associated with P^c $[P]$ as G-vector bundles.

2. When a G-spin^c-structure is given on E, the *associated complex line bundle* is the G-equivariant line bundle induced from the homomorphism

$$GL_n^+ \times_{\pm1} U(1) \to U(1), \quad (g,z) \mapsto z^2.$$

We write L_E for the associated complex line bundle.

3. When M is a manifold, a G-spin^c-*structure* [resp. G-spin-*structure*] *on* M is defined to be a G-spin^c-structure [resp. G-spin-structure] on TM. We write L_M for L_{TM}.

Suppose M is a G-spin or G-spin^c manifold endowed with a G-invariant Riemannian metric m. Then it is well known that we have the associated spinor bundle W_M which has the following properties [**9**]:

1. The Clifford algebra bundle of TM acts on W_M so that the action of each element of TM is skew hermitian.
2. Suppose $\dim M \equiv 0 \bmod 2$. Then W_M has a canonical G-invariant complex linear hermitian involution γ_M which anti-commutes with the Clifford multiplication of TM.

We write $W_M = W_M^0 \oplus W_M^1$ for the $\boldsymbol{Z}/2$-graded structure defined by γ_M.

3. Suppose M is a G-spin manifold satisfying $\dim M \equiv 2, 3$ or $4 \bmod 8$. Then W_M has a canonical G-invariant quaternionic structure J_M which preserves the metric on W_M and commutes with the Clifford multiplication.
4. Suppose M is a G-spin manifold.
   (a) When $\dim M \equiv 2 \bmod 8$, J_M has degree 1.
   (b) When $\dim M \equiv 4 \bmod 8$, J_M has degree 0.

When a $U(1)$-connection θ on L_W is given, making use of Levi-Civita connection as well, we have the Dirac operator

$$D : \Gamma(W_M) \to \Gamma(W_M) \tag{1}$$

as the composition of the covariant derivative $\Gamma(W_M) \to \Gamma(T^*M \otimes_{\boldsymbol{R}} W_M)$ and the Clifford multiplication.

In this note we mainly deal with n-dimensional manifolds with $n \equiv 2, 3, 4 \bmod 8$. In these cases we slightly generalize the definition of the Dirac operator:

Definition 3. Suppose M is a G-spin^c-manifold with a G-invariant metric. A first order G-equivariant differential operator D in (1) is a *Dirac-type operator* if it satisfies the following conditions.

1. D is formally self-adjoint.
2. The principal symbol of D coincides with the Clifford multiplication.
3. When $\dim M \equiv 0 \bmod 2$, we assume that D has degree 1.
4. When M is G-spin and $\dim M \equiv 2, 3, 4 \bmod 8$, we assume that D commutes with J_M.

Remark 4. 1. Two Dirac-type operators are linearly connected through Dirac-type operators, which implies that they have the same index when M is closed and even dimensional.
2. The above generalization of the definition of Dirac-type operator makes it easier to show the product formula later.

2.1.2. $\mathbf{ind}_{\boldsymbol{G}}^{\boldsymbol{R}/\boldsymbol{RSp}}$.

Let $RSp(G)$ be the Grothendieck group of finite dimensional quaternionic representations of G. The injective ring homomorphism $\boldsymbol{C} \to \boldsymbol{H}$ induces an injection $RSp(G) \to R(G)$ ([**1**]).

Definition-Lemma 5. Suppose Z is an $8k+2$-dimensional closed G-spin manifold. Fix a G-invariant Riemannian metric on M, and let D_Z be a (G-invariant) Dirac-type operator on M. Then the kernel $\operatorname{Ker} D_Z$ is a $\boldsymbol{Z}/2$-graded G-module. If we write $(\operatorname{Ker} D_Z)^0 \oplus (\operatorname{Ker} D_Z)^1$ for the decomposition,

$$\operatorname{ind}_G^{R/RSp} Z :\equiv (\operatorname{Ker} D_Z)^0 \bmod RSp(G) \in R(G)/RSp(G)$$

is independent of the choice of the metric and D_Z.

When the group action is trivial, the above well-definedness is well known [**5**].

Proof. Let $E_\lambda = E_\lambda^0 \oplus E_\lambda^1$ be the eigenspace of D_Z^2 for the eigenvalue λ. When $\lambda > 0$, we show that E_λ^0 and E_λ^1 have G-invariant quaternionic structures. Then the rest of the proof is parallel to the standard argument for the case of the trivial group action. Since Z is $8k+2$-dimensional, there exists a degree-1 G-invariant quaternionic structure J_M which commutes with D_Z. Hence J_M gives a degree-1 quaternionic structure on E_λ. If we define J_λ by

$$J_\lambda := \lambda^{-1/2} J_M D_Z : E_\lambda \to E_\lambda,$$

then J_λ is a degree-0 anti-linear map on E_λ which satisfies $J_\lambda^2 = \lambda^{-1} J_M^2 D_Z^2 = -1$. It implies that J_λ is the required degree-0 G-invariant quaternionic structure on E_λ. □

Remark 6. If Z is a closed spin manifold with $\dim Z \equiv 2 \bmod 4$, then $(\operatorname{Ker} D_Z)^0$ and $(\operatorname{Ker} D_Z)^1$ are two representation spaces of G which are complex conjugate to each other by J_M.

Definition-Lemma 7. Suppose X is an $8k+4$-dimensional compact G-spin^c manifold. We assume that a G-spin reduction of the G-spin^c structure is given on a neighborhood of the boundary ∂X of X. Fix a G-invariant Riemannian metric which is of the product form on a neighborhood of ∂X. Choose a G-equivariant Dirac-type operator D_X such that D_X is translation invariant on a neighborhood of ∂X. We define the G-equivariant index $\operatorname{ind}_G D_X$ of a Dirac-type operator D_X by using the Atiyah-Patodi-Singer boundary condition. Then

$$\operatorname{ind}_G^{R/RSp} X :\equiv \operatorname{ind}_G D_X \bmod RSp(G) \in R(G)/RSp(G)$$

is independent of the choice of the metric and D_X.

We will show the above claimed well-definedness in Section 6.

Remark 8. The Atiyah-Patodi-Singer boundary condition is equivalent to that with the following setting.

1. Let $\hat{X}$ be the union of X and $\partial X \times [0, \infty)$ glued along ∂X. Let $D_{\hat{X}}$ be the natural extension of D_X onto $\hat{X}$.
2. Let $\operatorname{Ker}_0 D_{\hat{X}}$ be the set of elements of $\operatorname{Ker} D_{\hat{X}}$ which are L^2-bounded.
3. Let $\operatorname{Ker}_b D_{\hat{X}}$ be the set of elements of $\operatorname{Ker} D_{\hat{X}}$ which converge to some limit values at ∞.
4. Then we have $\operatorname{ind}_G D_X := [(\operatorname{Ker}_0 D_{\hat{X}})^0] - [(\operatorname{Ker}_b D_{\hat{X}})^1] \in R(G)$.

In the above formulation $\operatorname{ind}_G^{R/RSp} X$ is defined by using analytic indices on an open manifold. Hence it is not straightforward to calculate $\operatorname{ind}_G^{R/RSp} X$ directly from the definition. We will use the above setting in the proof of the product formula later.

2.2. Properties of $\mathrm{ind}_G^{R/RSp} X$.

We use the notations in Definition-Lemma 7. We write Y for the boundary ∂X. We give two practical methods to calculate $\mathrm{ind}_G^{R/RSp} X$.

2.2.1. Characteristic submanifold.

Let L_X be the G-equivariant complex line bundle associated to the G-spin^c structure on TX. Then on the boundary Y, the reduction to G-spin structure gives a G-invariant non-vanishing section s of L_X. We assume that there exists an extension s_X of s to a section on X which satisfies the following conditions.

- s_X is G-invariant.
- s_X is transverse to the 0-section.

Let Z be the zero set $s^{-1}(0)$. Then Z has a canonical G-spin structure as follows.

Since the normal bundle of Z is isomorphic to the restriction $L_X|_Z$ of L_X, we have a decomposition $TX|Z = TZ \oplus L_X|_Z$. Introduce a spin^c structure on the line bundle $L_X|_Z$ over Z so that its associated line bundle is identified with $L_X|_Z$. Then the spin^c structures on $L_X|_Z$ and $TX|Z$ induce a spin^c structure on Z. From the multiplicative property of the associated line bundles, the associated line bundle of the spin^c structure on Z is canonically isomorphic to the trivial line bundle, which implies Z has a well-defined spin structure. Since the construction is canonical, if the spin^c structure on X is G-equivariant and the section s is G-invariant, then the spin structure on Z is G-equivariant.

Definition 9. We call Z a *G-characteristic submanifold.*

A main theorem of this paper is:

Theorem 10. *Suppose Z is a G-characteristic submanifold. Then we have*

$$\mathrm{ind}_G^{R/RSp} X = \mathrm{ind}_G^{R/RSp} Z.$$

Zhang gave two proofs of the above theorem when X is closed and the group action is trivial. Our proof is given in Section 7, by using a different technique from Zhang's.

Remark 11. When the group action is trivial, we always have such a transverse section s_X. However this is not the case for general group action.

When $G = 1$, let us write $\mathrm{ind}^{\boldsymbol{Z}/2} X$ for $\mathrm{ind}_G^{R/RSp} X \in \boldsymbol{Z}/2$. Then we obtain:

Corollary 12. *Suppose $Z = s_X^{-1}(0)$ is contained in the fixed point set X^G.*

1. *When the order of G is odd, we have* $\mathrm{ind}_G^{R/RSp} X = (\mathrm{ind}^{\boldsymbol{Z}/2} X)[1] \bmod RSp(G)$, *where* $[1]$ *is the class of the trivial one-dimensional complex representation.*
2. *When the order of G is even, and Z is connected, we have* $\mathrm{ind}_G^{R/RSp} X = (\mathrm{ind}^{\boldsymbol{Z}/2} X)$ $[\boldsymbol{C}_\rho] \bmod RSp(G)$, *for some class* $[\boldsymbol{C}_\rho]$ *of the one-dimensional complex representation* $\rho : G \to \{\pm 1\} \subset U(1)$.

Proof. Since the G-action on TZ is trivial, the G-action on the spin structure on Z factors through a representation $G \to \{\pm 1\}$ for each connected component of Z. If the order of G is odd, then the representation is trivial. If Z is connected, then we have a single representation ρ. □

2.2.2. Excision property.

Suppose X' is a compact G-spin manifold with boundary $-Y \coprod Y'$. Then $\tilde{X} := X' \cup_Y X$ is a compact G-spin^c manifold with boundary Y'. Then we have the following excision property.

Proposition 13.

$$\mathrm{ind}_G^{R/RSp} X \equiv \mathrm{ind}_G^{R/RSp} \tilde{X}.$$

We will prove Proposition 13 in Section 6.

Corollary 14. *In particular if $Y' = \emptyset$, then $\tilde{X}$ is a closed G-spin^c manifold and we have*

$$\mathrm{ind}_G^{R/RSp} X \equiv \mathrm{ind}_G \tilde{X} \bmod RSp(G).$$

Note that when $\tilde{X}$ is closed, the index $\mathrm{ind}\,\tilde{X}$ of the spin^c Dirac-type operator is calculated by using the ordinary G-equivariant index theorem ([**3**], [**4**]).

3. $R(G)/RO(G)$-valued equivariant indices.

We will define two variants of equivariant indices with values in $R(G)/RO(G)$. Since the arguments are parallel to $R(G)/RSp(G)$-case, we give only the statement, omitting their proofs.

Let $RO(G)$ be the Grothendieck group of finite dimensional real representations of G. The complexification induces an injection $RO(G) \to R(G)$.

Definition-Lemma 15. Suppose Z is an $8k+6$-dimensional closed G-spin manifold. Fix a G-invariant Riemannian metric on M, and let D_Z be a (G-invariant) Dirac-type operator. Then the kernel $\mathrm{Ker}\, D_Z$ is $\boldsymbol{Z}/2$-graded. If we write $(\mathrm{Ker}\, D_Z)^0 \oplus (\mathrm{Ker}\, D_Z)^1$ for the decomposition,

$$\mathrm{ind}_G^{R/RO} Z :\equiv (\mathrm{Ker}\, D_Z)^0 \bmod RO(G) \in R(G)/RO(G)$$

is independent of the choice of the metric and D_Z.

Definition-Lemma 16. Suppose X is an $8k+8$-dimensional compact G-spin^c manifold. We assume that a G-spin reduction of the G-spin^c structure is given on a neighborhood of the boundary ∂X of X. Fix a G-invariant Riemannian metric which is of the product form on a neighborhood of ∂X. Choose a G-equivariant Dirac-type operator D_X such that D_X is translation invariant on a neighborhood of ∂X. We define the G-equivariant index $\mathrm{ind}_G\, D_X$ of a Dirac-type operator D_X by using the Atiyah-

Patodi-Singer boundary condition. Then

$$\mathrm{ind}_G^{R/RO} X :\equiv \mathrm{ind}_G D_X \bmod RO(G) \in R(G)/RO(G).$$

is independent of the choice of the metric and D_X.

THEOREM 17. *Suppose Z is a G-characteristic submanifold. Then we have*

$$\mathrm{ind}_G^{R/RO} X = \mathrm{ind}_G^{R/RO} Z.$$

Suppose X' is a compact G-spin manifold with boundary $-Y \coprod Y'$. Then $\tilde{X} := X' \cup_Y X$ is a compact G-spin^c manifold with boundary Y'. Then we have the following excision property.

PROPOSITION 18.

$$\mathrm{ind}_G^{R/RO} X \equiv \mathrm{ind}_G^{R/RO} \tilde{X}.$$

COROLLARY 19. *In particular if $Y' = \emptyset$, then $\tilde{X}$ is a closed G-*spin^c *manifold and we have*

$$\mathrm{ind}_G^{R/RO} X \equiv \mathrm{ind}_G \tilde{X} \bmod RO(G).$$

4. Examples.

Let G be the order-3 subgroup of $U(1)$, and $t \in R(G)$ the class of standard complex 1-dimensional representation. Then we have

$$\begin{aligned}
R(G) &= \boldsymbol{Z}t \oplus \boldsymbol{Z} \oplus \boldsymbol{Z}t^{-1}, \\
RSp(G) &= \boldsymbol{Z}(t+t^{-1}) \oplus 2\boldsymbol{Z} \subset R(G), \\
RO(G) &= \boldsymbol{Z}(t+t^{-1}) \oplus \boldsymbol{Z} \subset R(G), \\
R(G)/RSp(G) &\xrightarrow{\cong} \boldsymbol{Z} \oplus \boldsymbol{Z}/2, \quad [at+b+ct^{-1}] \mapsto (a-c) \oplus (b \bmod 2), \\
R(G)/RO(G) &\xrightarrow{\cong} \boldsymbol{Z}, \qquad\qquad [at+b+ct^{-1}] \mapsto a-c.
\end{aligned}$$

4.1. *R/RSp*-cases.

2-dimensional example: T^2. Let ω be $e^{2\pi i/3}$ and Z_0 the flat torus $\boldsymbol{C}/(\boldsymbol{Z}+\boldsymbol{Z}\omega)$. The standard G-action on $\boldsymbol{C}$ induces a G-action on Z_0 with three fixed points. Introduce the translation-invariant spin structure on Z_0. Since G is an odd order cyclic group, there is a unique lift of the G-action to the spin structure. If we use the natural complex structure on TZ_0, then we have a canonical isomorphism $(W_{Z_0}^0)^{\otimes 2} \cong T^*Z_0$ and $(W_{Z_0}^1)^{\otimes 2} \cong \overline{T^*Z_0}$ for the spinor bundle $W_{Z_0} = W_{Z_0}^0 \oplus W_{Z_0}^1$ over Z_0. For this spin structure W_Z has non-vanishing flat sections $\phi_0 \in \Gamma(W^0_{Z_0})$ and $\phi_1 \in \Gamma(W^1_{Z_0})$ satisfying $\phi_0^{\otimes 2} = dz$ and $\phi_1^{\otimes 2} = \overline{dz}$. Since $g \cdot dz = d(g^{-1}z) = g^2 dz$, we obtain $g \cdot \phi_0 = g\phi_0$ and $g \cdot \phi_1 = g^{-1}\phi_1$. The kernel of the standard Dirac-type operator for the flat metric is spanned by ϕ_0 and ϕ_1, which implies $[(\mathrm{Ker}\, D_{Z_0})^0] = t$ and $[(\mathrm{Ker}\, D_{Z_0})^1] = t^{-1}$. Then $[(\mathrm{Ker}\, D_{Z_0})^0] \bmod RSp(G)$ gives

$$\mathrm{ind}_G^{R/RSp} Z_0 = 1 \oplus 0 \in R(G)/RSp(G) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2.$$

4-dimensional example: $T^2 \times S^2$. Endow $\boldsymbol{CP}^1$ with the Fubini-Study Kähler structure and the spin^c structure associated to its complex structure. Define a G-action on $\boldsymbol{CP}^1$ by rotation with two fixed points, and the action preserves these structures. Since $(\mathrm{Ker}\, D_{\boldsymbol{CP}^1})^0 \cong H^0(\boldsymbol{CP}^1, \mathcal{O})$ and $(\mathrm{Ker}\, D_{\boldsymbol{CP}^1})^1 \cong H^1(\boldsymbol{CP}^1, \mathcal{O})$, we have $[(\mathrm{Ker}\, D_{\boldsymbol{CP}^1})^0] = 1$ and $[(\mathrm{Ker}\, D_{\boldsymbol{CP}^1})^1] = 0$ in $R(G)$. Let X_0 be $Z_0 \times \boldsymbol{CP}^1$ with the product spin^c structure and the diagonal G-action. Then the multiplicative property of the kernel of the Dirac-type operators implies $[(\mathrm{Ker}\, D_{X_0})^0] = t$ and $[(\mathrm{Ker}\, D_{X_0})^1] = t^{-1}$, and hence $\mathrm{ind}\, D_{X_0} = t - t^{-1}$. Then $\mathrm{ind}\, D_{X_0} \bmod RSp(G)$ gives

$$\mathrm{ind}_G^{R/RSp} X_0 = 2 \oplus 0 \in R(G)/RSp(G) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2.$$

Example of Theorem 10. We check Theorem 10 for X_0. Let $\{0, \infty\}$ be the fixed point set of the G-action on $\boldsymbol{CP}^1$. We first show that $Z_0 \times \{0, \infty\}$ is a G-invariant characteristic submanifold of X_0 defined as the transversal zeros of a G-invariant section s_0 of L_{X_0}. Because L_{X_0} is the pullback of $L_{\boldsymbol{CP}^1}$ with respect to the projection $X_0 \to \boldsymbol{CP}^1$, it suffices to show that $\{0, \infty\}$ is a G-invariant characteristic submanifold of $\boldsymbol{CP}^1$ defined as the transversal zeros of a G-invariant section s_0' of $L_{\boldsymbol{CP}^1}$. Now, by using the canonical isomorphism $L_{\boldsymbol{CP}^1} \cong T\boldsymbol{CP}^1$, we can take $s_0' = zd/dz$.

Section 4.1 implies that the contribution of each of the two components of $Z_0 \times \{0, \infty\}$ to the right hand side of Theorem 10 is $1 \oplus 0$, while Section 4.1 implies that the left hand side is equal to $2 \oplus 0$, and the equality of Theorem 10 holds.

4.2. $\boldsymbol{R/RO}$-cases.

6-dimensional example: T^6. Let Z_1 be $Z_0 \times Z_0 \times Z_0$. Then the product Dirac-type operator for the product G-spin structure satisfies $[\mathrm{Ker}\, D_{Z_1}] = (t + t^{-1})^3 = 1 + 3t + 3t^{-1} + 1$. The first and the third terms give $[(\mathrm{Ker}\, D_{Z_1})^0]$ and the second and the fourth terms give $[(\mathrm{Ker}\, D_{Z_1})^1]$, and hence we have $[(\mathrm{Ker}\, D_{Z_1})^0] = 1 + 3t^{-1}$ and $[(\mathrm{Ker}\, D_{Z_1})^1] = 1 + 3t$ in $R(G)$. Then $(\mathrm{Ker}\, D_{Z_1})^0 \bmod RO(G)$ gives

$$\mathrm{ind}_G^{R/RO} Z_1 = -3 \in R(G)/RO(G) \cong \boldsymbol{Z}.$$

8-dimensional example: $T^6 \times S^2$. Let $\boldsymbol{CP}^1$ be the G-spin^c manifold in Section 4.1, and X_1 the product $Z_1 \times \boldsymbol{CP}^1$ with the product spin^c structure and the diagonal G-action. Then the multiplicative property of the index of the Dirac-type operators implies $\mathrm{ind}_G X_1 = (t - t^{-1})^3 = 3(t^{-1} - t)$. Then $\mathrm{ind}_G X_1 \bmod RO(G)$ gives

$$\mathrm{ind}_G^{R/RO} X_1 = -6 \in R(G)/RO(G) \cong \boldsymbol{Z}.$$

Example of Theorem 17. We check Theorem 17 for X_1. As in Section 4.1, we can show that $Z_1 \times \{0, \infty\}$ is a G-invariant characteristic submanifold of X_1 defined as the transversal zeros of a G-invariant section s_1 of L_{X_1}. Section 4.2 implies that the contribution of each of the two components of $Z_1 \times \{0, \infty\}$ to the right hand side of Theorem 17 is -3, while Section 4.2 implies that the left hand side is equal to -6, and the equality of Theorem 17 holds.

5. Sum formula and product formula.

We prepare a sum formula and a product formula for indices under the Atiyah-Patodi-Singer boundary condition.

5.1. Sum formula.

1. For $i = 0$ and 1, let X_i be an even dimensional compact G-spin^c manifold.
2. Fix a G-invariant Riemannian metric on X_i of the product form on a neighborhood of the boundary ∂X_i. Suppose that the boundary ∂X_i is the disjoint union of two closed G-manifolds Y_i^+ and Y_i^-. We assume that there exists a G-spin^c closed Riemannian manifold Y such that both Y_0^+ and $-Y_1^-$ are isomorphic to Y as G-spin^c closed Riemannian manifold Y.
3. Glue X_0 and X_1 along $Y_0^+ \cong Y \cong -Y_1^-$ to obtain a closed G-spin^c manifold X with a G-invariant Riemannian metric.

Proposition 20 (Sum formula). *Under the above setting, we have*

$$\mathrm{ind}_G\, D_{X_0} + [\mathrm{Ker}\, D_Y] + \mathrm{ind}_G\, D_{X_1} = \mathrm{ind}_G\, D_X \in R(G),$$

where D_Y is the Dirac-type operator on Y.

Proof. A direct proof of the above sum formula is given by "stretching neck" argument which is standard by now. For the case $G = \{1\}$, such a proof is described in [**7**, p. 367, Remark 11.33(2)]. The proof there is valid for equivariant case as well. □

Remark 21. If the Dirac-type operator is the honest composition of the covariant derivative and the Clifford multiplication, and moreover if $G = \{1\}$, then the sum formula is a direct consequence of the Atiyah-Patodi-Singer index formula: the APS index formulas for $\mathrm{ind}\, D_{X_0}$ and $\mathrm{ind}\, D_{X_1}$ are

$$\mathrm{ind}\, D_{X_0} = \int_{X_0} e^{c_1(L_{X_0})/2} \hat{A}(X_0) + \frac{\eta(D_Y) + \dim \mathrm{Ker}\, D_Y}{2}$$
$$\mathrm{ind}\, D_{X_1} = \int_{X_1} e^{c_1(L_{X_1})/2} \hat{A}(X_1) + \frac{-\eta(D_Y) + \dim \mathrm{Ker}\, D_Y}{2}.$$

Adding these two equations, we obtain the required relation. To show the sum formula along this line, it would be necessary to extend the Atiyah-Patodi-Singer index formula to a formula [**2**] for general Dirac-type operators and also to equivariant setting. For the honest Dirac operator, an equivariant version of the APS index formula is given by

Donnelly [**6**], and we can show the sum formula for this case. On the other hand, without group action, the APS index formula is extended by several people. It would be possible to show a Donnelly-type formula for equivariant setting. The authors, however, could not find any reference.

5.2. Product formula.

Let G and H be two compact Lie groups. Suppose we have the data Z, P, F and X with the next setting.

1. Z is an even dimensional closed G-spin^c manifold with a G-invariant Riemannian metric.
2. P is a G-equivariant principal H-bundle over Z.
3. F is an even dimensional compact $(G \times H)$-spin^c manifold with boundary ∂F. We fix a $(G \times H)$-invariant Riemannian metric on F which is of the product form on a neighborhood of ∂F.
4. X is the compact G-manifold with boundary Y that is defined by $X := P \times_H F$ and $Y := P \times_H \partial F$.

We endow X with a G-spin^c structure as follows. Let π be the projection from X to Z. Then we have a G-equivariant isomorphism

$$TX \cong \pi^* TZ \oplus (P \times_H TF)$$

which is well defined up to G-homotopy. Now we fix a G-spin^c structure on X as the direct sum of the next two G-spin^c structures:

1. The G-spin^c structure on $\pi^* TZ$ that comes from the G-spin^c structure on Z.
2. The G-spin^c structure on $P \times_H TF$ that comes from the $(G \times H)$-spin^c structure on F.

Let D_Z be a (G-invariant) Dirac-type operator on Z, and D_F a ($G \times H$-invariant) Dirac-type operator on F which is translation invariant on the neighborhood of ∂F.

Proposition 22 (Product formula). *There exist a G-invariant Riemannian metric m_X on X and a Dirac-type operator D_X for m_X which have the following properties.*

1. *m_X and D_X are translation invariant on a neighborhood of Y.*
2. *There exists a G-equivariant isometry*

$$TX \cong \pi^* TZ \oplus (P \times_H TF),$$

and also a G-equivariant isomorphism

$$W_X \cong \pi^* W_Z \otimes (P \times_H W_H)$$

as $\boldsymbol{Z}/2$-graded Clifford modules with respect to the identification of the Clifford multiplication

$$c(v \oplus w) = c(v) \otimes 1 + \gamma_Z \otimes c(w)$$

for $v \in TZ$ and $w \in P \times_H TF$, and the relation $\gamma_X \cong \gamma_Z \otimes id + id \otimes \gamma_F$.

3. (a) *When the H-action on $\mathrm{Ker}_0 D_F$ is trivial, the above isomorphism induces an isomorphism $\mathrm{Ker}_0 D_X \cong \mathrm{Ker}\, D_Z \otimes \mathrm{Ker}_0 D_F$ for some appropriate Riemannian metrics and Dirac-type operators.*
   (b) *When the H-action on $\mathrm{Ker}_b D_F$ and $\mathrm{Ker}\, D_{\partial F}$ is trivial, the above isomorphism induces an isomorphism $\mathrm{Ker}_b D_X \cong \mathrm{Ker}\, D_Z \otimes \mathrm{Ker}_b D_F$ for some appropriate Riemannian metrics and some G-invariant Dirac-type operators D_X on X.*

To show Proposition 22 we basically follow the argument in [**3**, Section 9]. There are, however, two differences. One is to use Dirac-type operators instead of pseudo-differential operators. The other is to use open manifolds instead of closed manifolds.

Proof of Proposition 22. Since $D_F : \Gamma(W_F) \to \Gamma(W_F)$ is $G \times H$-invariant, it gives rise to a G-equivariant differential operator

$$\tilde{D}_F : \Gamma(P \times_H W_F) \to \Gamma(P \times_H W_F)$$

which contains only the derivatives along fibers over Z. Then $\tilde{D}_F$ gives a differential operator

$$\gamma_Z \otimes \tilde{D}_F : \Gamma(\pi^* W_Z \otimes (P \times_H W_F)) \to \Gamma(\pi^* W_Z \otimes (P \times_H W_F)).$$

Next let $\{U_i\}$ be an open covering of Z such that the restriction of P on each U_i is trivial if we forget the G-action: $P\,|_{U_i} \cong U_i \times H$. From this trivialization we have a H-equivariant diffeomorphism $\pi^{-1}(U_i) \cong U_i \times F$. Let $\pi_i : \pi^{-1}(U_i) \to F$ be the projection map. Then we have

$$(\pi^* W_Z \otimes (P \times_H W_F))|_{\pi}^{-1}(U_i) \cong \pi^*(W_Z|_{U_i}) \otimes \pi_i^* W_F.$$

By using this tensor product structure, we lift the action of D_Z to define a differential operator $\tilde{D}_{Z,i}$ on $\pi^* W_Z \otimes (P \times_H W_F)\,|_{\pi^{-1}(U_i)}$. Let $\{\rho_i^2\}$ be a partition of unity for the open covering $\{U_i\}$. Then the sum $\sum \rho_i \tilde{D}_{Z,i} \rho_i$ is a differential operator on $\pi^* W_Z \otimes (P \times_H W_F)$. Take the average by using the G-action and the Haar measure of G to obtain a G-invariant differential operator $\tilde{D}_Z$. Then $\tilde{D}_Z$ anti-commutes with $\gamma_Z \otimes \tilde{D}_F$ (see [**3**]). Now we define

$$D_X := \tilde{D}_Z + \gamma_Z \otimes \tilde{D}_F,$$

and we have $D_X^2 = \tilde{D}_Z^2 + \tilde{D}_F^2$. We want to check that D_X is a Dirac-type operator for a G-invariant metric on X in the sense of Definition 3. The exact sequence

$$0 \to P \times_H TF \to TX \to \pi^* TZ \to 0$$

has a splitting $\theta_i : \pi^*TZ|_{U_i} \to TX|_{U_i}$ on each U_i which corresponds the trivialization of P over U_i. Then $\sum(\pi^*\rho_i^2)\theta_i$ gives a global splitting. Take the average by using the G-action and the Haar measure of G to obtain a G-invariant splitting $\theta : \pi^*TZ \to TX$. We define a G-invariant Riemannian metric m_X on X so that the G-equivariant isomorphism $TX \cong \pi^*TZ \oplus (P \times_H TF)$ induced from θ is an isometry. Now it is straightforward to check that the principal symbol of D_X coincides with the Clifford product for the metric m_X, which implies that D_X is a Dirac-type operator in the sense of Definition 3.

The rest of the proof is quite parallel to [**3**]. It suffices to show

$$\operatorname{Ker}_0 D_X = \operatorname{Ker} \tilde{D}_Z \cap \operatorname{Ker}_0 \gamma_Z \otimes \tilde{D}_F, \quad \operatorname{Ker}_b D_X = \operatorname{Ker} \tilde{D}_Z \cap \operatorname{Ker}_b \gamma_Z \otimes \tilde{D}_F.$$

It is obvious that the left-hand-side contains the right-hand-side in each equality. The converse is guaranteed by Lemma 25 which is stated and proved below.

Remark 23. More generalized version of Proposition 22 is explained in Section 6.3 of [**7**, pp. 181–182], in particular in Lemma 6.10, as well as details of its proof.

We formulate the necessary lemma under a slightly generalized setting. Let X be a compact Riemannian manifold with a spin^c structure, and Y the boundary of X with the induced spin^c structure. Assume that the metric of X is of the product form near Y. We write $\hat{X} := X \cup_Y Y \times [0, \infty)$ and $X_t := X \cup_Y Y \times [0, t]$ for $t \geq 0$.

Suppose D is a Dirac-type operator on $\hat{X}$ which is of the form $D = \gamma\partial_t + D_Y$ on the cylinder $Y \times [0, \infty)$ where t is the standard parameter of $[0, \infty)$, γ is the Clifford multiplication of ∂_t, and D_Y is a Dirac-type operator on Y.

Let ϕ be a section of the associated spinor bundle over $\hat{X}$.

Lemma 24. *Suppose $D^2\phi = 0$ and ϕ is bounded on $\hat{X}$. Then, as $t \to \infty$, $\phi|_{Y \times \{t\}}$ converges to some element of* $\operatorname{Ker} D_Y$*, and $\partial_t\phi$ converges to* 0*. Both convergences hold in any Sobolev norm.*

Proof. Expand ϕ as $\sum_\lambda \phi_\lambda$ so that $D_Y\phi_\lambda = \lambda\phi_\lambda$. Since $D_Y\gamma + \gamma D_Y = 0$, we have $D^2 = -\partial_t^2 + D_Y^2$ and hence $\partial_t^2\phi_\lambda = \lambda^2\phi_\lambda$. Since ϕ_λ is bounded for $t \geq 0$, we obtain $\phi_\lambda = e^{-|\lambda|t}(\phi_\lambda|_{Y \times \{0\}})$, which implies the lemma. □

Suppose we have the decomposition $D = D_0 + D_1$ such that D_0 and D_1 are formally self-adjoint operators on Y anti-commuting each other, and hence satisfying $D^2 = D_0^2 + D_1^2$. We assume that, on the cylinder, D_0 and D_1 are translation invariant and D_1 contains only the derivatives along the fibers $Y \times \{t\}$. Moreover we assume that D_0 has the decomposition

$$D_0 = \gamma\partial_t + D_0'$$

satisfying $D_Y = D_0' + D_1$, where D_0' is a formally self-adjoint operator on Y anti-commuting with D_1.

Lemma 25. *If $D^2\phi = 0$ and ϕ is bounded on $\hat{X}$, then we have $D_0\phi = D_1\phi = 0$.*

Proof. Stokes' theorem implies

$$\int_{X_t}(|D_0\phi|^2+|D_1\phi|^2)-\int_{X_t}(\phi,D^2\phi)=\int_{Y\times\{t\}}(\gamma\phi|_{Y\times\{t\}},(D_0\phi)|_{Y\times\{t\}}).$$

We show that the Sobolev norms of $(D_0\phi)|_{Y\times\{t\}}$ converge to 0 as $t\to\infty$. From Lemma 24 the limit of $\phi|_{Y\times\{t\}}$ under $t\to\infty$ is an element of $\operatorname{Ker} D_Y$, and the limit of $\partial_t\phi|_{Y\times\{t\}}$ is zero. Since $D_Y^2=(D_0')^2+D_1^2$, we have $\operatorname{Ker} D_Y=\operatorname{Ker} D_0'\cap\operatorname{Ker} D_1$. Hence we have

$$(D_0\phi)|_{Y\times\{t\}}=\gamma(\partial_t\phi)|_{Y\times\{t\}}+D_0'\phi|_{Y\times\{t\}}\to 0,$$

which implies the claim. □

6. Well-definedness and the excision property.

We show Definition-Lemma 7 and Proposition 13 using the sum formula.

6.1. Well-definedness of $\mathrm{ind}_G^{Z/2}X$.

We show the well-definedness of $\mathrm{ind}_G^{R/RSp}X$ (Definition-Lemma 7).

Suppose that X is an $8k+4$-dimensional compact G-spin^c manifold with boundary $Y=\partial X$. Let $L=L_X$ be the G-equivariant complex line bundle associated with the G-spin^c-structure. We fix a reduction of the G-spin^c structure to a G-spin structure on a neighborhood of Y. Then L has a canonical G-equivariant trivialization on a neighborhood of Y. In particular L has a trivial G-invariant flat connection on a neighborhood of Y. Fix a G-invariant connection θ on the whole L which is an extension of the flat connection. In the following argument we need both Riemannian metrics and such connections as background data to choose Dirac-type operators. We, however, suppress the notation for the connections.

1. Let m_+ and m_- be any two G-invariant Riemannian metrics on X which are of the product form on a neighborhood of Y.
2. Take a G-invariant Riemannian metric m_0 on $Y\times[-1,+1]$ which is of the product form on a neighborhood of the boundaries so that $m_0\,|_{Y\times\{-1\}}=m_-\,|_Y$ and $m_0\,|_{Y\times\{+1\}}=m_+\,|_Y$.
3. Let $D(m_+),D(m_-)$ be some Dirac-type operators on X for the two Riemannian metrics m_+ and m_-, and $D(m_0)$ be a Dirac-type operator on $Y\times[-1,+1]$ for the Riemannian metric m_0.
4. Let $D_{m_+|_Y}$ and $D_{m_-|_Y}$ be some Dirac-type operators on Y for the two Riemannian metrics $m_+\,|_Y$ and $m_-\,|_Y$.

Glue $-X\cup Y\times[-1,+1]\cup X$ to obtain an oriented closed G-manifold $\tilde X$. We define the Riemannian metric $\tilde m$ on $\tilde X$ by patching m_-,m_0 and m_+. We endow $-X$ with a G-spin^c-structure which comes from that of X. Then, since we have a reduction of the G-spin^c-structure to G-spin structure on Y, the G-spin^c-structures on $-X$ and X can be glued together with the G-spin structure on $Y\times[-1,+1]$ to define a G-spin^c-structure on $\tilde X$. Then the index of a Dirac-type operator $D(\tilde m)$ on $\tilde X$ for Riemannian metric $\tilde m$ is a topological invariant, which depends only on the spin^c-structure on $\tilde{X}$. Note that $\tilde{X}$ is topologically the double of $Y \times [0,1] \cup X$ and that the spin^c-structure on $\tilde{X}$ is reversed by the defining involution on the double. Hence we have $\operatorname{ind} D(\tilde{m}) = 0$.

On the other hand, from the sum formula, we have the following equality in $R(G)$:

$$\begin{aligned} 0 = [\operatorname{ind}_G D(\tilde{m})] = &-[\operatorname{ind}_G D(m_-)] + [\operatorname{Ker} D(m_- |_Y)] \\ &+ [\operatorname{ind}_G D(m_0)] + [\operatorname{Ker} D(m_+ |_Y)] + [\operatorname{ind}_G D(m_-)]. \end{aligned}$$

Since $D(m_0)$ is a Dirac-type operator on an $8k+4$-dimensional spin manifold, $(\operatorname{Ker} D(m_0))^0$ and $(\operatorname{Ker} D(m_0))^1$ have G-invariant quaternionic structures. Similarly, since $D(m_+ |_Y)$ and $D(m_- |_Y)$ are Dirac-type operators on an $8k+3$-dimensional spin manifold, $\operatorname{Ker} D(m_+ |_Y)$ and $\operatorname{Ker} D(m_- |_Y)$ have G-invariant quaternionic structures. Hence we have

$$0 = [\operatorname{ind}_G D(\tilde{m})] \equiv -[\operatorname{ind}_G D(m_-)] + [\operatorname{ind}_G D(m_+)] \bmod RSp(G),$$

which implies our claim.

6.2. Proof of Proposition 13.

Let us recall the assumption of Proposition 13. Suppose that X is a compact $8k+4$-dimensional G-spin^c-manifold which is G-spin on the boundary $Y = \partial X$. When X' is a compact G-spin manifold with the boundary $-Y \coprod Y'$, the glued union $\tilde{X} := X' \cup X$ is a G-spin^c manifold with the boundary Y'. Fix a G-invariant Riemannian metric on $\tilde{X}$ which is of the product form on a neighborhood of Y and Y'. Let $D_X, D_{X'}$ and D_Y be some Dirac-type operators on X, X' and Y which are defined for the above Riemannian metrics. The sum formula implies

$$[\operatorname{ind}_G D_{\tilde{X}}] = [\operatorname{ind}_G D_{X'}] + [(\operatorname{Ker} D_Y)^1] + [\operatorname{ind}_G D_X].$$

Since X' is an $8k+4$-dimensional G-spin manifold and Y is an $8k+3$-dimensional G-spin manifold, $[\operatorname{ind}_G D_{X'}]$ and $[(\operatorname{Ker} D_Y)^1]$ are elements of $RSp(G)$. Hence we have

$$\operatorname{ind}_G^{R/RSp} D_X \equiv \operatorname{ind}_G D_{\tilde{X}} \bmod RSp(G).$$

7. Proof of the main theorem.

Our main theorems are Theorems 10 and 17. Since their proofs are quite parallel, we state only the proof of Theorem 10. We will reduce our proof to the next calculation.

7.1. 2-dimensional disk.

When X is a 2-dimensional disk, we can directly calculate $\operatorname{ind} X$ with the Atiyah-Patodi-Singer boundary condition. We use the following setting.

1. Fix an $SO(2)$-invariant Riemannian metric on $\boldsymbol{R}^2$ whose restriction on the complement of the unit disk is isometric to $S^1 \times [0,\infty)$ with the product metric. We write D for this

Riemannian manifold with $SO(2)$-action. The associated conformal structure gives an $SO(2)$-invariant complex structure on D, and hence an $SO(2)$-spin^c structure on D.
2. On the cylinder $S^1 \times [0,\infty)$, the translation invariant harmonic spinors are the constant functions in even degree, and $const \cdot d\bar{z}$ in odd degree, where $z = -t + \theta i \bmod 2\pi i$ is the complex coordinate of the point $(e^{i\theta}, t) \in S^1 \times [0,\infty)$.
3. On $S^1 \times [0,\infty)$ the $SO(2)$-*spin*^{c} structure coincides with the product of the Lie group spin structure on S^1 and the unique spin structure on $[0,\infty)$.

The next lemma and the definition of ind (Remark 8) implies

$$\mathrm{ind}(D) = 0 \in R(SO(2)).$$

The following argument is parallel to the proof of Lemma 4.32 of [**7**].

LEMMA 26. *Let $\bar\partial$ be the Dolbeault operator on D. Then we have*:

$$\mathrm{Ker}_0\, \bar\partial = 0, \quad \mathrm{Ker}_b\, \bar\partial = 1 \in R(SO(2)),$$
$$\mathrm{Ker}_0\, \bar\partial^* = 0, \quad \mathrm{Ker}_b\, \bar\partial^* = 0.$$

PROOF. Let $\boldsymbol{CP}^1$ be the one-point compactification of D. If we use the coordinate $z \bmod 2\pi i$ on the cylinder defined before, a coordinate on a neighborhood of the infinity point is given by $w = \exp(-z)$.

1. Note that $\mathrm{Ker}_b\, \bar\partial$ is the set of bounded holomorphic functions on D, and hence is identified with the set of holomorphic functions on $\boldsymbol{CP}^1$ from Riemann's removable singularity theorem. It implies that $\mathrm{Ker}_b\, \bar\partial$ is the set of constant functions, and hence we have $\mathrm{Ker}_0\, \bar\partial = 0$ and $\mathrm{Ker}_b\, \bar\partial = 1$ in $R(SO(2))$.
2. Note that the complex conjugation of $\mathrm{Ker}_b\, \bar\partial^*$ is the set of bounded holomorphic 1-forms on D. If we write a bounded holomorphic 1-form as $f(z)dz = g(w)dw$ on the cylinder, $|f(z)|$ is bounded as the real part of z goes to $-\infty$. It implies that the singularity of

$$g(w) = \frac{f(-\log w)}{w}$$

at $w = 0$ is at most a pole with degree 1. Since the only meromorphic 1-form on $\boldsymbol{CP}^1$ that allows at most a pole with degree 1 is 0, we have $\mathrm{Ker}_0\, \bar\partial^* = 0$ and $\mathrm{Ker}_b\, \bar\partial^* = 0$.

□

7.2. Complex line bundles.

We can calculate the index of the Dirac-type operator on the total space of some complex line bundles by using the above calculation for the disk D.

We use the following setting.

1. Let Z be a closed G-spin manifold.
2. Let ν be a G-equivariant complex line bundle over Z. We write $Y := S(\nu)$ and

$X := D(\nu)$.

3. By using the projection $\pi : X \to Z$, we have a decomposition

$$TX = \pi^* TZ \oplus T_{fiber} X$$

defined only up to homotopy, where $T_{fiber}X$ is the tangent bundle along the fibers of π.
   (a) π^*TZ has a G-spin structure coming from the G-spin structure on Z.
   (b) When we fix a G-invariant metric on ν, each fiber of $T_{\mathrm{fiber}}X$ is diffeomorphic to D and the diffeomorphism is well defined up to the $SO(2)$-action on D. Hence $T_{\mathrm{fiber}}X$ has a G-spin^c structure whose restriction on each fiber is isomorphic to the $SO(2)$-spin^c structure on D.
   (c) On a neighborhood of the boundary Y of X, the G-spin^c structure has a canonical reduction to a G-spin structure.
   (d) Now we can define a G-spin^c structure on TX by taking the direct sum of the above two G-spin^c structures. The G-spin^c structure on TX has a reduction to a G-spin structure on a neighborhood of Y.

The product formula Proposition 22 immediately implies:

Proposition 27.

$$\mathrm{Ker}_0\, D_{X_0} \cong (\mathrm{Ker}\, D_Z)^0 \otimes \mathrm{Ker}_0\, \bar\partial \oplus (\mathrm{Ker}\, D_Z)^1 \otimes \ker_0 \bar\partial^*$$
$$\mathrm{Ker}_b\, D_{X_0} \cong (\mathrm{Ker}\, D_Z)^0 \otimes \mathrm{Ker}_b\, \bar\partial_D^* \oplus (\mathrm{Ker}\, D_Z)^1 \otimes \ker_b \bar\partial_D.$$

Using Proposition 27 we can directly check the next corollary which is a special case of Theorem 10.

Corollary 28. *When Z is $8k+2$-dimensional, we have*

$$\mathrm{ind}_G^{\boldsymbol{Z}/2} X = \mathrm{ind}_G^{R/RSp} Z.$$

Proof. Lemma 26 and Proposition 27 imply $\mathrm{ind}_G D_X = -[(\mathrm{Ker}\, D_Z)^1] \equiv [(\mathrm{Ker}\, D_Z)^0] \bmod RSp(G)$. □

7.3. Proof of Theorem 10.

Now we prove Theorem 10 for general cases.

Suppose Z is a closed G-characteristic submanifold of an $8k+4$-dimensional compact G-spin^c manifold X with G-spin boundary. Let ν be the normal bundle of Z and $D(\nu)$ the disk bundle of ν. Then $D(\nu)$ is identified with a neighborhood of Z. The excision property (Proposition 13) implies $\mathrm{ind}_G^{R/RSp} X = \mathrm{ind}_G^{R/RSp} D(\nu)$. On the other hand Corollary 28 implies $\mathrm{ind}_G^{R/RSp} D(\nu) = \mathrm{ind}_G^{R/RSp} Z$. From these two relations we obtain the required equality $\mathrm{ind}_G^{R/RSp} X = \mathrm{ind}_G^{R/RSp} Z$.

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