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SL$_2$-ORBITS AND DEGENERATIONS OF MIXED HODGE STRUCTURE

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Abstract

We extend Schmid's SL$_2$-orbit theorem to a class of variations of mixed Hodge structure which normal functions, logarithmic deformations, degenerations of 1-motives and archimedean heights. In particular, as a consequence of this theorem, we obtain a simple formula for the asymptotic behavior of the archimedean height of a flat family of algebraic cycles which depends only on the weight filtration and local monodromy.

1. Introduction

Let $f : X \to S$ be a smooth, projective morphism of complex, quasi-projective varieties. Then, by the work of Griffiths [**18**], the cohomology groups $\mathcal{V}_s = H^k(X_s)$ patch together to form a variation of Hodge structure $\mathcal{V}$ over S. Furthermore, as a consequence of Schmid's orbit theorems [**34**], [**7**], one has a complete local theory regarding how such variations of Hodge structure degenerate along the boundary of a (partial) compactification $S \hookrightarrow \bar{S}$.

Namely, by the work of Hironaka [**22**] and Borel [**11**], we can restrict our attention to the case where S is a product of punctured disks Δ^{*n} and the monodromy representation of $\mathcal{V}$ is given by a system of unipotent transformations $T_j = e^{-N_j}$. Schmid's nilpotent orbit theorem asserts that, after lifting the period map of $\mathcal{V}$ to a π_1-equivariant map

$$F : U^n \to \mathcal{D}$$

from a product of upper half-planes into the corresponding classifying space of polarized Hodge structure, there exists an associated nilpotent orbit

$$\theta(\mathbf{z}) = \exp\left(\sum_j z_j N_j\right).F_\infty$$

which is asymptotic to $F(\mathbf{z})$ with respect to a suitable metric on $\mathcal{D}$. Furthermore, the possible nilpotent orbits $\theta(\mathbf{z})$ which can arise in this way are, in turn, classified by the SL$_2$-orbit theorem [**34**], [**7**] which,

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roughly speaking, says that every such nilpotent orbit $\theta(\mathbf{z})$ is asymptotic to another nilpotent orbit $\hat{\theta}(\mathbf{z})$ which arises from a representation of $\mathrm{SL}_2(\mathbb{R})^n$.

More precisely, recall that the Lie group $\mathrm{G}_{\mathbb{R}}$ consisting of all real automorphisms of the polarization acts transitively on $\mathcal{D}$. Accordingly, a 1-variable nilpotent orbit $\hat{\theta}(z)$ is said to be an SL_2-orbit if there exists a base point $F_o \in \mathcal{D}$ and a Lie homomorphism $\psi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{G}_{\mathbb{R}}$ such that

$$\hat{\theta}\left(g.\sqrt{-1}\right) = \psi(g).F_o.$$

Schmid's 1-variable SL_2-orbit theorem then asserts that given any nilpotent orbit $e^{zN}.F$ of pure, polarized Hodge structure, there exists a SL_2-orbit $e^{zN}.\hat{F}$, and a distinguished real analytic function

$$g : (a, \infty) \to \mathrm{G}_{\mathbb{R}}$$

such that

(a) $e^{iyN}.F = g(y)e^{iyN}.\hat{F}$;
(b) $g(y)$ and $g^{-1}(y)$ have convergent series expansions about ∞ of the form $(1 + \sum_{k=1}^{\infty} A_k y^{-k})$ with $A_k \in \ker(\operatorname{ad} N)^{k+1}$.

Likewise, in the several variable case, every n-variable nilpotent orbit is asymptotic via a $g(y)$-like function to an SL_2^n-orbit over a suitable region of U^n.

In this article, we consider analogous questions for morphisms $f : X \to S$ which are no longer necessarily proper or smooth. In this context, the variations of pure Hodge structure considered above are replaced (cf. §3) by variations of graded-polarized mixed Hodge structure which are *admissible* in the sense of Steenbrink and Zucker [**37**].

In [**31**], we proved that for admissible variations over a 1-dimensional base S, one has a corresponding nilpotent orbit theorem. To state our main result, we recall (cf. §2) that the period map of a variation of graded-polarized mixed Hodge structure takes values in the quotient of a classifying space $\mathcal{M}$ of graded-polarized mixed Hodge structure upon which a Lie group G acts transitively by automorphisms. Furthermore [**25**], in this setting the natural analogs of the SL_2-orbits considered above are admissible nilpotent orbits $e^{zN}.\hat{F}$ for which the associated limiting mixed Hodge structure (cf. §3) is split over $\mathbb{R}$.

Accordingly, by virtue of the above remarks, it is natural to conjecture that given an admissible nilpotent orbit $e^{zN}.F$, there should exist a split orbit $e^{zN}.\hat{F}$ and a distinguished real analytic function

$$g : (a, \infty) \to \mathrm{G}$$

such that

(a) $e^{iyN}.F = g(y)e^{iyN}.\hat{F}$;
(b) $g(\infty) := \lim_{y \to \infty} g(y) \in \ker(\operatorname{ad} N)$;

(c) $g^{-1}(\infty)g(y)$ and $g^{-1}(y)g(\infty)$ have convergent series expansions about ∞ of the form $(1+\sum_{k>0} A_k y^{-k})$ with $A_k \in \ker(\mathrm{ad}N)^{k+1}$.

In §6–9, we prove the existence [Theorem (4.2)] of such a function $g(y)$ provided the Hodge numbers of the associated classifying space $\mathcal{M}$ belong to one of the following two subcases, each of which arises in a number of geometric settings (e.g., 1-motives [**12**], logarithmic deformations [**38**], moduli of curves [**20**]):

(I) $h^{p,q} = 0$ unless $p+q = k,\ k-1$;
(II) $h^{p,q} = 0$ unless $p+q = 2k-1$, or $(p,q) = (k,k),\ (k-1,k-1)$.

In particular, as a consequence of the SL_2-orbit theorem described above, we obtain a simple formula for the asymptotic behavior of the archimedean height [**1**], [**2**], [**16**]

$$h(s) = \langle Z_s, W_s\rangle_\infty$$

of a flat family of algebraic cycles Z_s, $W_s \subseteq X_s$ over a smooth curve S, which depends only on the weight filtration and local monodromy of the associated variation of mixed Hodge structure [**19**]. Applying this result to the case where X is the Jacobian bundle attached to a family of smooth projective curves and Z, W arise from the Ceresa cycle gives an alternate proof of some recent results of Hain and Reed [**20**] on the biextension line bundle over $\mathcal{M}_g$.

As in [**34**], [**7**], the proof of Theorem (4.2) boils down to the construction of an explicit solution to an associated system of "monopole equations" attached to the nilpotent orbit $e^{zN}.F$. More precisely (cf. §2), in each of the two subcases (I) and (II) considered above, there exists a natural subgroup H of G which acts transitively on the corresponding classifying space $\mathcal{M}$ by isometries. As such (cf. §6), each choice of base point $F_o \in \mathcal{M}$ defines an auxiliary principal bundle

$$\mathrm{H}^{F_o} \to \mathrm{H} \to \mathrm{H}/\mathrm{H}^{F_o}$$

P over $\mathcal{M}$. Accordingly, a choice of connection ∇ on P determines a lift of $e^{iyN}.F_\infty$ to an H-valued function $h(y)$ which is tangent to ∇. Moreover, as in [**34**], the resulting function $h(y)$ satisfies a differential equation [Theorem (6.11)] of the form

$$h^{-1}\frac{dh}{dy} = -\mathrm{L}\ \mathrm{Ad}\,(h^{-1}(y))N \tag{1.1}$$

relative to a suitable endomorphism L of $\mathfrak{h} = \mathrm{Lie}\,(\mathrm{H})$. In particular, as a consequence of equation (1.1), the Hodge components

$$\beta(y) = \beta^{1,-1}(y) + \beta^{0,0}(y) + \beta^{-1,1}(y) + \beta^{0,-1}(y) + \beta^{-1,0}(y)$$

of the function $\beta(y) = \mathrm{Ad}\,(h^{-1}(y))N$ associated to a nilpotent orbit $e^{iyN}.F$ of type (I) satisfy the following system of differential equations

$$\frac{d}{dy}\beta_0(y) = -[\beta_0(y), \mathrm{L}\,\beta_0(y)], \qquad \beta_0(y) = \sum_{r+s=0} \beta^{r,s}(y), \tag{1.2}$$

$$\frac{d}{dy}\begin{pmatrix}\beta^{-1,0}\\ \beta^{0,-1}\end{pmatrix} = \sqrt{-1}\begin{pmatrix}\mathrm{ad}\,\beta^{0,0} & -2\,\mathrm{ad}\,\beta^{-1,1}\\ 2\,\mathrm{ad}\,\beta^{1,-1} & -\mathrm{ad}\,\beta^{0,0}\end{pmatrix}\begin{pmatrix}\beta^{-1,0}\\ \beta^{0,-1}\end{pmatrix}. \tag{1.3}$$

Following [**34**], we then observe that equation (1.2) becomes equivalent to Nahm's equations [**23**]

$$-2\frac{d}{dy}X^+(y) = [Z(y), X^+(y)], \qquad 2\frac{d}{dy}X^-(y) = [Z(y), X^-(y)] \tag{1.4}$$
$$-\frac{d}{dy}Z(y) = [X^+(y), X^-(y)]$$

upon setting $X^+(y) = 2i\beta^{1,-1}(y)$, $Z(y) = 2i\beta^{0,0}(y)$ and $X^-(y) = -2i\beta^{-1,1}(y)$. Moreover, using the methods of [**7**], one can construct a series solution (cf. §7) to equation (1.4) in the form of a function

$$\Phi(y) : (a,\infty) \to \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}_{\mathbb{C}}), \qquad \Phi(y) = \sum_{n\geq 0} \Phi_n y^{-1-n/2} \tag{1.5}$$

such that $X^-(y) = \Phi(y)x^-$, $Z(y) = \Phi(y)\mathfrak{z}$, and $X^+(y) = \Phi(y)x^+$ where:

$$x^- = \frac{1}{2}\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix}, \qquad \mathfrak{z} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad x^+ = \frac{1}{2}\begin{pmatrix}1 & i\\ i & -1\end{pmatrix}. \tag{1.6}$$

Building upon the series solution (1.5), we then construct a similar series solution to (1.3) in §8. Taken with equation (1.1), such a series solution for $\beta(y)$ then allows us to compute $h(y)$ modulo left multiplication by an element $h_o \in \mathrm{H}$. Imposing the boundary condition

$$\lim_{y\to\infty} e^{-iyN}h(y).F_o = F$$

then determines h_o. Having computed $h(y)$, the desired function $g(y)$ is then given by the formula

$$h(y) = g(y)y^{-H/2}$$

where $H = \Phi_0(x^+ + x^-)$.

To illustrate how the SL_2-orbit theorem described above works in the context of a geometric example, let X be a compact Riemann surface and

$$c_1 = c_{12} - c_{11}, \qquad c_2 = c_{22} - c_{21} \tag{1.7}$$

be a pair of disjoint 0-cycles on X. Then (up to an additive constant), there exists a unique harmonic function $f : X - |c_2| \to \mathbb{R}$ such that

$$\Omega = \frac{1}{2\pi}(*df - i\,df) \tag{1.8}$$

is a holomorphic 1-form on $X - |c_2|$ with simple poles along $|c_2| = \{c_{22}, c_{21}\}$ and residues

$$\mathrm{Res}_{c_{22}}(\Omega) = \frac{1}{2\pi i}, \qquad \mathrm{Res}_{c_{21}}(\Omega) = -\frac{1}{2\pi i}.$$

The archimedean height of c_1 and c_2 is then defined to be

$$\langle c_1, c_2 \rangle = 2\pi\, \mathrm{Im}\left(\int_{c_{11}}^{c_{12}} \Omega \right). \tag{1.9}$$

To bring in the mixed Hodge structures, we now recall [**12**] that the elements of $H^1(X-|c_2|)$ can be decomposed according to (mixed) Hodge type. Furthermore, with respect to this decomposition, Ω generates the classes of type $(1,1)$. As such, the integral (1.9) can be viewed as a period of $H^1(X - |c_2|)$ with respect to c_1. Therefore, upon varying the triple (X, c_1, c_2), the integral (1.9) defines a "period map" whose asymptotic behavior is governed by Theorem (4.2). In particular [Theorem (5.19)], near a degenerate point $s = 0$,

$$\langle c_1(s), c_2(s) \rangle \approx -\mu \log |s|$$

where μ is a constant which depends only on the local monodromy of the associated variation of mixed Hodge structure.

More concretely, let $E \to \Delta^*$ be the family of elliptic curves

$$E_s = \mathbb{C}/(\mathbb{Z} \oplus \tau(s)\mathbb{Z})$$

defined by the function $\tau(s) = \frac{1}{\pi i}\log(s)$ and

$$h(s) = \langle e_3 - e_0, e_2 - e_1 \rangle$$

be the height function determined by the 2-torsion points

$$e_0 = 0, \quad e_1 = \frac{1}{2}, \quad e_2 = \frac{\tau}{2}, \quad e_3 = \frac{1}{2}(1+\tau).$$

Then, a short calculation shows that

$$h(s) = -\log \left| \frac{\vartheta^2(e_2)}{\vartheta^2(e_1)} \right| + \frac{1}{2} \log |\exp(-2\pi i e_3)|$$

where ϑ is Riemann's theta function, and hence $h(s) \approx -\frac{1}{2}\log|s|$ as $s \to 0$.

To illustrate another application of the SL_2-orbit theorem, let

$$F : U \to \mathcal{M}$$

be the period map of a non-constant, admissible variation of type (I). Then, as a consequence of Theorem (4.2), the holomorphic sectional curvature of $F(z)$ is negative, and bounded away from zero as $\mathrm{Im}(z) \to \infty$ [Theorem (4.9)].

Heuristically, the proof of this fact boils down to replacing $F(z)$ by the corresponding split orbit $\hat{\theta}(z) = e^{zN}.\hat{F}$ and then noting that split orbits of type (I) are actually SL_2-orbits. More precisely, by virtue of the above remarks,

$$\|F_*(d/dz)\|_{F(z)} \approx \|\hat{\theta}_*(d/dz)\|_{\hat{\theta}(z)}.$$

Accordingly, since $\hat{\theta}(z)$ is a nilpotent orbit, $\hat{\theta}_*(\frac{d}{dz})$ is basically just N, and hence (up to a constant scalar factor)

$$\|F_*(d/dz)\|_{F(z)} \approx \|N\|_{\hat{\theta}(z)}.$$

Therefore (cf. §2), since the real elements of G act on $\mathcal{M}$ by isometries, it then follows that

$$\|N\|_{\hat{\theta}(z)} = \|N\|_{e^{xN}e^{iyN}.\hat{F}} = \|N\|_{e^{iyN}.\hat{F}}.$$

Consequently, since $\hat{\theta}(z)$ is actually an SL_2-orbit,

$$e^{iyN}.\hat{F} = \exp\left(-\frac{1}{2}\log(y)H\right)e^{iN}.\hat{F}$$

where H is real and $[H,N] = -2N$. Thus,

$$\begin{aligned}\|F_*(d/dz)\|_{F(z)} &\approx \|N\|_{e^{iyN}.\hat{F}_\infty} = \|N\|_{\exp(-\frac{1}{2}\log(y)H)e^{iN}.\hat{F}_\infty} \\ &= \|\mathrm{Ad}\left(\exp\left(\frac{1}{2}\log(y)H\right)\right)N\|_{e^{iN}.\hat{F}_\infty} \\ &= (1/y)\|N\|_{e^{iN}.\hat{F}_\infty}\end{aligned}$$

and hence the pullback of the metric of $\mathcal{M}$ along F is asymptotic to a constant multiple of the Poincaré metric.

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2. Preliminary Remarks

In this section, we recall the construction of the period map of a variation of graded-polarized mixed Hodge structure, and discuss the geometry of the associated classifying spaces of graded-polarized mixed Hodge structure [**24**], [**30**], [**38**].

Definition 2.1. Let S be a complex manifold. Then, a variation of graded-polarized mixed Hodge structure $\mathcal{V}$ over S consists of the following data:

(1) A finite rank, $\mathbb{Q}$-local system $\mathcal{V}_{\mathbb{Q}}$ over S;
(2) A rational, increasing filtration $\cdots \subseteq \mathcal{W}_k \subseteq \mathcal{W}_{k+1} \subseteq \cdots$ of $\mathcal{V}_{\mathbb{C}} = \mathcal{V}_{\mathbb{Q}} \otimes \mathbb{C}$ by sublocal systems;

(3) A decreasing filtration $\cdots \subseteq \mathcal{F}^p \subseteq \mathcal{F}^{p-1} \subseteq \cdots$ of $\mathcal{V}_{\mathbb{C}} \otimes \mathcal{O}_S$ by holomorphic subbundles;

(4) A collection of non-degenerate bilinear forms

$$Q_k : Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}) \otimes Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}) \to \mathbb{Q}$$

of alternating parity $(-1)^k$;

subject to the following two conditions:

(a) $\mathcal{F}$ is horizontal with respect to the Gauss–Manin connection ∇ of $\mathcal{V}$, i.e., $\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S$;

(b) For each index k, $(Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}), \mathcal{F}Gr^{\mathcal{W}}_k, Q_k)$ is a variation of pure, polarized Hodge structure of weight k.

Remark. The rational structure of $\mathcal{V}$ plays no role in either the statement or the proof of the SL_2-orbit theorem. With the exception of the material on Arakelov geometry in §5, all of the results in this paper are valid in the category of real variations of graded-polarized mixed Hodge structure.

In analogy with the pure case [**34**], the isomorphism class of a variation of graded-polarized mixed Hodge structure $\mathcal{V} \to S$ is determined by its period map

$$\varphi : S \to \Gamma\backslash\mathcal{M}, \qquad \Gamma = \mathrm{Image}(\rho) \tag{2.2}$$

and its monodromy representation $\rho : \pi_1(S, s_0) \to GL(V)$ on a fixed reference fiber $V = \mathcal{V}_{s_o}$. More precisely, let W and $Q = \{Q_k\}$ denote the specialization of the weight filtration and graded-polarizations of $\mathcal{V}$ to V. Define X to be the flag variety consisting of all decreasing filtrations F of $V_{\mathbb{C}}$ such that

$$\dim(F^p) = \mathrm{rank}(\mathcal{F}^p)$$

and let $\mathcal{M}$ denote the classifying space [**30**] consisting of all filtrations $F \in X$ such that (F, W) is a mixed Hodge structure which is graded-polarized by Q. Then, the period map (2.2) is obtained by simply pulling back the Hodge filtration $\mathcal{F}$ of $\mathcal{V}$ to $\mathcal{V}_{s_o}$ via the Gauss–Manin connection.

As in the pure case, the classifying spaces $\mathcal{M}$ defined above are complex manifolds upon which a real Lie group acts transitively by complex automorphisms. In the subsections below, we shall introduce a certain "maximally homogeneous" hermitian metric on $\mathcal{M}$, and compute its curvature.

Theorem 2.3 ([**30**]). *The classifying space $\mathcal{M}$ is a complex manifold upon which the real Lie group*

$$\mathrm{G} = \left\{ g \in GL(V_{\mathbb{C}})^W \mid Gr(g) \in \mathrm{Aut}_{\mathbb{R}}(Q) \right\}$$

acts transitively by automorphisms, where $GL(V_{\mathbb{C}})^W$ *denotes the stabilizer of* W *in* $GL(V_{\mathbb{C}})$*, and* $Gr(g)$ *denotes the induced action of* $g \in GL(V_{\mathbb{C}})$ *on* Gr^W*.*

Proof. That G acts transitively on $\mathcal{M}$ is a matter of simple linear algebra. In particular, since G acts transitively on $\mathcal{M}$, the orbit $\check{\mathcal{M}} \subseteq X$ of $F_o \in \mathcal{M}$ under the action of the complex Lie group

$$\mathrm{G}_{\mathbb{C}} = \left\{ g \in GL(V_{\mathbb{C}})^W \mid Gr(g) \in \mathrm{Aut}_{\mathbb{C}}(Q) \right\}$$

is well defined, independent of F_o. Therefore, in order to show that $\mathcal{M}$ is a complex manifold on which G acts by automorphisms, it is sufficient to show (cf. [**30**]) that $\mathcal{M}$ is an open subset of $\check{\mathcal{M}} \cong \mathrm{G}_{\mathbb{C}}/\mathrm{G}_{\mathbb{C}}^{F_o}$, i.e., for every $F \in \mathcal{M}$, there exists a neighborhood U of 1 in $\mathrm{G}_{\mathbb{C}}$ such that

$$g_{\mathbb{C}} \in U \implies g_{\mathbb{C}}.F \in \mathcal{M}.$$

q.e.d.

Warning. $\mathrm{G}_{\mathbb{C}}$ is the complexification of $\mathrm{G}_{\mathbb{R}} = G \cap GL(V_{\mathbb{R}})$. In general, $\mathrm{G} \neq \mathrm{G}_{\mathbb{R}}$.

In order to construct a hermitian metric on $\mathcal{M}$, we now recall the following result of Deligne [**12**]:

Theorem 2.4. *Let* (F, W) *be a mixed Hodge structure. Then, there exists a unique, functorial bigrading*

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q} \tag{2.5}$$

of the underlying complex vector space $V_{\mathbb{C}}$ *such that*

(a) $F^p = \oplus_{a \geq p} I^{a,b}$*;*
(b) $W_k = \oplus_{a+b \leq k} I^{a,b}$*;*
(c) $\overline{I^{p,q}} = I^{q,p} \mod \oplus_{r<q,s<p} I^{r,s}$*.*

Corollary 2.6. *Each choice of graded-polarization* $Q = \{Q_k\}$ *of* (F, W) *determines a unique, functorial mixed Hodge metric* h_F *on* $V_{\mathbb{C}}$ *such that*

(i) *The decomposition* (2.5) *is orthogonal with respect to* h_F*;*
(ii) $u,\ v \in I^{p,q} \implies h_F(u, v) = i^{p-q} Q_{p+q}([u], [\bar{v}])$*.*

Accordingly, via the standard identification of $T_F(\mathcal{M})$ *with a subspace of*

$$T_F(X) = \bigoplus_p \mathrm{Hom}(F^p, V_{\mathbb{C}}/F^p)$$

the mixed Hodge metric (2.6) *extends to a hermitian metric* h *on* $T(\mathcal{M})$*.*

Remark. Equivalently, the induced metric (2.6) on $T(\mathcal{M})$ can be described as follows: Let F be a point in $\mathcal{M}$. Then, application of

Theorem (2.4) to the mixed Hodge structure $(F^{\cdot}\mathfrak{g}_{\mathbb{C}}, W_{\cdot}\mathfrak{g}_{\mathbb{C}})$ defines a functorial bigrading

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{r+s\leq 0} \mathfrak{g}^{r,s}_{(F,W)} \tag{2.7}$$

such that

$$\mathfrak{t}_F = \bigoplus_{r<0} \mathfrak{g}^{r,s}_{(F,W)}$$

is a vector space complement to the isotopy algebra $\mathfrak{g}^F_{\mathbb{C}}$ of F in $\mathfrak{g}_{\mathbb{C}}$. Consequently,

$$T_F(\mathcal{M}) \cong \mathfrak{t}_F \tag{2.8}$$

via the differential of the exponential map

$$e : \mathfrak{t}_F \to \check{\mathcal{M}}, \qquad e(u) = \exp(u).F.$$

Moreover, relative to the isomorphism (2.8), $h_F(\alpha,\beta) = \mathrm{Tr}\,(\alpha\beta^*)$..

In the pure case, the metric (2.6) can be identified with a G-invariant metric on the corresponding classifying space of pure, polarized Hodge structure $\mathcal{D}$. In contrast, in the mixed case, the action of G on $\mathcal{M}$ usually has non-compact isotopy, and hence there usually do not exist any G-invariant metrics on $\mathcal{M}$. Nonetheless, both the decomposition (2.5) and the metric (2.6) are maximally homogeneous in the following sense:

Theorem 2.9 ([**24**])**.** *Let $F \in \mathcal{M}$, $G_{\mathbb{R}} = \mathrm{G} \cap GL(V_{\mathbb{R}})$ and*

$$\Lambda^{-1,-1}_{(F,W)} = \bigoplus_{r,s<0} \mathfrak{g}^{r,s}_{(F,W)}.$$

Then,

$$\mathcal{M} = \mathrm{G}_{\mathbb{R}} \exp(\Lambda^{-1,-1}_{(F,W)}).F. \tag{2.10}$$

Moreover, given any element $g \in \mathrm{G}_{\mathbb{R}} \cup \exp(\Lambda^{-1,-1}_{(F,W)})$:

(i) $I^{p,q}_{(g.F,W)} = g.I^{p,q}_{(F,W)}$;
(ii) *The induced map $L_{g*} : T_F(\mathcal{M}) \to T_{g.F}(\mathcal{M})$ is an isometry.*

To compute the curvature of $T(\mathcal{M})$ with respect to the mixed Hodge metric, let us fix a point $F \in \mathcal{M}$. Then, on account of equation (2.10), every element $g_{\mathbb{C}} \in \mathrm{G}_{\mathbb{C}}$ such that $g_{\mathbb{C}}.F \in \mathcal{M}$ admits a factorization of the form:

$$g_{\mathbb{C}} = g_{\mathbb{R}} e^{\lambda} f \tag{2.11}$$

where $g \in \mathrm{G}_{\mathbb{R}}$, $e^{\lambda} \in \exp(\Lambda^{-1,-1}_{(F,W)})$ and $f \in \mathrm{G}^F_{\mathbb{C}}$. Moreover, (cf. [**30**]) by restricting the possible values of λ and $\log(f)$ one can define a distinguished real-analytic factorization of the form (2.11) over a neighborhood of $1 \in \mathrm{G}_{\mathbb{C}}$. Accordingly, by combining this factorization with Theorem (2.9), we can then calculate the curvature of $\mathcal{M}$ following [**11**]:

Theorem 2.12 ([**29**]). *Let $F \in \mathcal{M}$, and $\mathfrak{g}_{\mathbb{C}} = \eta_+ \oplus \eta_0 \oplus \eta_- \oplus \Lambda^{-1,-1}$ denote the decomposition of $\mathfrak{g}_{\mathbb{C}}$ defined by the subalgebras*

$$\eta_+ = \bigoplus_{r\geq 0,\, s<0} \mathfrak{g}^{r,s}_{(F,W)} \qquad \eta_- = \bigoplus_{r<0,\, s\geq 0} \mathfrak{g}^{r,s}_{(F,W)}$$
$$\eta_0 = \mathfrak{g}^{0,0}_{(F,W)} \qquad \Lambda^{-1,-1} = \bigoplus_{r,s<0} \mathfrak{g}^{\rho,s}_{(F,W)}.$$

Let π_+, π_0, π_- and π_Λ denote the corresponding projection operators form $\mathfrak{g}_{\mathbb{C}}$ onto η_+, η_0, η_- and $\Lambda^{-1,-1}$. Then, relative to the identification (2.8)*, the hermitian holomorphic curvature of $T(\mathcal{M})$ at F with respect to the mixed Hodge metric* (2.6) *is given by the formula:*

$$R(u,v) = S(u,\bar{v}) - S(v,\bar{u})$$

where

$$S(u,\bar{v}) = \pi_{\mathfrak{t}}\,\mathrm{ad}\left(\left(\pi_+[\bar{v},u] + \frac{1}{2}\pi_0[\bar{v},u]\right) + \left(\pi_+[\bar{u},v] + \frac{1}{2}\pi_0[\bar{u},v]\right)^*\right)$$
$$+ [\pi_{\mathfrak{t}}\,\mathrm{ad}\pi_+(\bar{v}), \pi_{\mathfrak{t}}\,\mathrm{ad}\pi_+(\bar{u})^*]$$

and $\pi_{\mathfrak{t}}$ denotes orthogonal projection from $gl(V_{\mathbb{C}})$ onto $\mathfrak{t}_F$ with respect to h_F.

Corollary 2.13. *The holomorphic sectional curvature of $\mathcal{M}$ along $u \in T_F(\mathcal{M})$ is given by the formula $R(u) = h_F(S(u,\bar{u})u,u)/h_F^2(u,u)$.*

Remark. Unlike the pure case, the mixed Hodge metric h need not have negative holomorphic sectional curvature along horizontal directions. The underlying reason for this is that G need not be semisimple, and hence one can construct holomorphic, horizontal maps $F : \mathbb{C} \to \mathcal{M}$.

Following [**24**], in order to address the fact that G usually acts with non-compact isotopy on $\mathcal{M}$, we now construct a natural fibration $\mathcal{M} \to \mathcal{M}_{\mathbb{R}}$ such that:

(i) $\mathrm{G}_{\mathbb{R}}$ acts transitively by isometries on $\mathcal{M}_{\mathbb{R}}$;
(ii) The fiber over $\hat{F}$ is isomorphic to the subalgebra

$$\Lambda^{-1,-1}_{(\hat{F},W)} \cap Lie(\mathrm{G}_{\mathbb{R}})$$

via the map $\lambda \mapsto e^{i\lambda}.\hat{F}$.

To this end, we recall that a grading of an increasing filtration W of a finite dimensional vector space V is a semisimple endomorphism Y of V such that W_k is the direct sum of W_{k-1} and the k-eigenspace $E_k(Y)$ for each index k. In particular, by Theorem (2.4), each mixed Hodge structure (F,W) induces a functorial grading $Y = Y_{(F,W)}$ on the underlying weight filtration W via the rule:

$$E_k(Y) = \bigoplus_{p+q=k} I^{p,q}. \tag{2.14}$$

Accordingly, a mixed Hodge structure (F, W) is said to be *split over* $\mathbb{R}$ if and only if the associated grading (2.14) is defined over $\mathbb{R}$, i.e., $\overline{I^{p,q}} = I^{q,p}$.

Theorem 2.15 ([**24**])**.** *The locus of points $F \in \mathcal{M}$ such that (F, W) is split over $\mathbb{R}$ is a C^∞ submanifold of $\mathcal{M}$ on which $\mathrm{G}_\mathbb{R}$ acts transitively by isometries.*

To continue [**24**], let $\pi : \mathcal{M} \to \mathcal{M}_\mathbb{R}$ be a C^∞ fibration such that:
(a) $\pi(F) \in \exp(\Lambda^{-1,-1}_{F,W}).F$;
(b) $g \in \mathrm{G}_\mathbb{R} \implies \pi(g.F) = g.\pi(F)$;
(c) $F \in \mathcal{M}_\mathbb{R} \implies \pi(F) = F$.

Then, on account of the fact that

$$\exp\left(\Lambda^{-1,-1}_{(F,W)}\right) \cap \mathrm{G}^F = 1,$$

the equation

$$\pi(F) = e(F)^{-1}.F$$

defines a C^∞ function $e : \mathcal{M} \to \mathrm{G}$ such that
(1) $e(F) \in \exp(\Lambda^{-1,-1}_{(F,W)})$;
(2) $\hat{F} := e(F)^{-1}.F \in \mathcal{M}_\mathbb{R}$;
(3) $g \in \mathrm{G}_\mathbb{R} \implies e(g.F) = \mathrm{Ad}\,(g)e(F)$;
(4) $F \in \mathcal{M}_\mathbb{R} \implies e(F) = 1$.

Conversely, given a C^∞ function $e : \mathcal{M} \to \mathrm{G}$ which satisfies conditions (1)–(4), the above process can be inverted to define a corresponding fibration $\pi : \mathcal{M} \to \mathcal{M}_\mathbb{R}$ as above. Thus, as a consequence of the next result, there exists a unique fibration $\mathcal{M} \to \mathcal{M}_\mathbb{R}$ such that

$$\overline{e(F)} = e(F)^{-1}.$$

Theorem 2.16 ([**7**])**.** *Let (F, W) be a mixed Hodge structure. Then there exists a unique, real element*

$$\delta \in \Lambda^{-1,-1}_{(F,W)} = \bigoplus_{r,s<0} gl(V_\mathbb{C})^{r,s}_{(F,W)}$$

such that $(\hat{F}, W) = (e^{-i\delta}.F, W)$ is split over $\mathbb{R}$.

Proof. Let $Y = Y_{(F,W)}$ denote the grading (2.14) of W. Then, by virtue of Theorem (2.4),

$$\bar{Y} = Y \mod \Lambda^{-1,-1}_{(F,W)}.$$

Consequently (cf. [**7**]), there exists a unique real element δ of $\Lambda^{-1,-1}_{(F,W)}$ such that

$$\bar{Y} = e^{-2i\delta}.Y.$$

Therefore, by virtue of part (i) of Theorem (2.9), $(\hat{F}, W) = (e^{-i\delta}.F, W)$ is split over $\mathbb{R}$, with grading $Y_{(\hat{F},W)} = e^{-i\delta}.Y_{(F,W)}$. q.e.d.

In particular, since both the mixed Hodge metric and the splitting operation (2.16) depend upon the Deligne–Hodge decomposition (2.5), the complexity of the fibration $\mathcal{M} \to \mathcal{M}_{\mathbb{R}}$ provides a measure of the failure of G to act on $\mathcal{M}$ by isometries. As such, the next result implies that the geometry of the classifying spaces considered in §1 should be "simple" [cf. Theorem (2.19)]:

Theorem 2.17. *Let $\mathcal{M}$ be a classifying space of type* (I) *or* (II) (cf. §1)*, and*

$$\mathrm{Lie}_{-r}(W) = \{\alpha \in gl(V_{\mathbb{C}}) \mid \alpha(W_k) \subseteq W_{k-r}\}.$$

Then, the fibration $\mathcal{M} \to \mathcal{M}_{\mathbb{R}}$ defined by Theorem (2.16) *is isomorphic to the trivial fibration*

$$\mathcal{M} \cong \mathbb{R}^d \times \mathcal{M}_{\mathbb{R}}$$

where $d = \dim_{\mathbb{C}} \mathrm{Lie}_{-2}(W)$.

Proof. If $\mathcal{M}$ is type (I) then $d = 0$ and every point $F \in \mathcal{M}$ is split over $\mathbb{R}$ due to the short length of W. Similarly, if $\mathcal{M}$ is type (II) then

$$\Lambda^{-1,-1}_{(F,W)} = \mathfrak{g}^{-1,-1} = \bigoplus_{p+q=-2} \mathfrak{g}^{r,s} = \mathrm{Lie}_{-2}(W) \tag{2.18}$$

due to the Hodge numbers of $\mathcal{M}$. Consequently, in this case, the fibration $\mathcal{M} \to \mathcal{M}_{\mathbb{R}}$ is given by the formula

$$e^{i\lambda}.F \mapsto F, \qquad \lambda \in \mathrm{Lie}_{-2}(W) \cap gl(V_{\mathbb{R}}), \quad F \in \mathcal{M}_{\mathbb{R}}.$$

q.e.d.

Theorem 2.19. *Let $\mathcal{M}$ be a classifying space of type* (I) *or* (II)*. Then, the subgroup*

$$\mathrm{H} = \{g \in \mathrm{G} \mid Gr(g) \in \mathrm{Aut}_{\mathbb{R}}(W_k/W_{k-2})\}$$

of G *consisting of those elements $g \in \mathrm{G}$ which induce real automorphisms of W_k/W_{k-2} for all k, acts transitively on $\mathcal{M}$ by isometries.*

Proof. If $\mathcal{M}$ is type (I) then $\mathcal{M} = \mathcal{M}_{\mathbb{R}}$ and $\mathrm{H} = \mathrm{G}_{\mathbb{R}}$, so we're done by Theorem (2.15). Suppose therefore that $\mathcal{M}$ is type (II). Then, since H contains the subgroups $\mathrm{G}_{\mathbb{R}}$ and

$$\exp\left(\Lambda^{-1,-1}_{(F,W)}\right) = \exp(\mathrm{Lie}_{-2}(W))$$

for every point $F \in \mathcal{M}$, it then follows from Theorem (2.9) that H acts transitively on $\mathcal{M}$. To see that H acts by isometries, recall [**7**] that the set $\mathcal{Y}(W)$ consisting of all gradings Y of W is an affine space upon which $\exp(\mathrm{Lie}_{-1}(W))$ acts simply transitively by the rule

$$g.Y = \mathrm{Ad}\,(g)Y. \tag{2.20}$$

Accordingly, given any element $g \in \mathrm{G}$ and any grading $Y \in \mathcal{Y}(W)$, there exist unique elements $g^Y \in \mathrm{G}^Y$ and $g_{-1} \in \exp(Lie_{-1}(W))$ such that

$$g = g_{-1}g^Y \tag{2.21}$$

and $g.Y = g_{-1}.Y$.

Suppose now that $Y = \bar{Y}$. Then, since every element of G acts by real automorphisms on Gr^W, the corresponding factor g^Y appearing in (2.21) actually belongs to $\mathrm{G}_{\mathbb{R}}$. Furthermore, since $\mathcal{M}$ is type (II), $g_{-1} = e^{\alpha}$ can be factored as

$$g_{-1} = (1 + \alpha_{-1})(1 + \alpha_{-2}) \tag{2.22}$$

where $\alpha_{-j} \in E_{-j}(\mathrm{ad}Y)$. In particular, if $g \in \mathrm{H}$ then $\alpha_{-1} \in gl(V_{\mathbb{R}})$ since $g = g_{-1}g^Y$ acts by real automorphisms on W_k/W_{k-2}. Consequently,

$$g = g_{-1}g^Y = \left\{(1 + \alpha_{-1})g^Y\right\}\left\{(g^Y)^{-1}(1 + \alpha_{-2})g^Y\right\} \tag{2.23}$$

where the first term in curly braces on the right hand side of (2.23) belongs to $\mathrm{G}_{\mathbb{R}}$, while the second term belongs $\exp(\mathrm{Lie}_{-2}(W))$. Therefore, by Theorem (2.9) and equation (2.18),

$$L_{g*} : T_F(\mathcal{M}) \to T_{g.F}(\mathcal{M})$$

is an isometry for all $F \in \mathcal{M}$. q.e.d.

Remark. The proof of Theorem (2.19) implies the following additional fact: If $\mathcal{M}$ is type (I) or (II) then $h \in \mathrm{H}$, $F \in \mathcal{M} \implies I^{p,q}_{(h.F,W)} = h.I^{p,q}_{(F,W)}$.

3. Limits of Mixed Hodge Structure

Let $\mathcal{V} \to \Delta^*$ be a variation of graded-polarized mixed Hodge structure. Then, in contrast to the pure case, the period map of $\mathcal{V}$ can have irregular singularities at the origin. The source of this apparent disparity lies in the geometry of the associated classifying spaces. Namely, unlike the pure case [**34**], the classifying spaces of graded-polarized mixed Hodge structure $\mathcal{M}$ discussed in §2 need not have negative holomorphic sectional curvature along horizontal directions.

Nevertheless, by comparison with the ℓ-adic case, Deligne conjectured in [**13**] that the period map of a variation of mixed Hodge structure arising from a family of complex algebraic varieties should not have such irregular singularities. Furthermore, according to [**13**], there should exist a category of "good" variations of mixed Hodge structures which both contains all of the geometric variations and possesses the following salient features of the pure case:

(a) The existence of the limiting mixed Hodge structure;
(b) In the geometric case, the limiting Hodge structure (a) should admit a de Rham theoretic construction in terms of the log complex of the underlying morphism $f : X \to \Delta$;
(c) The existence of a functorial mixed Hodge structure on the cohomology $H^*(X, \mathcal{V})$ of a good variation $\mathcal{V} \to X$;

(d) Nilpotent Orbit Theorem: The period map of a good variation of mixed Hodge structure should be asymptotic to the corresponding nilpotent orbit.

In [**37**], Steenbrink and Zucker formulated the following definition of a good variation:

Definition 3.1. A variation of graded-polarized mixed Hodge structure $\mathcal{V} \to \Delta^*$ with unipotent monodromy is admissible if

(i) The limiting Hodge structure F_∞ of $\mathcal{V}$ exists;
(ii) The relative weight filtration ${}^rW = {}^rW(N, W)$ exists.

The first evidence that this is indeed the correct definition is Deligne's proof in the appendix to [**37**] that conditions (i) and (ii) already imply that (a) the pair $(F_\infty, {}^rW)$ is a mixed Hodge structure, relative to which N is a $(-1,-1)$-morphism. Additional evidence is provided by the following two results [**15**], [**33**], special cases of which are proven in [**37**]:

— Every geometric variation is admissible, and admits a de Rham theoretic construction (b) of its limiting mixed Hodge structure $(F_\infty, {}^rW)$;
— The cohomology $H^*(X, \mathcal{V})$ of an admissible variation $\mathcal{V} \to X$ admits a functorial mixed Hodge structure (c).

In this section, we consider the singularities (d) of the period map

$$\varphi : \Delta^* \to \Gamma\backslash\mathcal{M} \tag{3.2}$$

of an admissible variation $\mathcal{V} \to \Delta^*$ with unipotent monodromy. To this end, let $p : U \to \Delta^*$ denote the universal cover of the punctured disk by the upper half-plane, and (s, z) be a pair of coordinates relative to which p assumes the form $s = e^{2\pi i z}$. Then, by virtue of the local liftablity of φ, there exists a holomorphic, horizontal map $F : U \to \mathcal{M}$ which makes the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{F} & \mathcal{M} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow \\ \Delta^* & \xrightarrow{\varphi} & \Gamma\backslash\mathcal{M}. \end{array} \tag{3.3}$$

Consequently, by the commutativity of (3.3), the function

$$\psi(z) := e^{-zN}.F(z) \tag{3.4}$$

descends to a well defined map $\psi(s) : \Delta^* \to \check{\mathcal{M}}$. Moreover, we have the following result:

Lemma 3.5. *$\mathcal{V}$ is admissible if and only if both the relative weight filtration ${}^rW$ and the limiting Hodge filtration*

$$F_\infty = \lim_{s\to 0} \psi(s)$$

exist.

Thus, by the theorem of Deligne [**37**] quoted above, given an admissible variation $\mathcal{V} \to \Delta^*$, each choice of coordinates (s, z) as above defines an associated limiting mixed Hodge structure $(F_\infty, {}^rW)$. Furthermore, just as in §2, the pair $(F_\infty, {}^rW)$ induces a functorial decomposition

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{r,s} \mathfrak{g}^{r,s}_{(F_\infty, {}^rW)} \tag{3.6}$$

such that

$$\mathfrak{t}_\infty = \bigoplus_{r<0} \mathfrak{g}^{r,s}_{(F_\infty, {}^rW)}$$

is a vector space complement to the isotopy algebra $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$ in $\mathfrak{g}_{\mathbb{C}}$. As such, near $s = 0$,

$$\psi(s) = e^{\Gamma(s)}.F_\infty$$

relative to a unique $\mathfrak{t}_\infty$-valued holomorphic function $\Gamma(s)$ such that $\Gamma(0) = 0$. Accordingly, by the definition of $\psi(s)$,

$$F(z) = e^{zN} e^{\Gamma(s)}.F_\infty \tag{3.7}$$

for $\mathrm{Im}(z) \gg 0$. Moreover, just as in the pure case the period map $F(z)$ is asymptotic to the associated nilpotent orbit $\theta(z) = e^{zN}.F_\infty$ obtained by setting $\Gamma(s) = 0$ in equation (3.7):

Definition 3.8. An admissible, 1-variable nilpotent orbit is a holomorphic map $\theta : \mathbb{C} \to \check{\mathcal{M}}$ of the form

$$\theta(z) = e^{zN}.F$$

where $F \in \check{\mathcal{M}}$ and N is a nilpotent element of $\mathfrak{g}_{\mathbb{R}}$ such that

— $N(F^p) \subseteq F^{p-1}$;
— $\theta(z) \in \mathcal{M}$ for $\mathrm{Im}(z) \gg 0$;
— (Admissibility): The relative weight filtration ${}^rW(N, W)$ exists.

Theorem 3.9 (Nilpotent Orbit Theorem [**31**])**.** *Let $\mathcal{V} \to \Delta^*$ be an admissible variation of graded-polarized mixed Hodge structure with unipotent monodromy. Then,*

(1) *$\theta(z) = e^{zN}.F_\infty$ is an admissible nilpotent orbit;*
(2) *There exist non-negative constants α, β and K such that* $\mathrm{Im}(z) > \alpha \implies \theta(z) \in \mathcal{M}$ *and*

$$d_{\mathcal{M}}(F(z), \theta(z)) < K\mathrm{Im}(z)^\beta e^{-2\pi \mathrm{Im}(z)}.$$

The proof of Theorem (3.9) depends upon the following results [**34**], [**7**], [**14**] about split orbits which play a fundamental role in §4–9:

Definition 3.10. A split orbit is an admissible nilpotent orbit $(e^{zN}.\hat{F}, W)$ for which the associated limiting mixed Hodge structure $(\hat{F}, {}^rW)$ is split over $\mathbb{R}$.

In the pure case, the notion of split and SL_2-orbit coincide.Therefore, by [**34**], [**7**] we have the following classification of such orbits:

Definition 3.11. Let H be a pure Hodge structure of weight k, and $e = (1,0)$ and $f = (0,1)$ denote the standard basis of $\mathbb{C}^2$. Define $S(1)$ to be the standard representation of $sl_2(\mathbb{C})$ on $\mathbb{C}^2$ equipped with the pure Hodge structure of weight one obtained by declaring

$$\nu_+ = e + if, \qquad \nu_- = e - if \tag{3.12}$$

to be of type $(1,0)$ and $(0,1)$ respectively. Then, a representation of $sl_2(\mathbb{C})$ on H is Hodge if it induces a morphism of Hodge structures from $sl_2(\mathbb{C}) \subset S(1) \otimes S(1)^*$ to $\mathrm{End}\,(H) = H \otimes H^*$.

Theorem 3.13 ([**6**], [**7**]). *Let $\mathcal{D}$ be a classifying space of pure Hodge structure, $F_o \in \mathcal{D}$ and $\psi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{G}_{\mathbb{R}}$ be a representation of $\mathrm{SL}_2(\mathbb{R})$. Then,*

$$\theta(g.\sqrt{-1}) = \psi(g).F_o$$

is an SL_2-orbit if and only if $\rho = \psi_*$ is Hodge with respect to F_o.

Theorem 3.14 ([**34**]). *Let H be a Hodge representation and $S(k) = \mathrm{Sym}^k(S(1))$. Then, H can be decomposed into a direct sum of irreducible Hodge submodules. Furthermore, every irreducible Hodge representation is isomorphic to one of the following types*[1]

(a) *$H(d) \otimes S(m)$, $m \geq 0$;*
(b) *$E(p,q) \otimes S(n)$, $p - q > 0$, $n \geq 0$;*

where $H(d) = \mathbb{C}$ and $E(p,q) = \mathbb{C}^2$ denote the following Hodge structures, equipped with the trivial action of $sl_2(\mathbb{C})$:

— *$H(d)$ is weight $-2d$ and type $(-d,-d)$;*
— *$E(p,q)$ is weight $p+q$, ν_+ of type (p,q) and ν_- of type (q,p).*

Remark. Let H be a Hodge representation, and Q be a polarization of H which is compatible with the given action of $sl_2(\mathbb{C})$. Then, the decomposition of Theorem (3.14) can be chosen to be orthogonal with respect to Q. Furthermore, each irreducible summand is isomorphic to one of the standard tensor products (a) (b) equipped with the following polarizations:

— $H(d) : Q(1,1) = 1$;
— $S(1) : Q(e,f) = 1$, $S(k) = \mathrm{Sym}^k(S(1))$;
— $E(p,q) : Q(e,f) = i^{q-p+1}$.

In the mixed case, a split orbit $\theta(z) = e^{zN}.\hat{F}$ induces SL_2-orbits on Gr^W. Accordingly, each choice of grading Y of W defines a corresponding lift of the associated representations of sl_2 on Gr^W to a representation

$$\rho_Y : sl_2(\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}.$$

[1] By convention $S(0) = H(0)$.

In [**14**], Deligne showed how to use the limiting mixed Hodge structure of $\theta(z)$ to make a distinguished choice of grading Y such that the associated representation ρ_Y has a number of very special properties. To state Deligne's result, let

$$\mathfrak{n}_o = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathfrak{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathfrak{n}_o^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.15}$$

denote the standard generators of $sl_2(\mathbb{C})$ and ${}^rY$ denote the grading (2.14) of the relative weight filtration ${}^rW$ defined by the $I^{p,q}$'s of the limiting mixed Hodge structure of θ.

Theorem 3.16 ([**14**]). *Let $\theta(z) = e^{zN}.\hat{F}$ be a split orbit. Then, there exists a unique, functorial $\mathbb{R}$-grading Y of W such that*

(1) $[{}^rY, Y] = 0$*;*
(2) $[N - \rho_Y(n_o), \rho_Y(n_o^+)] = 0$.

Furthermore, if

$$N = N_0 + N_{-1} + N_{-2} + \cdots \tag{3.17}$$

denotes the decomposition of N with respect to the eigenvalues of ad$Y$ *and*

$$N_0 = \rho(\mathfrak{n}_o), \qquad H = \rho(\mathfrak{h}), \qquad N_0^+ = \rho(\mathfrak{n}_o^+) \tag{3.18}$$

denotes the sl_2-triple defined by the representation $\rho = \rho_Y$, then:

(a) *For $k > 0$, N_{-k} is either zero or a highest weight vector for ρ of weight $k-2$;*
(b) $H = {}^rY - Y$*;*
(c) $e^{zN_0}.\hat{F}$ *is an* SL$_2$*-orbit* (Data: $F_o = e^{iN_0}.\hat{F}$, $\psi_* = \rho$);
(d) *Y preserves $\hat{F}$, $Y_{(e^{iyN_0}.\hat{F},W)} = Y$, and $Y_{(e^{zN}.\hat{F},W)} = e^{zN}.Y$.*

In particular, as consequence of (a), $N_{-1} = 0$ and $[N_0, N_{-2}] = 0$.

Proof. See [**25**], [**31**], [**35**]. q.e.d.

Remark. More generally, in [**14**] Deligne proved the following result: Let ${}^rY$ be a grading of the relative weight filtration such that $[{}^rY, N] = -2N$. Assume ${}^rY$ preserves W. Then, there exists a system of graded representations $Gr(\rho)$ and a unique functorial $\mathbb{C}$-grading

$$Y = Y(N, {}^rY) \tag{3.19}$$

of W which satisfies conditions (1)–(2) and (a)–(b) of Theorem (3.16). Accordingly, if $(e^{zN}.F, W)$ is an admissible nilpotent orbit then application of (3.19) to N and ${}^rY = Y_{(F,{}^rW)}$ defines a corresponding grading

$$Y = Y(F, W, N) \tag{3.20}$$

of W which preserves F.

4. $\mathbf{SL_2}$-Orbit Theorem

Let X be a complex algebraic variety. Then, by [**12**, III, §8.2] the hodge numbers $h^{p,q}$ of the mixed Hodge structure attached to $H^n(X,\mathbb{C})$ satisfy the following numerical conditions:

(i) $h^{p,q}=0$ unless $0\leq p,q\leq n$;
(ii) If X is proper, then $h^{p,q}=0$ unless $p+q\leq n$;
(iii) If X is smooth, then $h^{p,q}=0$ unless $p+q\geq n$;
(iv) If $N=\dim(X)$ and $n\geq N$, then $h^{p,q}=0$ unless $n-N\leq p,q\leq N$.

Accordingly, by conditions (i) and (iv), given any complex algebraic variety X, the mixed Hodge structures attached to $H^1(X;\mathbb{Z}(1))$ and $H^{2N-1}(X;\mathbb{Z}(N))$ are of the form

$$H_{\mathbb{C}} = I^{0,0}\oplus I^{0,-1}\oplus I^{-1,0}\oplus I^{-1,-1} \tag{4.1}$$

with Gr^W_{-1} polarizable, and hence determine [**12**, III, §10.1] a corresponding pair of 1-motives, called the Picard and Albanese 1-motives of X. Likewise, given a family $f:X\to S$ of complex algebraic varieties, the local systems $Pic=R^1_{f*}(\mathbb{Z}(1))$ and $Alb=R^{2n-1}_{f*}(\mathbb{Z}(n))$ support admissible variations of 1-motives of type (II) over a Zariski open subset of S. Moreover, by conditions (ii) and (iii), Pic and Alb reduce to variations of type (I) whenever the generic fiber of f is either proper or smooth.

Returning now to the context of abstract variations, our main result can be stated as follows: [proof occupies §6 –9.]

Theorem 4.2 (SL_2-Orbit Theorem)**.** *Let $e^{zN}.F$ be an admissible nilpotent orbit of type* (I) *or* (II)*, with relative weight filtration ${}^rW={}^rW(N,W)$ and δ-splitting* [*cf. Theorem* (2.16)]

$$(F,{}^rW)=(e^{i\delta}.\hat{F},{}^rW).$$

Define $\mathfrak{h}=Lie(\mathrm{H})$ [*cf. Theorem* (2.19)]*. Then, there exists an element*

$$\zeta\in\mathfrak{h}\cap\ker(N)\cap\Lambda^{-1,-1}_{(\hat{F},{}^rW)}$$

and a distinguished real analytic function $g:(a,\infty)\to\mathrm{H}$ such that

(a) $e^{iyN}.F=g(y)e^{iyN}.\hat{F}$*;*
(b) *$g(y)$ and $g^{-1}(y)$ have convergent series expansions about ∞ of the form*

$$g(y)=e^{\zeta}(1+g_1y^{-1}+g_2y^{-2}+\cdots)$$
$$g^{-1}(y)=(1+f_1y^{-1}+f_2y^{-2}+\cdots)e^{-\zeta}$$

with $g_k,\ f_k\in\ker(\mathrm{ad}N_0)^{k+1}\cap\ker(\mathrm{ad}N_{-2})$;

(c) *δ, ζ and the coefficients g_k are related by the formula*

$$e^{i\delta} = e^{\zeta}\left(1 + \sum_{k>0} \frac{1}{k!}(-i)^k (\mathrm{ad}\, N_0)^k\, g_k\right)$$

where (N_0, H, N_0^+) denotes the sl_2 triple attached to $e^{zN}.\hat{F}$ by Theorem (3.16), and $N = N_0 + N_{-2}$ is the corresponding decomposition of N. Moreover, ζ can be expressed as a universal Lie polynomial over $\mathbb{Q}(\sqrt{-1})$ in the Hodge components $\delta^{r,s}$ of δ with respect to $(\hat{F}, {}^rW)$. Likewise, the coefficients g_k and f_k can be expressed as universal, non-commuting polynomials over $\mathbb{Q}(\sqrt{-1})$ in $\delta^{r,s}$ and $\mathrm{ad}\, N_0^+$.

By way of applications of this result, we now state three general consequences of Theorem (4.2). To this end, we note that, in conjunction with the nilpotent orbit theorem discussed in §3, one expects to be able to reduce many questions regarding the asymptotic behavior of an admissible variation $\mathcal{V} \to \Delta^*$ to the case of split orbits via Theorem (4.2). More precisely, one has:

Corollary 4.3. *Let $\mathcal{V} \to \Delta^*$ be an admissible variation of type* (I) *or* (II)*, with period map $F(z) : U \to \mathcal{M}$ and nilpotent orbit $e^{zN}.F$. Then, adopting the notation of Theorem* (4.2)*, there exists a distinguished, real–analytic function $\gamma(z)$ with values in $\mathfrak{h}$ such that, for* $\mathrm{Im}\,(z)$ *sufficiently large,*

(i) $F(z) = e^{xN} g(y) e^{iyN_{-2}} y^{-H/2} e^{\gamma(z)}.F_o$;

(ii) $|\gamma(z)| = O(\mathrm{Im}\,(z)^{\beta} e^{-2\pi \mathrm{Im}\,(z)})$ *as $y \to \infty$ and x restricted to a finite subinterval of $\mathbb{R}$, for some constant $\beta \in \mathbb{R}$*

where $F_o = e^{iN_0}.\hat{F}$.

Proof. By equation (3.7), we can write

$$F(z) = e^{zN} e^{\Gamma(s)}.F_{\infty}, \qquad s = e^{2\pi i z}$$

relative to a distinguished $\mathfrak{g}_{\mathbb{C}}$–valued holomorphic function $\Gamma(s)$ which vanishes at $s = 0$. Therefore,

$$\begin{aligned} F(z) &= e^{zN} e^{\Gamma(s)}.F = e^{xN} e^{iyN} e^{\Gamma(s)}.F \\ &= e^{xN} e^{iyN} e^{\Gamma(s)} e^{-iyN} e^{iyN}.F = e^{xN} e^{\Gamma_1(z)} e^{iyN}.F \end{aligned}$$

where $\Gamma_1(z) = e^{iyN} e^{\Gamma(s)} e^{-iyN}$. By Theorem (4.2),

$$e^{iyN}.F = g(y) e^{iyN}.\hat{F} = g(y) e^{iyN_{-2}} y^{-H/2}.F_o$$

since $y^{-H/2}.F_o = e^{iyN_0}.\hat{F}$. Consequently, if $h(y) = g(y) e^{iyN_{-2}} y^{-H/2}$ then

$$\begin{aligned} (4.4) \qquad F(z) &= e^{xN} e^{\Gamma_1(z)} e^{iyN}.F = e^{xN} e^{\Gamma_1(z)} h(y).F_o \\ &= e^{xN} h(y) h^{-1}(y) e^{\Gamma_1(z)} h(y).F_o = e^{xN} h(y) e^{\Gamma_2(z)}.F_o \end{aligned}$$

where $\Gamma_2(z) = h^{-1}(y)e^{\Gamma_1(z)}h(y)$. Also,

$$|\Gamma_2(z)| = O(\mathrm{Im}\,(z)^{\beta}e^{-2\pi\mathrm{Im}\,(z)}) \tag{4.5}$$

since $\Gamma(s)$ is a holomorphic function such that $\Gamma(0) = 0$, e^{iyN} and $e^{iyN_{-2}}$ are polynomial in y, $g(y) = O(1)$ and $y^{H/2}$ acts as multiplication by an integral power of $y^{1/2}$ on the eigenspaces of H.

To complete the proof, we now recall that by equation (2.11), we may write

$$e^{\Gamma_2(z)} = g_{\mathbb{R}}(z)e^{\lambda(z)}f(z) \tag{4.6}$$

where each factor is real–analytic, and

$$g_R(z) \in \mathrm{G}_{\mathbb{R}}, \qquad \lambda(z) \in \Lambda^{-1,-1}_{(F_o,W)}, \qquad f(z) \in \mathrm{G}^{F_o}_{\mathbb{C}}.$$

Accordingly, for $\mathrm{Im}\,(z)$ sufficiently large, there exists a unique $\mathfrak{h}$-valued function $\gamma(z)$ such that

$$e^{\gamma(z)} = g_{\mathbb{R}}(z)e^{\lambda(z)}.$$

By equation (4.4), $\gamma(z)$ satisfies (i) since $f(z)$ takes values in $\mathrm{G}^{F_o}_{\mathbb{C}}$. Likewise, $\gamma(z)$ satisfies condition (ii) by virtue of equation (4.5) and the fact that the decomposition (4.6) is real–analytic. q.e.d.

Remark. For variations of type (I), $N = N_0$. For variations of type (II), $N = N_0 + N_{-2}$ and $\ker(N) = \ker(N_0) \cap \ker(N_{-2})$.

Our first application of Theorem (4.2) is the following analog of the 1-variable norm estimates [**34**, Theorem (6.6)]:

Theorem 4.7. *Let $\mathcal{V} \to \Delta^*$ be an admissible variation of type* (I) *or* (II) *with weight filtration $\mathcal{W}$ and relative weight filtration ${}^r\mathcal{W}$. Then, adopting the notation of Theorem* (4.2)*,*

- (a) *The norm $\|\sigma(s)\|$ of a flat, global section of $\mathcal{V}$ remains bounded as $s \to 0$;*
- (b) *Over any angular sector A of Δ^*, a flat section σ of ${}^r\mathcal{W}_k$ satisfies the estimate*
$$\|\sigma(s)\| = O((-\log|s|)^{\frac{k}{2}})$$
provided $\mathcal{W}_\ell = 0$ for $\ell < 0$.

More generally, if $F(z) : U \to \mathcal{M}$ denotes the period map of $\mathcal{V}$ then, for $x = Re(z)$ restricted to a finite subinterval of $\mathbb{R}$,

$$v \in E_k(H) \cap \ker(N_{-2}) \implies \|v\|_{F(z)} = O(y^{\frac{k}{2}}) \tag{4.8}$$

as $y \to \infty$.

Proof. The estimate (4.8) implies items (a) and (b). Indeed, by the previous remark, after pulling back $\mathcal{V}$ to the upper half-plane, a flat global section of $\mathcal{V}$ is represented by a constant vector

$$v \in \ker(N) = \ker(N_0) \cap \ker(N_{-2}).$$

Therefore, upon decomposing v into its isotypical components with respect to the representation of sl_2 defined by (N_0, H, N_0^+), it then follows that [since N_{-2} commutes with (N_0, H, N_0^+)] each such component is also contained in $\ker(N_0) \cap \ker(N_{-2})$, and hence belongs to $E_k(H) \cap \ker(N_{-2})$ for some index $k \leq 0$. Consequently, by (4.8), $\|v\|_{F(z)}$ is bounded.

Likewise, over any angular sector, a flat section of ${}^r\mathcal{W}_k$ is represented by a constant vector $v \in {}^rW_k$. Therefore, recalling (3.16b) that

$$H = {}^rY - Y$$

where ${}^rY$ is a grading of ${}^rW$ and Y is a grading of W which commutes with ${}^rY$, it then follows that

$$W_\ell = 0 \text{ for } \ell < 0 \implies {}^rW_k \subseteq \bigoplus_{j \leq k} E_j(H).$$

Invoking (4.8), one then obtains (b).

To establish (4.8), suppose that $\mathcal{V}$ is a split orbit, i.e., $F(z) = e^{zN}.\hat{F}$. Then, given a vector $v \in E_k(H) \cap \ker(N_{-2})$,

$$\|v\|_{e^{zN}.\hat{F}} = \|v\|_{e^{xN}e^{iyN}.\hat{F}} = \|e^{-xN}v\|_{e^{iyN}.\hat{F}} = \|v + v'(x)\|_{e^{iyN}.\hat{F}}$$

where

$$v'(x) \in \bigoplus_{j \leq k-2} E_j(H)$$

since $N_0 : E_a(H) \to E_{a-2}(H)$, $N_{-2}(v) = 0$, and $e^{xN} = e^{xN_0}e^{xN_{-2}}$ as $[N_0, N_{-2}] = 0$. Accordingly, it suffices to show that

$$v \in E_k(H) \cap \ker(N_{-2}) \implies \|v\|_{e^{iyN}.\hat{F}} = y^{\frac{k}{2}}\|v\|_{e^{iN}.\hat{F}}.$$

However, since $e^{zN}.\hat{F}$ is a split orbit,

$$e^{iyN}.\hat{F} = e^{iyN_{-2}}y^{-H/2}e^{iN_0}.\hat{F}.$$

Therefore, as $H \in \mathfrak{g}_{\mathbb{R}}$ via (3.14), $N_{-2} \in \Lambda^{-1,-1}_{(\tilde{F},W)}$ for all $\tilde{F} \in \mathcal{M}$ by (2.18), and $v \in \ker(N_{-2})$,

$$\begin{aligned}\|v\|_{e^{iyN}.\hat{F}} &= \|v\|_{e^{iyN_{-2}}y^{-H/2}e^{iN_0}.\hat{F}} = \|e^{-iyN_{-2}}v\|_{y^{-H/2}e^{iN_0}.\hat{F}} \\ &= \|v\|_{y^{-H/2}e^{iN_0}.\hat{F}} = \|y^{H/2}v\|_{e^{iN_0}.\hat{F}} = y^{\frac{k}{2}}\|v\|_{e^{iN_0}.\hat{F}}.\end{aligned}$$

More generally, given an admissible variation $\mathcal{V} \to \Delta^*$ of type (I) or (II), one can replicate the above argument mutatis mutandis using Corollary (4.3). The only trick is to note that since $f_k \in \ker(\operatorname{ad} N_0)^{k+1}$, the term $\operatorname{Ad}(y^{H/2})(f_k\, y^{-k})$ is at worst $O(1)$ in y, and $[N_{-2}, g^{-1}(y)] = 0$ since all the terms of the series expansion of $g^{-1}(y)$ belong to $\ker(\operatorname{ad} N_{-2})$. q.e.d.

Theorem (4.7) shows that admissible variations of type (I) satisfy norm estimates which are identical to the pure case. The next result makes a similar assertion regarding the holomorphic sectional curvature:

Theorem 4.9. *Let $\mathcal{V} \to \Delta^*$ be an admissible variation of type* (I) *with non-trivial monodromy logarithm N, and period map $F(z) : U \to \mathcal{M}$. Then, the holomorphic sectional curvature of $\mathcal{M}$ along $F(z)$ is negative, and bounded away from zero for* $\mathrm{Im}\,(z)$ *sufficiently large.*

Proof. By Corollary (2.13), the holomorphic sectional curvature of $\mathcal{M}$ along $u \in T_F(\mathcal{M})$ is given by a formula of the form

$$R(u) = \frac{h_F(S_F(u,\bar{u})u,u)}{h_F^2(u,u)}$$

relative to a $\mathrm{G}_{\mathbb{R}}$-invariant tensor field S. Consequently, upon writing $F(z)$

$$F(z) = e^{xN}g(y)y^{-H/2}e^{\gamma(z)}.F_o$$

as per Corollary (4.3), one finds that [via the $\mathrm{G}_{\mathbb{R}}$-invariance of S]

$$R(F_*(d/dz)) = \frac{h_{F_o}(S_{F_o}(\theta(z),\bar{\theta}(z))\theta(z),\theta(z))}{h_{F_o}(\theta(z),\theta(z))} \tag{4.10}$$

where

$$\theta(z) = \mathrm{Ad}\,(e^{-\gamma(z)})(\beta^{-1,1}(y) + \beta^{-1,0}(y)) \tag{4.11}$$

and $\beta^{-1,1}(y)$ and $\beta^{-1,0}(y)$ denote the Hodge components of the function

$$\beta(y) = \mathrm{Ad}\,(h^{-1}(y))N, \qquad h(y) = g(y)y^{-H/2} \tag{4.12}$$

with respect to the base point $F_o = e^{iN}.\hat{F}$. In particular, as a consequence of the proof of Theorem (4.2) for nilpotent orbits of type (I) given in §8, $\beta(y)$ admits a series expansion about infinity of the form

$$\beta(y) = \sum_{n\geq 0} \beta_n y^{-1-n/2}$$

with leading order term $\beta_0 = N$. Therefore, by equations (4.10)–(4.12),

$$\lim_{\mathrm{Im}\,(z)\to\infty} R(F_*(d/dz)) = \frac{h_{F_o}(S_{F_o}(\xi,\bar{\xi})\xi,\xi)}{h_{F_o}^2(\xi,\xi)} \tag{4.13}$$

where

$$\xi = N^{-1,1} = \frac{1}{4}(iH + N_0 + N_0^+). \tag{4.14}$$

On the other hand, by Theorem (2.12),

$$h_{F_o}(S_{F_o}(\xi,\bar{\xi})\xi,\xi) = -h_{F_o}([\bar{\xi},\xi],[\bar{\xi},\xi]).$$

Thus,

$$\lim_{\mathrm{Im}\,(z)\to\infty} R(F_*(d/dz)) < 0.$$

q.e.d.

Remark. Theorem (4.9) is false for variations of type (II). In particular, if $\mathcal{V}$ is Hodge–Tate then $R(F_*(d/dz)) = 0$ for all z.

To put the next result in context, we recall that in the pure case, Schmid's nilpotent orbit theorem asserts the existence of the limiting Hodge filtration of a variation of pure polarized Hodge structure $\mathcal{V} \to \Delta^*$. In the mixed case, this existence of the limiting Hodge filtration is assumed. Less clear in the mixed case, however, is how the corresponding grading

$$\mathcal{Y}(s) = Y_{(\mathcal{F}(s),\mathcal{W})}$$

of $\mathcal{W}$ behaves as $s \to 0$.

Theorem 4.15. *Let $\mathcal{V} \to \Delta^*$ be an admissible variation of type* (I) *or* (II) *with period map $F(z) : U \to \mathcal{M}$. Then, the limiting grading*

$$Y_\infty = \lim_{\mathrm{Im}\,(z)\to\infty} e^{-zN}.Y_{(F(z),W)}$$

exists, and coincides with the grading $Y(F_\infty, W, N)$ defined by equation (3.20).

Proof. By Corollary (4.3),

$$\begin{aligned} F(z) &= e^{xN} g(y) e^{iyN_{-2}} y^{-H/2} e^{\gamma(z)}.F_o \\ &= e^{xN} g(y) e^{\gamma_1(z)} e^{iyN_{-2}} y^{-H/2}.F_o = e^{xN} g(y) e^{\gamma_1(z)} e^{iyN}.\hat{F} \end{aligned}$$

where

$$\gamma_1(z) = \mathrm{Ad}\,(e^{iyN_{-2}} y^{-H/2})\gamma(z)$$

is a $\mathfrak{h}$-valued function of order $\mathrm{Im}\,(z)^\beta e^{-2\pi \mathrm{Im}\,(z)}$, and $F_o = e^{iN_0}.\hat{F}$. Consequently, if $Y = Y(\hat{F}, W, N)$ then

(4.16)

$$e^{-zN}.Y_{(F(z),W)} = e^{-iyN} g(y) e^{\gamma_1(z)}.Y_{(e^{iyN}.\hat{F},W)} = e^{-iyN} g(y) e^{\gamma_1(z)} e^{iyN}.Y$$

since $Y_{(e^{iyN}.\hat{F},W)} = e^{iyN}.Y$ by Theorem (3.16d). Setting

$$\gamma_2(z) = \mathrm{Ad}\,(e^{-iyN})\gamma_1(z) \tag{4.17}$$

it then follows from equations (4.16) and (4.17) that

$$\lim_{\mathrm{Im}\,(z)\to\infty} e^{-zN}.Y_{(F(z),W)} = \lim_{\mathrm{Im}\,(z)\to\infty} e^{-iyN} g(y) e^{iyN} e^{\gamma_2(z)}.Y. \tag{4.18}$$

Therefore, by part (b) of Theorem (4.2),

$$\begin{aligned} e^{-iyN} g(y) e^{iyN} &= e^{\zeta} e^{-iy\,\mathrm{ad}\,N} \left(1 + \sum_{k>0} g_k y^{-k}\right) \\ &= e^{\zeta} \left(1 + \sum_{k>0} \sum_{j=0}^{k} \frac{1}{j!} (-i)^j (\mathrm{ad} N_0)^j\, g_k y^{j-k}\right) \end{aligned}$$

since $N = N_0 + N_{-2}$, $[N_0, N_{-2}] = 0$, $g_k \in \ker(\mathrm{ad}\, N_0)^{k+1} \cap \ker(\mathrm{ad}\, N_{-2})$ and $\zeta \in \ker(\mathrm{ad}\, N_0) \cap \ker(\mathrm{ad}\, N_{-2})$. Consequently, by part (c) of Theorem (4.2),

$$\lim_{y\to\infty} e^{-iyN} g(y) e^{iyN} = e^{\zeta}\left(1 + \sum_{k>0} \frac{1}{k!}(-i)^k (\mathrm{ad} N_0)^k\, g_k\right) = e^{i\delta}. \tag{4.19}$$

On the other hand, by equation (4.17), $\gamma_2(z)$ is also of order $\mathrm{Im}\,(z)^{\beta} e^{-2\pi \mathrm{Im}\,(z)}$ for some constant β. Therefore,

$$\lim_{\mathrm{Im}\,(z)\to\infty} e^{\gamma_2(z)} = 1. \tag{4.20}$$

Inserting equations (4.19) and (4.20) into equation (4.18), it then follows that

$$Y_\infty = \lim_{\mathrm{Im}\,(z)\to\infty} e^{-zN}.Y_{(F(z),W)} = e^{i\delta}.Y = Y(F_\infty, W, N)$$

since $Y(F_\infty, W, N) = e^{i\delta}.Y(\hat{F}_\infty, W, N) = e^{i\delta}.Y$ by the functoriality of Y (cf. [**31**]). q.e.d.

Remark. By [**25**], Theorem (4.15) is also true for unipotent variations (e.g., the variations attached to fundamental group of a smooth variety [**21**]) and variations for which the limiting mixed Hodge structure is split over $\mathbb{R}$ in some suitable coordinate system (e.g., the A–model variation considered in mirror symmetry [**30**]).

5. Arakelov Geometry

Let M be a graded-polarized mixed Hodge structure. Then, motivated by the construction of [**19**] described below, we define the height of M to be

$$h(M) = 2\pi\|\delta\| \tag{5.1}$$

where δ denotes the splitting of M defined in §2, and $\|*\|$ denotes the mixed Hodge norm of M.

To relate the height functional (5.1) to the standard archimedean height pairing defined by [**1**], [**2**], [**16**], let X be a non-singular complex projective variety of dimension n, and Z and W be a pair of algebraic cycles in X of dimensions $d = \dim(Z)$ and $e = \dim(W)$ such that

(i) Z and W are homologous to zero in X;
(ii) $d + e = n - 1$;
(iii) $|Z| \cap |W| = \emptyset$.

Then, by §3 of [**19**], the mixed Hodge structure on $H_{2d+1}(X - |W|, |Z|; \mathbb{Z}(-d))$ carries a canonical subquotient $B = B_{Z,W}$ with graded pieces

$$Gr^W_0 \cong \mathbb{Z}(0), \quad Gr^W_{-1} \cong H_{2d+1}(X; \mathbb{Z}(-d)), \quad Gr^W_{-2} \cong \mathbb{Z}(1) \tag{5.2}$$

such that

$$h(B_{Z,W}) = |\langle Z, W\rangle| \tag{5.3}$$

where $\langle Z, W\rangle$ denotes the archimedean height of the pair (Z, W).

More precisely, via the cycles Z and W, one obtains canonical positive generators $1 \in Gr^W_0(B) \cong \mathbb{Z}(0)$ and $1^\vee \in Gr^W_{-2}(B) \cong \mathbb{Z}(1)$. Moreover, as a consequence of Proposition (3.2.13) in [**19**],

$$\delta(1) = \frac{1}{2\pi}\langle Z, W\rangle 1^\vee \tag{5.4}$$

from which one then obtains (5.3) via the definition of the mixed Hodge metric.

Likewise, given a smooth, proper morphism $\pi : X \to S$ of relative dimension n, and a pair of flat, algebraic cycles Z and W in X of relative dimensions d and e such that, for generic $s \in S$, X_s is smooth and the triple (X_s, Z_s, W_s) satisfies conditions (i)–(iii) above, one obtains a corresponding height function

$$h(s) = \langle Z_s, W_s\rangle \tag{5.5}$$

over a Zariski dense open subset S' of S.

Let D be a normal crossing divisor contained in the boundary of a smooth partial compactification $\overline{S'}$ of S'. In [**19**], Hain analyzed the asymptotic behavior of (5.5) near D under the assumption that the associated variation of mixed Hodge structure

$$\mathcal{V} \to S', \qquad \mathcal{V}_s = B_{Z_s, W_s} \tag{5.6}$$

induced constant variation of pure Hodge structure on $Gr^{\mathcal{W}}$. In [**27**], Lear computed the asymptotic behavior of (5.5) under the assumption that S is a curve using the theory of normal functions.

In this section, we consider the asymptotic behavior of (5.5) near a normal crossing divisor D about which $\mathcal{V}$ degenerates with unipotent monodromy by applying Theorem (4.2) to the 1-parameter degenerations $f^*(\mathcal{V})$ obtained by pulling back $\mathcal{V}$ along a holomorphic map f from the unit disk Δ into $\overline{S'}$.

To this end, let us assume for the moment that $\dim(S) = 1$ and p is a point about which $\mathcal{V}$ degenerates with unipotent monodromy. By (5.4), the corresponding height function (5.5) is then given by the formula

$$\delta(1) = \frac{1}{2\pi}h(s)1^\vee \tag{5.7}$$

where δ denotes the section of $\mathcal{V} \otimes \mathcal{V}^*$ defined by the pointwise application of the splitting (2.16) to the fibers of $\mathcal{V}$, and 1 and $1^\vee$ denote the generators of $Gr^W_0(\mathcal{V}) \cong \mathbb{Z}(0) \otimes \mathcal{O}_{S'}$ and $Gr^W_{-2}(\mathcal{V}) \cong \mathbb{Z}(1) \otimes \mathcal{O}_{S'}$ respectively.

As usual, for the purpose of calculating the asymptotic behavior of (5.5) near p, we replace $\mathcal{V}$ by the corresponding period map $F : U \to \mathcal{M}$

obtained restricting $\mathcal{V}$ to a deleted neighborhood Δ^* of p. Using the nilpotent orbit theorem discussed in §3, we can then replace $F(z)$ by the corresponding nilpotent orbit

$$\theta(z) = e^{zN}.F_\infty \tag{5.8}$$

since we are only interested in calculating the leading order terms of (5.5). Invoking Theorem (4.2), we can then calculate the asymptotic behavior of $h(s)$ modulo terms that remain bounded as $s \to 0$ (recall $s = e^{2\pi i z}$) by replacing $\theta(z)$ by the corresponding split orbit $\hat{\theta}(z) = e^{zN}.\hat{F}_\infty$.

Indeed, by Corollary (4.3), for any admissible period map $F(z)$ of type (II) with unipotent monodromy, the corresponding gradings $Y_{(F(z),W)}$ and $Y_{(\hat{\theta}(z),W)}$ are related by an equation of the form

$$Y_{(F(z),W)} = Y_{(\hat{\theta}(z),W)} + \epsilon(z) \tag{5.9}$$

where $\epsilon(z)$ is a real analytic function which remains bounded as $y = \mathrm{Im}(z) \to \infty$ and $x = \mathrm{Re}(z)$ restricted to any finite subinterval of $\mathbb{R}$. Moreover, by (3.16d),

$$Y_{(\hat{\theta}(z),W)} = e^{zN}.Y = Y + 2zN_{-2} \tag{5.10}$$

where Y is a real grading of W, and hence

$$\delta_{(\hat{\theta}(z),W)} = yN_{-2} \tag{5.11}$$

since

$$Y_{(F,W)} - \overline{Y}_{(F,W)} = 4i\delta_{(F,W)} \tag{5.12}$$

for any mixed Hodge structure of type (II). Therefore, by equation (5.9)–(5.12),

$$\delta_{(F(z),W)} = yN_{-2} + \frac{1}{2}\mathrm{Im}\,(\epsilon(z)). \tag{5.13}$$

Inserting equation (5.13) into (5.7), it then follows that, near $s = 0$,

$$h(s) = -\mu \log|s| + \eta(s) \tag{5.14}$$

where $N_{-2}(1) = \mu\, 1^\vee$ and $\eta(s)$ is a real analytic function which remains bounded as $s \to 0$.

Remark. More generally, it follows from (5.9)–(5.13) that if $h_{\mathcal{V}}(s)$ denotes the height function (5.1) attached to an admissible variation $\mathcal{V} \to \Delta^*$ of type (II) with unipotent monodromy, then

$$h_{\mathcal{V}}(s) = -\mu \log|s| + \eta(s)$$

where $\mu = \|N_{-2}\|_{F_o}$ denotes the norm of N_{-2} with respect to the base point $F_o \in \mathcal{M}$ defined in Corollary (4.3), and $\eta(s)$ is once again a real-valued analytic function which remains bounded as $s \to 0$.

Now, according to the above recipe, in order to calculate the asymptotic behavior of the height paring $\langle Z_s, W_s\rangle$, it would seem that one must compute N, W, and F_∞, along with the corresponding splittings and gradings. This is not necessary. Indeed, these auxiliary object appear in equation (5.14) only via the decomposition

$$N = N_0 + N_{-2} \tag{5.15}$$

which can computed directly from the pair (N, W) as follows: Let Y be the grading appearing in (5.10), relative to which N decomposes as (5.15) according to the eigenvalues of $ad\, Y$, and Y' be any other grading of W. Then, since $Lie_{-1}(W)$ acts transitively on the set of all gradings of W,

$$Y' = Y + \alpha_{-1} + \alpha_{-2} \tag{5.16}$$

where α_j belongs to the j eigenspace $E_j(\mathrm{ad}\, Y)$ of $\mathrm{ad}\, Y$. Furthermore, because N_0 acts trivially on $E_0(Y)$ and $E_{-2}(Y)$,

$$[N_0, \alpha_{-2}] = 0. \tag{5.17}$$

Therefore,

$$[Y', N] = [Y + \alpha_{-1} + \alpha_{-2}, N_0 + N_{-2}] = -2N_{-2} + [\alpha_{-1}, N_0] \tag{5.18}$$

by virtue of equation (5.17) and the short length of W, which forces both $[\alpha_{-1}, N_{-2}]$ and $[\alpha_{-2}, N_{-2}] = 0$. Accordingly, if Y' is any grading of W such that $[Y', N]$ lowers W by 2 (i.e., $[\alpha_{-1}, N_0] = 0$) then $N_{-2} = -\frac{1}{2}[Y', N]$. Thus, in summary, we obtain the following result:

Theorem 5.19. *Let $h(s)$ denote the height function* (5.5) *attached to flat family of algebraic cycles Z_s, $W_s \subseteq X_s$ over a smooth curve S. Let p be a point at which the corresponding variation $\mathcal{V}$ defined by equation* (5.6) *degenerates with unipotent monodromy. Let N denote the local monodromy of $\mathcal{V}$ about p, and Y' be any grading of the weight filtration W of $\mathcal{V}$ such that $[Y', N]$ lowers W by 2. Define $N_{-2} = -\frac{1}{2}[Y', N]$. Then, near $s = 0$,*

$$h(s) = -\mu \log |s| + \eta(s)$$

where $N_{-2}(1) = \mu\, 1^\vee$ and $\eta(s)$ is a real analytic function which remains bounded as $s \to 0$.

Proof. It remains only to justify (5.9), from which Theorem (5.19) then follows from equations (5.10)–(5.17) and accompanying arguments. To verify (5.9), recall that by Corollary (4.3), near the given puncture, the period map $F(z)$ of the variation (5.6) assumes the form

$$F(z) = e^{xN} g(y) e^{iyN_{-2}} y^{-H/2} e^{\gamma(z)}.F_o \tag{5.20}$$

where $H \in \mathfrak{g}_{\mathbb{R}}$ commutes with the grading $Y = Y_{(F_o, W)}$ appearing in equation (5.10), and $\gamma(z)$ is a real analytic, $\mathfrak{h}$-valued function which is of order $y^{\beta}e^{-2\pi y}$ as $y \to \infty$ and x is restricted to a finite subinterval of $\mathbb{R}$. Therefore,

$$(5.21) \qquad \begin{aligned} Y_{(F(z),W)} &= e^{xN}g(y)e^{iyN_{-2}}y^{-H/2}e^{\gamma(z)}.Y_{(F_o,W)} \\ &= e^{xN}g(y)e^{iyN_{-2}}(y^{-H/2}e^{\gamma(z)}y^{H/2})y^{-H/2}.Y_{(F_o,W)} \\ &= e^{xN}g(y)e^{iyN_{-2}}e^{\gamma_1(z)}.Y_{(F_o,W)} \end{aligned}$$

where $\gamma_1(z) = \mathrm{Ad}\,(y^{-H/2})\gamma(z)$ is a real analytic function of order $y^{\beta'}e^{-2\pi y}$ for some constant $\beta' \in \mathbb{R}$. Accordingly,

$$(5.22) \qquad e^{\gamma_1(z)}.Y_{(F_o,W)} = e^{\gamma_1(z)}.Y = Y + \gamma_2(z)$$

where $\gamma_2(z)$ is again of order $y^{\beta'}e^{-2\pi y}$. Inserting (5.22) into (5.21), it then follows that

(5.23)

$$\begin{aligned} Y_{(F(z),W)} &= e^{xN}g(y)e^{iyN_{-2}}.(Y + \gamma_2(z)) \\ &= e^{xN}g(y).(Y + 2iyN_{-2} + \gamma_3(z)) \\ &= (e^{xN}g(y)e^{-xN})e^{xN}.(Y + 2iyN_{-2} + \gamma_3(z)) \\ &= (e^{xN}g(y)e^{-xN}).(Y + 2zN_{-2} + \gamma_4(z)) \\ &= (e^{xN}g(y)e^{-xN}).(Y + \gamma_4(z)) + (e^{xN}g(y)e^{-xN}).(2zN_{-2}) \end{aligned}$$

where, for some constant $\beta'' \in \mathbb{R}$, $\gamma_3(z)$ and $\gamma_4(z)$ are real analytic functions of order $y^{\beta''}e^{-2\pi y}$ as $y \to \infty$ with x restricted to a finite subinterval of $\mathbb{R}$. Moreover, by Theorem (4.2), the function $g(y)$ admits a convergent series expansion near $y = \infty$ of the form

$$(5.24) \qquad g(y) = e^{\zeta}(1 + g_1 y^{-1} + g_2 y^{-2} + \cdots)$$

where $\zeta, g_1, g_2, \dots \in \ker(\mathrm{ad}\, N_{-2})$, and hence

$$(5.25) \qquad g(y).N_{-2} = \mathrm{Ad}\,(g(y))N_{-2} = N_{-2}.$$

Therefore,

$$(5.26) \qquad (e^{xN}g(y)e^{-xN}).(2zN_{-2}) = 2zN_{-2}$$

since $N = N_0 + N_{-2}$ and $[N_0, N_{-2}] = 0$. Likewise, because of the series expansion (5.24) and the fact that $\zeta \in \ker(N_0) \cap \ker(N_{-2})$ by Theorem (4.2),

$$(5.27) \qquad \lim_{y \to \infty} e^{xN}g(y)e^{-xN} = e^{\zeta}$$

independent of x. Consequently,

$$(5.28) \qquad (e^{xN}g(y)e^{-xN}).(Y + \gamma_4(z)) = Y + \epsilon(z)$$

where $\epsilon(z)$ is a real analytic function which remains bounded as $y \to \infty$ and is x restricted to a finite subinterval of $\mathbb{R}$. Inserting (5.26) and

(5.28) into (5.23), it then follows that

$$Y_{(F(z),W)} = Y + 2zN_{-2} + \epsilon(z) = Y_{(\hat{\theta}(z),W)} + \epsilon(z)$$

as required. q.e.d.

Lemma 5.29. *Under the hypothesis of Theorem* (5.19)*,* $\mu \in \mathbb{Q}$*.*

Proof. It is sufficient to show that

$$V_{\mathbb{Q}} = \mathbb{Q}f_0 \oplus U_{\mathbb{Q}} \oplus \mathbb{Q}f_{-2} \tag{5.30}$$

where

(a) f_0 projects to $1 \in Gr_0^W$;
(b) $U_{\mathbb{Q}}$ is an N-invariant subspace of $W_{-1}(V_{\mathbb{Q}})$;
(c) f_{-2} projects to $1^{\vee} \in Gr_{-2}^W$.

Indeed, suppose that such a decomposition exists, and let Y' be any grading of W such that $[Y', N]$ lowers W by 2. Then, shows that

$$N(e_0) = \mu\, f_{-2} \tag{5.31}$$

where e_0 is the element of $E_0(Y')$ which projects to $1 \in Gr_0^W$. On the other hand, since e_0 and f_0 have the same image in Gr_0^W, we can write

$$e_0 = f_0 + u_{\mathbb{C}} + bf_{-2} \tag{5.32}$$

where $u_{\mathbb{C}} \in U_{\mathbb{Q}} \otimes \mathbb{C}$ and $b \in \mathbb{C}$. Inserting equation (5.32) into (5.31) and recalling that f_{-2} generates $W_{-2} \subseteq \ker(N)$, it then follows that

$$N(f_0) = -N(u_{\mathbb{C}}) + \mu\, f_{-2}.$$

Since both N and f_0 are rational, it then follows from (5.30) and the N-invariance of $U_{\mathbb{Q}}$ that both $-N(u_{\mathbb{C}})$ and $\mu\, f_{-2}$ belong to $V_{\mathbb{Q}}$. In particular, $\mu \in \mathbb{Q}$ since $f_{-2} \in V_{\mathbb{Q}}$.

To prove the existence of a decomposition (5.30) of $V_{\mathbb{Q}}$ with properties (a)–(c), observe that it is sufficient to show that

$$f_{-2} \notin N(W_{-1}(V_{\mathbb{Q}})). \tag{5.33}$$

Indeed, since N is nilpotent, its restriction to any N-invariant subspace of $V_{\mathbb{Q}}$ can be put in Jordan normal form. In particular, there exists a Jordan basis of cycles $\gamma_1 \cup \cdots \cup \gamma_r$ for the action of N on $W_{-1}(V_{\mathbb{Q}})$. Since $N(f_{-2}) = 0$, equation (5.33) implies that $\gamma_j = \{c\, f_{-2}\}$ for index some j and some rational coefficient c. The remaining cycles $\gamma_{k\neq j}$ generate the desired subspace $U_{\mathbb{Q}}$ of $W_{-1}(V_{\mathbb{Q}})$.

To complete the proof, we now verify (5.33) using the admissibility of $\mathcal{V}$. More precisely, by the previous remarks, we know that there exists a real grading Y of W such that $N = N_0 + N_{-2}$ relative to the eigenvalues of $\operatorname{ad} Y$. Since W is of type (II), it then follows that

$$W_{-1}(V_{\mathbb{C}}) = E_{-1}(Y) \oplus W_{-2}(V_{\mathbb{C}})$$

is an N-invariant splitting of $W_{-1}(V_{\mathbb{C}})$. Therefore, since $f_{-2} \in \ker(N)$ generates $W_{-2}(V_{\mathbb{C}})$ and $E_{-1}(Y)$ is N-invariant, $f_{-2} \notin N(W_{-1}(V_{\mathbb{C}}))$.
q.e.d.

Returning now to the general setting (5.5), let D be a normal crossing divisor about which $\mathcal{V}$ degenerates with unipotent monodromy. Let $(s_1, \dots, s_m)$ be local coordinates on $\overline{S'}$ relative to which D assumes the form $s_1 \cdots s_m = 0$ and $f : \Delta \to \overline{S'}$ be a holomorphic map of the form

$$f(t) = (t^{a_1} f_1(t), \dots, t^{a_m} f_m(t)) \tag{5.34}$$

where $a_1, \dots, a_m$ are nonnegative integers which are not all 0 and $f_1, \dots, f_m$ are nonvanishing holomorphic functions on Δ. Let N_j denote the monodromy logarithm of $\mathcal{V}$ about $s_j = 0$ and N denote the monodromy of $f^*(\mathcal{V})$ about $t = 0$. Then,

$$N = \sum_{j=1}^{m} a_j N_j \tag{5.35}$$

and hence

$$f^*(h)(t) = -\mu_{a_1,\dots,a_m} \log|t| + \eta(t)$$

where $\eta(t)$ is a real analytic function which remains bounded as $t \to 0$ and

$$\mu_{a_1,\dots,a_m} 1^{\vee} = -\frac{1}{2}[Y', N](1) \tag{5.36}$$

for any grading Y' of W such that $[Y', N]$ lowers W by 2.

Theorem 5.37. *Let $\mu = \mu_{a_1,\dots,a_m}$. Then, μ belongs to $\mathbb{Q}(a_1, \dots, a_m)$ and is homogeneous of degree 1 in $a_1, \dots, a_m$.*

Proof. That μ is homogeneous of degree 1 in $a_1, \dots, a_m$ follows immediately from equations (5.35) and (5.36). To see that μ is rational in $a_1, \dots a_m$, recall that the asymptotic behavior of the height depends only on the asymptotic behavior of the δ-splitting of the approximating split orbit $F(z) = e^{zN}.\hat{F}$. Therefore, since $F(z)$ depends polynomially on z and N, and the δ-splitting is determined by taking sums and intersections of $F(z)$, $\bar{F}(z)$ and W, it then follows that μ depends rationally on $a_1, \dots, a_m$. Finally, since by Lemma (5.29) μ assumes rational values at every m-tuple of positive integers, μ is defined over $\mathbb{Q}$. q.e.d.

In light of Theorem (5.37) the simplest possible asymptotic behavior that h can exhibit as s approaches D along various curves of the form (5.34) is for μ to be a linear function of $a_1, \dots, a_r$. In this case, we shall say that $h(s)$ has no jumps along D.

By Theorem (5.19), a sufficient condition for $h(s)$ to have no jumps along D is the existence of a grading Y of W such that $[Y, N_j]$ lowers

W by 2 for all j. Indeed, in this case

$$\mu_{a_1,\dots,a_r}\, 1^\vee = -\frac{1}{2}\left[Y, \sum_j a_j N_j\right](1).$$

The next result gives a sufficient condition for the existence of such a grading Y which depends only on the monodromy of the local system

$$Gr^{\mathcal{W}}_{-1}(\mathcal{V}_{\mathbb{Z}}) = [R^{2d+1}_{\pi *}(\mathbb{Z}(d))]^*$$

defined by the morphism $\pi : X \to S$, and not the particular choice of flat cycles Z and $W \subseteq X$.

Theorem 5.38. *Let $\tilde{T}$ denote the monodromy of $R^{2d+1}_{\pi *}(\mathbb{Z})$ around a holomorphic disk $f(\Delta^*)$ of type (5.34) with all coefficients $a_j > 0$. Suppose that $\tilde{T}$ has no Jordan blocks of rank 2. Then, there exists a grading Y of W such that $[Y, N_j]$ lowers W by 2 for all j, and hence the corresponding height function $h(s)$ has no jumps along D.*

Proof. The stated condition on $\tilde{T}$ is equivalent to the following condition on the monodromy cone $\mathcal{C} = \{a_1 N_1 + \cdots + a_m N_m \mid a_1, \dots, a_m > 0\}$: Let $N \in \mathcal{C}$ and $\tilde{N}$ denote the induced action of N on $Gr^{\mathcal{W}}_{-1}$. Then,

$$\ker(\tilde{N}) \cap \mathrm{Im}\,(\tilde{N}^2) = \ker(\tilde{N}) \cap \mathrm{Im}\,(\tilde{N}).$$

Next, recall that since $\mathcal{V}$ is of geometric origin, the data $(F_\infty, W, N_1, \dots, N_m)$ define an infinitesimal mixed Hodge module in the sense of Kashiwara [**26**], and hence every element N of $\mathcal{C}$ defines the same relative weight filtration

$${}^rW = {}^rW(N, W).$$

Furthermore, $(F_\infty, {}^rW)$ is a mixed Hodge structure with respect to which each N_j is a $(-1,-1)$-morphism. Let Y be the grading of W defined by application of (3.19) to N and ${}^rY = Y_{(F_\infty, {}^rW)}$. Then, relative to $\mathrm{ad}\, Y$, $N = N_0 + N_{-2}$. Likewise, due to the short length of W, each N_j decomposes as

$$N_j = (N_j)_0 + (N_j)_{-1} + (N_j)_{-2}$$

relative to $\mathrm{ad}\, Y$. Accordingly, the condition $[Y, N_j](W_0) \subseteq W_{-2}$ is equivalent to the assertion that $(N_j)_{-1} = 0$ for each j.

To complete the proof let $Y = Y(F, W, N)$ denote the grading (3.20) of W attached to the nilpotent orbit $e^{zN}.F_\infty$ and ρ be the corresponding representation of $sl_2(\mathbb{C})$ defined by the sl_2-pair N_0 and $H = {}^rY - Y$. Let $V(k)$ denote the isotypical component of ρ generated by the linear span of all irreducible submodules of highest weight k. Then, by the above remarks, it is sufficient to show that

(a) $\ker(\tilde{N}) \cap \mathrm{Im}\,(\tilde{N}^2) = \ker(\tilde{N}) \cap \mathrm{Im}\,(\tilde{N}) \implies V(1) = 0$;
(b) $V(1) = 0 \implies (N_j)_{-1} = 0$.

To verify (a), observe that since the action ρ preserves the eigenspaces of Y we have

$$V(k) = \bigoplus_j V(k) \cap E_j(Y).$$

Furthermore, since N acts trivially on $Gr^W_0 \cong E_0(Y)$ and $Gr^W_{-2} \cong E_{-2}(Y)$, the equality

$$\ker(N_0) \cap \mathrm{Im}\,(N_0^2) = \ker(N_0) \cap \mathrm{Im}\,(N_0) \tag{5.39}$$

holds on $E_0(Y)$ and $E_{-2}(Y)$. On $E_{-1}(Y)$ condition (5.39) is equivalent to the stated condition on $\tilde{N}$. Therefore, it is sufficient to prove that condition (5.39) implies that $V(1) = 0$, or equivalently, if $V(1) \neq 0$ then (5.39) fails. Accordingly, suppose that U is an irreducible representation of highest weight 1. Then $N_0^2(U) = 0$ whereas $N_0(U)$ is non-zero and contained in $\ker(N_0)$, which violates (5.39).

To establish (b), observe that $(N_j)_{-1} \in \ker(N_0) \cap E_{-1}(\mathrm{ad}\, H)$ since $[N, N_j] = 0$, $H = {}^rY - Y$ and $[{}^rY, N_j] = -2N_j$. Therefore, if e_0 is a generator of $E_0(Y) \subset V(0)$ then

$$u = (N_j)_{-1}(e_0) \in \ker(N_0) \tag{5.40}$$

because ρ acts trivially on $E_0(Y)$, and hence

$$N_0(u) = N_0(N_j)_{-1}(e_0) = [N_0, (N_j)_{-1}]e_0 = 0.$$

Likewise,

$$u \in E_{-1}(H) \tag{5.41}$$

since

$$H(u) = H(N_j)_{-1}(e_0) = [H, (N_j)_{-1}]e_0 = -(N_j)_{-1}(e_0) = -u.$$

Combining (5.40) and (5.41), it then follows that $u \in V(1)$.

Similarly, since ρ acts trivially on $E_{-2}(Y) \subset V(0)$, if $v \in E_\ell(H) \cap E_{-1}(Y)$ and $(N_j)_{-1}(v)$ is non-zero then $\ell = 1$ since

$$-(N_j)_{-1}(v) = [H, (N_j)_{-1}]v = -(N_j)_{-1}H(v) = -\ell(N_j)_{-1}(v).$$

Furthermore, $(N_j)_{-1}(v) \neq 0$ implies that $v \notin \mathrm{Im}(N_0)$ since

$$v = N_0(v') \implies (N_j)_{-1}(v) = (N_j)_{-1}N_0(v') = [N_0, (N_j)_{-1}]v' = 0$$

and hence $v \in V(1)$. Thus, $V(1) = 0 \implies (N_j)_{-1} = 0$. q.e.d.

Corollary 5.42. *If the local monodromy of $Gr^{\mathcal{W}}_{-1}(\mathcal{V}_{\mathbb{Z}})$ about D is trivial then the corresponding height function* (5.5) *has no jumps along D.*

A special case of (5.42), originally considered by Richard Hain [**19**], is when X_s remains smooth and only the cycles Z_s and W_s degenerate. More recently [**20**], Hain and Reed have used the height of the Ceresa cycle $C - C_-$ to study the Arakelov geometry of the moduli space $\mathcal{M}_g$ of smooth complex projective curves of genus $g > 2$. Briefly, given a curve $C \in \mathcal{M}_g$ and a pair of positive integers a and b such $a+b = g-1$, define

$$h(C) = \left\langle C^{(a)} - C_-^{(a)}, C^{(b)} - C_-^{(b)} \right\rangle$$

to be the height pairing attached to the Ceresa cycles in $Jac(C)$ determined by the a'th and b'th symmetric power of C. The height function $h(C)$ can be used to construct a metric on the $(8g+4)$'th power of the determinant line bundle $\mathcal{L}$ over $\mathcal{M}_g$. Comparison of this metric to the standard Hodge metric on $\mathcal{L}^{\otimes(8g+4)}$ then defines [modulo an additive constant] a function $\beta_g : \mathcal{M}_g \to \mathbb{R}$ which is an analog of Faltings delta function δ_g. To prove that $d\beta_g$ and $d\delta_g$ are linearly independent, Hain and Reed compute the asymptotic behavior of β_g and δ_g along the boundary divisor Δ of $\mathcal{M}_g$ in $\overline{\mathcal{M}}_g$. Recall that Δ is a union of components Δ_h such that, for $h > 0$, the generic point of Δ_h corresponds to a reducible curve C_0 with 1-node with components of genera h and $g-h$. In particular, for $g > 1$ it is well known that the geometric monodromy of a family of curves C_t degenerating to C_0 at $t = 0$ acts trivially on $H^1(C_t)$. Applying Corollary (5.42) it then follows that the metric considered by Hain and Reed extends continuously to $\widetilde{\mathcal{M}}_g = \overline{\mathcal{M}}_g - \Delta_0$, where Δ_0 is the divisor for which the generic point represents an irreducible curve with a normalization of genus $g-1$. Theorem (5.19) also implies that this metric extends continuously to any holomorphic arc meeting Δ_0 transversely (cf. Theorem 3 in [**20**]).

Remark. More generally, Theorem (5.19) implies the unpublished result from Lear's thesis [**27**] stated above, which appears as a crucial lemma in §8 of [**20**]. In [**3**], the author and P. Brosnan use Theorem (5.19) to compute the asymptotics of the height of the Ceresa cycle along the divisor $\Delta_0 \subset \overline{\mathcal{M}}_g$.

A further source of families of varieties for which the height does not jump are furnished by smooth complete intersections in $\mathbb{P}^n$ of even dimension: By the Lefschetz theorems, all of the odd cohomology groups of such a variety are zero. A general formula for the ranks of the Jordan blocks of the local monodromy of the middle cohomology of a semistable degeneration may be found in [**28**].

For 1-cycles in a family of hypersurfaces of degree d in $\mathbb{P}^4$ we have the following results: For $d = 1, 2$ there are no jumps since $H^3 = 0$. For $d = 3, 4$ we have $h^{3,0} = 0$, $h^{2,1} \neq 0$, and hence Theorem (5.38) is applicable only if the monodromy operator T is of finite order. For $d \geq 6$ the height does not jump as a consequence of Mark Green's result [**17**] that for a generic smooth hypersurface in $\mathbb{P}^4$ of degree ≥ 6 the image of the Abel-Jacobi map from the Chow group of 1-cycles which are homologous to zero mod rational equivalence into the third intermediate Jacobian is contained in the torsion subgroup. This leaves the quintic threefolds, which are Calabi-Yau manifolds. Although much has been said about the degenerations of such manifolds in general, specific examples of 2-parameter degenerations appear to be relatively rare in the literature. Of the examples considered in [**5**] and [**9**] only the family of quintics

$$(y_1^5 + \cdots + y_y^5) - ay_4^3y_5^2 - by_4^2y_5^3 = 0 \tag{5.43}$$

has the correct monodromy to avoid jumps.

To close this section, we now present two related examples which show that when the hypothesis of Theorem (5.38) is violated the height may or may not jump:

Example 5.44. Let $V_{\mathbb{Z}}$ be an integral lattice of rank 4, with basis $\{e_0, e, f, e_{-2}\}$, and N_1, N_2 denote the endomorphisms of $V_{\mathbb{Z}}$ defined by the matrices

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

Then, the nilpotent orbit $\varphi(s_1, s_2) = e^{\frac{1}{2\pi i}(\log(s_1)N_1 + \log(s_2)N_2)}.F_\infty$, defined by the filtrations

$$\begin{aligned} W_0(V_{\mathbb{Z}}) &= V_{\mathbb{Z}} \\ W_{-1}(V_{\mathbb{Z}}) &= \mathbb{Z}e \oplus \mathbb{Z}f \oplus \mathbb{Z}e_{-2} \\ W_{-2}(V_{\mathbb{Z}}) &= \mathbb{Z}e_{-2} \\ W_{-3}(V_{\mathbb{Z}}) &= 0 \end{aligned} \qquad \begin{aligned} F_\infty^{-1} &= V_{\mathbb{Z}} \otimes \mathbb{C} \\ F_\infty^0 &= \mathbb{C}e_0 \oplus \mathbb{C}e \\ F_\infty^1 &= 0 \end{aligned}$$

is admissible, and graded-polarizable. Direct calculation shows that the associated height function (5.5) is given by the formula

$$h(s_1, s_2) = \frac{(\log|s_1/s_2|)^2 - (\log|s_1 s_2|)^2}{\log|s_1 s_2|}.$$

Setting $(s_1, s_2) = (t^{a_1}, t^{a_2})$, it then follows that

$$\mu = \frac{4a_1 a_2}{a_1 + a_2}$$

and hence $h(s_1, s_2)$ jumps along D.

Example 5.45. In Example (5.44), redefine

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

Then,

$$h(s_1, s_2) = -\log|s_1 s_2|$$

and hence $\mu = a_1 + a_2$. Accordingly, $h(s_1, s_2)$ has no jumps along D.

The following property distinguishes Examples (5.44) and (5.45): By Theorem (5.19) a sufficient condition for the height not to jump is the existence of a common vector $e_o \in V$ such that

(a) v_o projects to $1 \in Gr^W_0$;
(b) $N_j \in W_{-2}$ for each j.

In this case,

$$(a_1 N_1 + \cdots + a_r N_r)(v_o) = \mu_{a_1,\ldots,a_m} 1^\vee.$$

In the case of Example (5.45) the vector e_o satisfies conditions (a)–(b) and $\mu = a_1 + a_2$. In contrast, there is no such vector v_o for the monodromy cone of Example (5.44).

Remark. Conditions (a) and (b) have the following cohomological interpretation: Let $x_1, \ldots, x_m$ be commuting variables and A be a left $\mathbb{Q}[x_1, \ldots, x_m]$-module. Then, the $\mathbb{Q}$-vector spaces

$$B^p = \bigoplus_{1 \le j_1 < \cdots < j_p \le m} x_{j_1} \ldots x_{j_p}(A)$$

form a complex with respect to the differential which maps the summands of B^p to the summands of B^{p+1} via the rule

$$d = (-1)^{s-1} x_{j_s} : x_{j_1} \cdots \hat{x}_{j_s} \cdots x_{j_p}(A) \to x_{j_1} \cdots x_{j_p}(A).$$

When $A = H^k(X_t)$ is the typical fiber of a variation of Hodge structure $\mathcal{V}$ with monodromy logarithms $N_1, \ldots, N_m$ along a divisor D and $x_j(a) = N_j(a)$ then $H^*(B^\bullet)$ coincides [**8**] with the local intersection cohomology of $\mathcal{V}$ along D. In the setting of the previous paragraph with $A = W_0/W_{-2}$ and x_j the induced action of N_j, the desired vector v_o is simply an element of $H^0(B^\bullet)$ which projects to a generator of Gr^W_0. This suggests the asymptotic behavior of the height is controlled by the local intersection cohomology of W_0/W_{-2} along the boundary divisor.

6. Nahm's Equation

Let K be a compact real Lie group. Then, Nahm's equation for K is the system of ordinary differential equations given by the gradient flow of the 3-form

$$\phi(T_1, T_2, T_3) = \langle T_1, [T_2, T_3] \rangle \tag{6.1}$$

on $\kappa = \mathrm{Lie}\,(K)$ defined by a choice of bi-invariant metric $\langle \cdot, \cdot \rangle$ on K. Equivalently, a triple of κ-valued functions (T_1, T_2, T_3) is a solution of Nahm's equation if and only if

$$\frac{dT_i}{dy} + [T_j, T_k] = 0 \tag{6.2}$$

for every cyclic permutation $(i\ j\ k)$ of $(1\ 2\ 3)$.

More generally, given a complex Lie algebra $\mathfrak{a}$, a triple of $\mathfrak{a}$-valued functions (T_1, T_2, T_3) is said to be a solution of Nahm's equation provided they satisfy the system of differential equations (6.2). Solutions to Nahm's equation are related to representations of $sl_2(\mathbb{C})$ as follows: Let $\{\tau_1, \tau_2, \tau_3\}$ be a basis of $sl_2(\mathbb{C}) = su_2 \otimes \mathbb{C}$ such that

$$\tau_i = [\tau_j, \tau_k] \tag{6.3}$$

for every cyclic permutation $(i\ j\ k)$ of $(1\ 2\ 3)$ and $\rho : sl_2(\mathbb{C}) \to \mathfrak{a}$ be a Lie algebra homomorphism. Then, the triple

$$T_i(y) = \rho(\tau_i) y^{-1}$$

is a solution of (6.2). Conversely, given a solution (T_1, T_2, T_3) of Nahm's equation which has a simple pole at $y = 0$, the linear map $\rho : sl_2(\mathbb{C}) \to \mathfrak{a}$ defined by setting

$$\rho(\tau_i) = \mathrm{Res}(T_i)$$

is a Lie algebra homomorphism.

In [**34**], Schmid showed that a nilpotent orbit of pure, polarized Hodge structure gives rise to a solution

$$\Phi : (a, \infty) \to \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}_{\mathbb{C}}), \qquad \Phi(y)\tau_i = T_i(y) \tag{6.4}$$

of Nahm's equation. In this section, we show that a nilpotent orbit

$$\theta(z) = e^{zN}.F_\infty \tag{6.5}$$

of graded-polarized mixed Hodge structure gives rise to a solution of a generalization of Nahm's equation which encodes how the extension data of $\theta(z)$ interacts with the nilpotent orbits of pure Hodge structure induced by $\theta(z)$ on Gr^W.

To this end, let $\mathcal{M}$ be a classifying space of graded-polarized mixed Hodge structure. Define $\mathcal{D}$ to be the direct sum of classifying spaces of pure, polarized Hodge structure onto which $\mathcal{M}$ projects via the map

$$F \mapsto FGr^W.$$

Let $\mathcal{Y}_{-2}(W)$ be the affine space consisting of all gradings Y of W such that [cf. Theorem (2.17)]:

$$Y - \bar{Y} \in Lie_{-2}(W) \tag{6.6}$$

and ι_Y denote the isomorphism $Gr^W \cong V_{\mathbb{C}}$ associated to $Y \in \mathcal{Y}_{-2}(W)$. Then:

Theorem 6.7. *The space $\mathcal{X} = \mathcal{D} \times \mathcal{Y}_{-2}(W)$ is a complex manifold upon which the Lie group* H [*cf.* Theorem (2.19)] *acts transitively by automorphisms. Furthermore, the correspondence*

$$F = \pi(\{H^{r,s}\}, Y) \iff F^p = \bigoplus_{a \geq p} \iota_Y(H^{a,b}) \tag{6.8}$$

defines a H*–equivariant projection map* $\pi : \mathcal{X} \to \mathcal{M}$ *with real analytic section*

$$\sigma(F) = \left(FGr^W, Y_{(F,W)}\right). \tag{6.9}$$

Proof. The only subtle point is the assertion that σ is a real–analytic section. To prove this, observe that by part (c) of Theorem (2.4), the grading $Y_{(F,W)}$ defined by the $I^{p,q}$'s of (F,W) takes values in $\mathcal{Y}_{-2}(W)$. Consequently, equation (6.9) defines a section of $\mathcal{X}$. To prove that σ is real-analytic, recall [**7**] that

$$I^{p,q} = F^p \cap W_{p+q} \cap \left(\bar{F}^q \cap W_{p+q} + \sum_{j>0} \bar{F}^{q-j} \cap W_{p+q-1-j}\right)$$

and hence the decomposition (2.5) is real-analytic with respect to the point $F \in \mathcal{M}$. q.e.d.

Next, following [**34**], we note that each choice of base point F_o defines a principal bundle P over $\mathcal{X}$ with connection ∇:

Theorem 6.10. *Let* $F_o \in \mathcal{M}_{\mathbb{R}}$ *and* $x_o = \sigma(F_o)$. *Then, the vector space* [*cf. Theorem* (2.12)]

$$\mathfrak{h}' = (\eta_+ \oplus \Lambda^{-1,-1} \oplus \eta_-) \cap \mathfrak{h}$$

is an $\mathrm{Ad}\,(\mathrm{H}^{x_o})$*–invariant complement to* $\mathfrak{h}^{x_o}$ *in* $\mathfrak{h}$, *and hence defines a connection* ∇ *on the principal bundle*

$$\mathrm{H}^{x_o} \to \mathrm{H} \to \mathrm{H}/\mathrm{H}^{x_o}$$

over $\mathcal{X} \cong \mathrm{H}/\mathrm{H}^{x_o}$.

Proof. Direct calculation shows that since $F_o \in \mathcal{M}_{\mathbb{R}}$, $\mathfrak{h}^{x_o} = \eta_0 \cap \mathfrak{h}$ and hence $\mathfrak{h}'$ is a vector space complement to $\mathfrak{h}^{x_0}$ in $\mathfrak{h}$. To see that $\mathfrak{h}'$ is invariant under the action of $\mathrm{Ad}\,(H^{x_0})$, let $h \in H^{x_o}$. Then, h preserves F_o since

$$h.F_o = h.\pi(x_o) = \pi(h.x_o) = \pi(x_o) = F_o.$$

Likewise, $h = \bar{h}$ since h acts by real automorphisms on Gr^W and preserves the real grading $Y_{(F_o,W)}$. Consequently, h is a morphism of (F_o, W) and hence preserves each summand appearing in the definition of $\mathfrak{h}'$. q.e.d.

Thus, by virtue of the above remarks, each choice of base point $F_o \in \mathcal{M}_{\mathbb{R}}$ defines a lift of $\theta(iy)$ to a function $h(y) : (a,\infty) \to \mathrm{H}$ such that:

(a) $h(y).x_o = \sigma(\theta(iy))$;
(b) h is tangent to ∇.

Theorem 6.11. *Let* L *denote the endomorphism of* $\mathfrak{h}$ *defined by the rule:*

$$\mathrm{L}\,|_{\eta_+} = +i, \qquad \mathrm{L}\,|_{\eta_0} = 0, \qquad \mathrm{L}\,|_{\eta_- \oplus \Lambda^{-1,-1}} = -i.$$

Then, the function $h(y)$ defined above satisfies the differential equation

$$h^{-1}(y)\frac{d}{dy}h(y) = -\mathrm{L}\ \mathrm{Ad}\,(h^{-1}(y))N. \tag{6.12}$$

Proof. Schmid's original derivation [**34**, Lemma (9.8)] of Nahm's equation for nilpotent orbits of pure, polarized Hodge structure shows that equation (6.12) holds modulo $\mathrm{Lie}_{-1}(W)$. Consequently, it is sufficient to verify that equation (6.12) holds modulo the subalgebra $\mathfrak{g}_{\mathbb{C}}^Y = \mathrm{Lie}\,(\mathrm{G}_{\mathbb{C}}^Y)$, $Y = Y_{(F_o,W)}$ since

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^Y \oplus \mathrm{Lie}_{-1}(W).$$

To this end, note that by definition $Y_{e^{iyN}.F_\infty} = \mathrm{Ad}\,(h(y))Y$. Upon differentiating both sides of this equation with respect to y and simplifying the result, it then follows that:

$$\mathrm{Ad}\,(h^{-1}(y))\frac{d}{dy}\,Y_{(e^{iyN}.F_\infty,W)} = \left[h^{-1}(y)\frac{d}{dy}h(y), Y\right]. \tag{6.13}$$

Therefore, if $z = x + iy$:

$$\begin{aligned}&\mathrm{Ad}\,(h^{-1}(y))\frac{d}{dy}Y_{(e^{iyN}.F_\infty,W)} \\ &\quad = i\ \mathrm{Ad}\,(h^{-1}(y))\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)Y_{(e^{zN}.F_\infty,W)}\Big|_{z=iy}.\end{aligned} \tag{6.14}$$

To compute $\frac{\partial}{\partial z}\,Y_{(e^{zN}.F_\infty,W)}$ and $\frac{\partial}{\partial \bar{z}}\,Y_{(e^{zN}.F_\infty,W)}$, we observe that as a consequence of equation (5.19) in [**30**]:

$$\begin{aligned}\frac{\partial}{\partial w}Y_{(e^{w\xi}.F,W)}\Big|_{w=0} &= [\pi_{\mathfrak{t}}(\xi), Y_{(F,W)}], \\ \frac{\partial}{\partial \bar{w}}Y_{(e^{w\xi}.F,W)}\Big|_{w=0} &= [\pi_+(\overline{\pi_{\mathfrak{t}}(\xi)}), Y_{(F,W)}]\end{aligned} \tag{6.15}$$

for any point $F \in \mathcal{M}$ and any element $\xi \in \mathrm{Lie}\,(\mathrm{G}_{\mathbb{C}})$, where π_+ and $\pi_{\mathfrak{t}}$ denote the projection operators[2] with respect to F defined in Theorem (2.12). In particular, upon setting $F = e^{iyN}.F_\infty$ it then follows from equations (6.14) and (6.15) that:

(6.16)

$$\mathrm{Ad}\,(h^{-1}(y))\frac{d}{dy}\,Y_{e^{iyN}.F_\infty} = i\,\mathrm{Ad}\,(h^{-1}(y))[\pi_{\mathfrak{t}}(N) - \pi_+(\overline{\pi_{\mathfrak{t}}(N)}), Y_{e^{iyN}.F_\infty}].$$

On the other hand, if π_0 denotes projection onto η_0 with respect to $F = e^{iyN}.F_\infty$ then

$$N = \pi_+(N) + \pi_0(N) + \pi_{\mathfrak{t}}(N).$$

Consequently, since N is defined over $\mathbb{R}$:

$$N = \bar{N} = \overline{\pi_+(N)} + \overline{\pi_0(N)} + \overline{\pi_{\mathfrak{t}}(N)}$$

[2] In [**30**], we used the alternative notation $\mathfrak{t}_F = q_F$ and $\pi_{\mathfrak{t}} = \pi_q$.

and hence $\pi_+(N) = \pi_+(\overline{\pi_{\mathfrak{t}}(N)})$. Accordingly, equation (6.16) may be rewritten as

$$\begin{aligned} \text{(6.17)} \qquad & \mathrm{Ad}\,(h^{-1}(y))\frac{d}{dy}\,Y_{e^{iyN}.F_\infty} \\ &= i\,\mathrm{Ad}\,(h^{-1}(y))[\pi_{\mathfrak{t}}(N) - \pi_+(N), Y_{e^{iyN}.F_\infty}] \\ &= i[\mathrm{Ad}\,(h^{-1}(y))\{\pi_{\mathfrak{t}}(N) - \pi_+(N)\}, \mathrm{Ad}\,(h^{-1}(y))Y_{e^{iyN}.F_\infty}] \\ &= i[\mathrm{Ad}\,(h^{-1}(y))\{\pi_{\mathfrak{t}}(N) - \pi_+(N)\}, Y] \end{aligned}$$

since $Y_{(e^{iyN}.F_\infty,W)} = Ad(h(y))Y$.

By construction:

$$\text{(6.18)} \qquad h(y).I^{p,q}_{(F_0,W)} = I^{p,q}_{(e^{iyN}.F_\infty,W)}$$

and hence $\mathrm{Ad}\,(h(y)) : \mathfrak{g}^{r,s}_{(F_0,W)} \to \mathfrak{g}^{r,s}_{(e^{iyN}.F_\infty,W)}$. Consequently,

$$\begin{aligned} & i\,\mathrm{Ad}\,(h^{-1}(y))\{\pi_{\mathfrak{t}}(N) - \pi_+(N)\} \\ &= i\,\hat{\pi_{\mathfrak{t}}}(\mathrm{Ad}\,(h^{-1}(y))N) - i\,\hat{\pi_+}(\mathrm{Ad}\,(h^{-1}(y)N)) \\ &= -\mathrm{L}\ \mathrm{Ad}\,(h^{-1}(y))N \quad \mathrm{mod}\ \mathrm{Lie}\,(\mathrm{G}^{\mathrm{Y}}_{\mathbb{C}}) \end{aligned}$$

where $\hat{\pi_{\mathfrak{t}}}$ and $\hat{\pi_+}$ denote projection with respect to $F_o \in \mathcal{M}_{\mathbb{R}}$. Therefore, by equation (6.17),

$$\text{(6.19)} \qquad \mathrm{Ad}\,(h^{-1}(y))\frac{d}{dy}\,Y_{(e^{iyN}.F_\infty,W)} = [-\mathrm{L}\ \mathrm{Ad}\,(h^{-1}(y))N, Y].$$

Accordingly, upon comparing equation (6.19) with equation (6.13), it then follows that

$$[-\mathrm{L}\ \mathrm{Ad}\,(h^{-1}(y))N, Y] = \left[h^{-1}(y)\frac{d}{dy}h(y), Y\right]$$

and hence $-\mathrm{L}\ \mathrm{Ad}\,(h^{-1}(y))N = h^{-1}(y)\frac{d}{dy}h(y)$ mod $\mathfrak{g}^Y_{\mathbb{C}}$ as required. q.e.d.

Example 6.20. Let $\theta(z) = e^{zN}.\hat{F}$ be a split orbit. Then, the function

$$h(y) = e^{iyN}e^{-iyN_0}y^{-H/2}$$

[cf. Theorem (3.16) for notation] is a solution of equation (6.12) with respect to the base point $F_o = e^{iN_0}.\hat{F} \in \mathcal{M}_{\mathbb{R}}$.

To prove this, equip $sl_2(\mathbb{C})$ with the standard Hodge structure (3.11) and $\mathfrak{g}_{\mathbb{C}}$ with the usual mixed Hodge structure induced by (F_o, W). Then, as a consequence of Theorem (3.13) and the fact [Theorem (3.16), part (c)] that $e^{zN_0}.\hat{F}$ is and SL$_2$-orbit with data $(F_o, \psi_* = \rho)$, the representation

$$\text{(6.21)} \qquad \rho : sl_2(\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$$

defined in Theorem (3.16) is a morphism of Hodge structure.

By direct calculation:

$$h^{-1}\frac{dh}{dy} = -\frac{H}{2y} + i\mathrm{Ad}\left(y^{H/2}\right)\mathrm{Ad}\left(e^{iyN_0}\right)\left(\sum_{k\geq 2} N_{-k}\right) \tag{6.22}$$

$$\mathrm{Ad}\left(h^{-1}(y)\right)N = \frac{N_0}{y} + \mathrm{Ad}\left(y^{H/2}\right)\mathrm{Ad}\left(e^{iyN_0}\right)\left(\sum_{k\geq 2} N_{-k}\right).$$

Similarly, a small computation in $sl_2(\mathbb{C})$ shows that the basis (1.6) satisfies the Hodge conditions:

$$x^- \in sl_2(\mathbb{C})^{-1,1}, \quad \mathfrak{z} \in sl_2(\mathbb{C})^{0,0}, \quad x^+ \in sl_2(\mathbb{C})^{1,-1}. \tag{6.23}$$

Therefore, since ρ is a morphism of Hodge structures, the image (X^+, Z, X^-) of the basis (1.6) under ρ satisfy the analogous conditions

$$X^- \in \mathfrak{g}^{-1,1}, \quad Z \in \mathfrak{g}^{0,0}, \quad X^+ \in \mathfrak{g}^{1,-1} \tag{6.24}$$

at (F_o, W). Comparing (1.6) and (3.15), it then follows that

$$N_0 = \frac{1}{2i}(X^+ - X^- + Z), \qquad N_0^+ = \frac{1}{2i}(X^+ - X^- - Z) \tag{6.25}$$
$$H = (X^+ + X^-).$$

Consequently,

$$\mathrm{L}\,(N_0) = \frac{1}{2i}\mathrm{L}\,(X^+ - X^- + Z) = \frac{1}{2i}(iX^+ + iX^-) = \frac{1}{2}H. \tag{6.26}$$

To continue, we now recall that by [**14**], [**25**]

$$(\mathrm{ad}N_0)^j N_{-k} \in \Lambda^{-1,-1}_{(F_o,W)} \tag{6.27}$$

and hence the function $\mathrm{Ad}\,(y^{H/2})Ad(e^{iyN_0})(\sum_{k\geq 2} N_{-k})$ takes values in $\Lambda^{-1,-1}_{(F_o,W)}$. Therefore, by equations (6.22) and (6.26):

$$\begin{aligned}
&- \mathrm{L}\,\mathrm{Ad}\,(H^{-1}(y))N \\
&= -\frac{\mathrm{L}\,(N_0)}{y} - \mathrm{L}\,\mathrm{Ad}\,(y^{H/2})Ad(e^{iyN_0})\left(\sum_{k\geq 2} N_{-k}\right) \\
&= -\frac{H}{2y} + i\mathrm{Ad}\,(y^{H/2})Ad(e^{iyN_0})\left(\sum_{k\geq 2} N_{-k}\right) = h^{-1}\frac{dh}{dy}.
\end{aligned}$$

To relate equation (6.12) with Nahm's equation, we now decompose

$$\beta(y) = \mathrm{Ad}\,(h^{-1}(y))N \tag{6.28}$$

according to its Hodge components with respect to (F_o, W). To this end, observe that as a consequence of equation (6.18), the Hodge decomposition of $\beta(y)$ with respect to (F_o, W) has the same form as the Hodge decomposition of N with respect to $(e^{zN}.F_\infty, W)$. Therefore, by the next lemma, the Hodge decomposition of $\beta(y)$ with respect to (F_o, W) is of the form

$$\beta(y) = \beta^{1,-1}(y) + \beta^{0,0}(y) + \beta^{-1,1}(y) + \beta_+(y) + \beta_-(y) \tag{6.29}$$

where

$$\beta_+(y) = \sum_{k>0} \beta^{0,-k}(y), \qquad \beta_-(y) = \sum_{k>0} \beta^{-1,1-k}(y). \tag{6.30}$$

Lemma 6.31. *Let $e^{zN}.F$ be a nilpotent orbit. Then, with respect to $(e^{zN}.F, W)$, the Hodge decomposition of N assumes the form:*

$$N = N^{-1,1} + N^{0,0} + N^{1,-1} + \left(\sum_{k>0} N^{-1,1-k}\right) + \left(\sum_{k>0} N^{0,-k}\right). \tag{6.32}$$

Proof. The fact the N is horizontal at $e^{zN}.F$ implies that

$$N = N^{-1,1} + \sum_{k>0} N^{-1,1-k} \mod \bigoplus_{r\geq 0} \mathfrak{g}^{r,s}. \tag{6.33}$$

Define

$$N_{-k} = \bigoplus_{r+s=-k} N^{r,s}.$$

Then, the horizontality (6.33) of N coupled with the fact that $N = \bar{N}$ implies that

$$N_0 = N^{-1,1} + N^{0,0} + N^{1,-1}. \tag{6.34}$$

Suppose that (6.32) is false and let k be the smallest integer such that N_{-k} violates (6.32). By (6.34), $k > 0$. As such, by equation (6.33)

$$N_{-k} = N^{-1,1-k} + N^{0,-k} + N^{p,-p-k} + \cdots$$

for some integer $p > 0$. By, Theorem (2.4):

$$\overline{\mathfrak{g}^{r,s}} = \mathfrak{g}^{s,r} \mod \bigoplus_{a<s,b<r} \mathfrak{g}^{a,b}. \tag{6.35}$$

Accordingly, $\overline{N^{p,-k-p}}$ is of Hodge type $(-k-p, p)$ modulo lower order terms. Consequently, since $N = \bar{N}$ and elements of type $(-k-p, p)$ are not horizontal, $\overline{N^{p,-k-p}}$ must be annihilated by part of the fallout of the complex conjugate of some Hodge component $N^{r,s}$ with $r+s > -k$. On the other hand, by the definition of k, all such components $N^{r,s}$ satisfy (6.32). Therefore, by equation (6.35), there is no way for $\overline{N^{r,s}}$ to annihilate $\overline{N^{p,-k-p}}$ since $p > 0$. q.e.d.

Following [**34**] and [**7**], define

$$\alpha(y) = -2h^{-1}(y)\frac{dh}{dy}. \tag{6.36}$$

Then, by virtue of equation (6.12),

$$\alpha(y) = \alpha^{1,-1}(y) + \alpha^{-1,1}(y) + \alpha_+(y) + \alpha_-(y)$$

where

$$\begin{aligned} \alpha^{1,-1} &= 2i\beta^{1,-1}, & \alpha_+ &= 2i\beta_+ \\ \alpha^{-1,1} &= -2i\beta^{-1,1} & \alpha_- &= -2i\beta_-. \end{aligned} \tag{6.37}$$

On the other hand, differentiation of equation (6.28) shows that

$$-2\frac{d\beta}{dy} = [\beta(y), \alpha(y)]. \tag{6.38}$$

Inserting equation (6.37) into (6.38) and taking Hodge components, we then obtain the following result:

Theorem 6.39. *Let $h(y)$ be a solution to equation* (6.12)*. Then,*

$$\frac{d}{dy}\beta_0(y) = -[\beta_0(y), \mathrm{L}\,\beta_0(y)], \qquad \beta_0(y) = \sum_{r+s=0} \beta^{r,s}(y) \tag{6.40}$$

and

$$\frac{d}{dy}\begin{pmatrix}\beta_- \\ \beta_+\end{pmatrix} = i\begin{pmatrix} \mathrm{ad}\,\beta^{0,0} & -2\,\mathrm{ad}\,\beta^{-1,1} \\ 2\,\mathrm{ad}\,\beta^{1,-1} & -\mathrm{ad}\,\beta^{0,0}\end{pmatrix}\begin{pmatrix}\beta_- \\ \beta_+\end{pmatrix} + 2i\begin{pmatrix}[\beta_+, \beta_-] \\ 0\end{pmatrix}. \tag{6.41}$$

In particular, as a consequence of equation (6.40), we obtain the following relationship between nilpotent orbits and solutions to Nahm's equation:

Corollary 6.42. *Let $h(y)$ be a solution of equation* (6.12)*, and*

$$X^-(y) = -2i\beta^{-1,1}(y), \quad Z(y) = 2i\beta^{0,0}(y), \quad X^+(y) = 2i\beta^{1,-1}(y). \tag{6.43}$$

The function $\Phi : (a, \infty) \to \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}_{\mathbb{C}})$ defined by setting

$$\Phi(y)x^+ = X^+(y), \quad \Phi(y)\mathfrak{z} = Z(y), \quad \Phi(y)x^- = X^-(y) \tag{6.44}$$

is a solution (6.4) *of Nahm's equation.*

Proof. The assertion that Φ is a solution to Nahm's equation is equivalent to the system of equations:

$$-2\frac{dX^+}{dy} = [Z(y), X^+(y)], \qquad 2\frac{dX^-}{dy} = [Z(y), X^-(y)] \tag{6.45}$$

$$-\frac{dZ}{dy} = [X^+(y), X^-(y)].$$

To verify that the triple (6.43) satisfies equation (6.45), one simply expands out equation (6.40) in terms of the Hodge components of β_0. q.e.d.

The remaining Hodge components of α and β determine the extension data $\theta(z)$. To relate these components to solution of Nahm's equation (6.44), let

$$A = \begin{pmatrix} \frac{1}{2}\mathrm{ad}\, Z(y) & -\mathrm{ad}\, X^-(y) \\ -\mathrm{ad}\, X^+(y) & -\frac{1}{2}\mathrm{ad}\, Z(y) \end{pmatrix} \tag{6.46}$$

and define

$$\tau_{-k} = \sum_{r>0, s>0, r+s=k} [\alpha^{0,r}, \alpha^{-1,1-s}]. \tag{6.47}$$

Then, equation (6.41) is equivalent to the hierarchy of differential equations:

$$\frac{d}{dy}\begin{pmatrix} \alpha^{-1,1-k} \\ \alpha^{0,-k} \end{pmatrix} = A \begin{pmatrix} \alpha^{-1,1-k} \\ \alpha^{0,-k} \end{pmatrix} + \begin{pmatrix} \tau_{-k} \\ 0 \end{pmatrix}, \qquad k = 1, 2, \dots. \tag{6.48}$$

Accordingly, equation (6.48) can be viewed as a system of equations relating the evolution of the extension data of $\theta(z)$ to the nilpotent orbits of pure Hodge structure induced by $\theta(z)$ on Gr^W.

7. Nilpotent Orbits of Pure Hodge Structure

The relation between nilpotent orbits and solutions of the generalized Nahm's equation presented in Theorem (6.11) can be inverted as follows:

Theorem 7.1. *Let $F_o \in \mathcal{M}_{\mathbb{R}}$, and suppose that $\beta(y)$ is an $\mathfrak{h}$-valued function which satisfies the Lax equation*

$$\frac{d\beta}{dy} = -[\beta(y), \mathrm{L}\, \beta(y)]. \tag{7.2}$$

Then, there exists an $\mathfrak{h}$-valued function $h(y)$, an element $\tilde{N} \in \mathfrak{h}$ and a point $\tilde{F} \in \check{\mathcal{M}}$ such that

(a) $h^{-1}(y)\frac{dh}{dy} = -\mathrm{L}\, \beta(y)$, $\beta(y) = \mathrm{Ad}\,(h^{-1}(y))\tilde{N}$;
(b) $h(y).F_o = e^{iy\tilde{N}}.\tilde{F}$.

Proof. The differential equation

$$h^{-1}(y)\frac{dh}{dy} = -\mathrm{L}\, \beta(y) \tag{7.3}$$

completely determines $h(y)$ up to a choice of initial value $h_o \in \mathrm{H}$. Likewise, by virtue of equations (7.2) and (7.3),

$$\mathrm{Ad}\,(h^{-1}(y))\frac{d}{dy}\mathrm{Ad}\,(h(y))\beta(y) = 0.$$

Therefore, $\beta(y) = \mathrm{Ad}\,(h^{-1}(y))\tilde{N}$ for some fixed element $\tilde{N} \in \mathfrak{h}$. Similarly, by virtue of equation (7.3),

$$h^{-1}(y)e^{iy\tilde{N}}\frac{d}{dy}e^{-iy\tilde{N}}h(y) = h^{-1}(y)e^{iy\tilde{N}}\left(-i\tilde{N}e^{-iy\tilde{N}}h(y) + e^{iy\tilde{N}}\frac{dh}{dy}\right)$$
$$= -i\mathrm{Ad}\,(h^{-1}(y))\tilde{N} + h^{-1}\frac{dh}{dy}$$
$$= -i\beta(y) - \mathrm{L}\,\beta(y) \in \mathfrak{g}_{\mathbb{C}}^{F_o}.$$

Accordingly,

$$e^{-iy\tilde{N}}h(y) = g_{\mathbb{C}}f(y)$$

for some $\mathrm{G}_{\mathbb{C}}^{F_o}$-valued function $f(y)$ and some fixed element $g_{\mathbb{C}} \in \mathrm{G}_{\mathbb{C}}$. Thus,

$$h(y).F_o = e^{iy\tilde{N}}g_{\mathbb{C}}f(y).F_o = e^{iy\tilde{N}}.\tilde{F}$$

where $\tilde{F} = g_{\mathbb{C}}.F_o$. q.e.d.

Remark. In order for $e^{z\tilde{N}}.\tilde{F}$ to be a proper nilpotent orbit in the sense of Definition (3.8), $\tilde{N}$ must be real and $\beta(y)$ must be horizontal with respect to F_o. In this case, we can then introduce a spectral parameter into equation (7.2) by simply replacing $\beta(y)$ by $\beta_\lambda(y) = \sum_{p,q}\lambda^p\beta^{p,q}(y)$.

In §6 of [**7**], Cattani, Kaplan and Schmid proved the SL_2-orbit theorem for nilpotent orbits of pure Hodge structure $\theta(z) = e^{zN}.F$ by constructing a series solution

$$\beta(y) = \sum_{n\geq 0}\beta_n y^{-1-n/2}$$

of equation (7.2) such that $(\tilde{N},\tilde{F}) = (N,F)$. In this section, we summarize this approach in some detail in preparation for the proof Theorem (4.2) presented in §8–9.

To this end, let $\mathfrak{a}$ be a complex Lie algebra and $\mathfrak{U}$ be a representation of $sl_2(\mathbb{C})$. Then, contraction against the Casimir element

$$\Omega = 2x^+x^- + 2x^-x^+ + \mathfrak{z}^2 \tag{7.4}$$

of $sl_2(\mathbb{C})$ defines a pairing

$$Q : \mathrm{Hom}(sl_2(\mathbb{C}),\mathfrak{a}) \otimes \mathrm{Hom}(\mathfrak{U},\mathfrak{a}) \to \mathrm{Hom}(\mathfrak{U},\mathfrak{a}) \tag{7.5}$$

via the rule

(7.6)
$$Q(A,B)(u) = 2[A(x^+),B(x^-.u)] + 2[A(x^-),B(x^+.u)] + [A(\mathfrak{z}),B(\mathfrak{z}.u)].$$

Furthermore, a short calculation shows that, relative to the adjoint representation $\mathfrak{U}$ of $sl_2(\mathbb{C})$, Nahm's equation (6.2) is equivalent to the differential equation

$$-8\frac{d\Phi}{dy} = Q(\Phi, \Phi). \tag{7.7}$$

Following [**7**], suppose that Φ has a convergent series expansion about infinity of the form

$$\Phi = \sum_{n\geq 0} \Phi_n y^{-1-n/2}$$

and let $Q = 8Q_o$. Then, equation (7.7) is equivalent to the recursion relations

$$\Phi_0 = Q_o(\Phi_0, \Phi_0) \tag{7.8}$$

and

$$(1+n/2)\Phi_n - 2Q_o(\Phi_0, \Phi_n) = \sum_{0<k<n} Q_o(\Phi_k, \Phi_{n-k}), \qquad n > 0. \tag{7.9}$$

Equation (7.8) implies that Φ_0 is either zero or an embedding of $sl_2(\mathbb{C})$ in $\mathfrak{g}_{\mathbb{C}}$. If $\Phi_0 = 0$ then $\Phi_n = 0$ for all n by induction. If $\Phi_0 \neq 0$ then a short calculation [**7**, 6.14] shows that

$$Q_o(\Phi_0, T) = \frac{1}{16}(\ell(\Omega) - \Omega T + 8T)$$

where $\ell(\Omega)T$ and ΩT respectively denote the left and diagonal action of the Casimir element (7.4) on $T \in \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}_{\mathbb{C}}) \cong \mathfrak{g}_{\mathbb{C}} \otimes sl_2(\mathbb{C})^*$.

To continue, we now recall [**7**, 6.18] that relative to the sl_2 module structure induced on $\mathfrak{g}_{\mathbb{C}}$ by Φ_0, we can decompose

$$\mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}_{\mathbb{C}}) = \sum_{r\geq 0}\sum_{\epsilon=-1}^{1} \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}(r))^{\epsilon} \tag{7.10}$$

where $\mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}(r))^{\epsilon}$ is the isotypical component of consisting of the span of all irreducible submodules of $\mathfrak{g}_{\mathbb{C}} \otimes sl_2(\mathbb{C})^*$ which are of highest weight r with respect to the left module structure and highest weight $r + 2\epsilon$ with respect to the diagonal structure.

Relative to the bigrading (7.10), the recursion relation (7.9) reduces to the equation [**7**, 6.20]

$$(n + \epsilon^2 + \epsilon(r+1))\Phi_n^{r,\epsilon} = 2\sum_{0<k<n} Q_o(\Phi_k, \Psi_{n-k})^{r,\epsilon}. \tag{7.11}$$

Therefore, subject to the compatibility condition

$$\sum_{0<k<n} Q_o(\Phi_k, \Psi_{n-k})^{n,-1} = 0 \tag{7.12}$$

equation (7.11) completely determines every component $\Phi_n^{r,\epsilon}$ except $\Phi_n^{n,-1}$ in terms of $\Phi_0, \dots, \Phi_{n-1}$. The verification of the compatibility condition (7.12) in turn reduces a standard weight argument (cf. [**7**, 6.21]). Thus, given a collection of elements

$$T_n \in \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}(n))^{-1} \tag{7.13}$$

there exists a unique series solution Φ of equation (7.7) such that

(a) $\Phi_n \in \oplus_{r\leq n,\, r\equiv n \mod 2} \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}(r))$;
(b) $\Phi_n^{n,-1} = T_n$;
(c) $\Phi_n^{n,0} = \Phi_n^{n,1} = 0$.

In particular, $\Phi_1 = 0$ since it must have highest weight -1 with respect to the diagonal action of $sl_2(\mathbb{C})$.

Imposing the condition that Φ should be horizontal and map $sl_2(\mathbb{R})$ into $\mathfrak{h} = \mathfrak{g}_{\mathbb{R}}$, it then follows that each T_n must also be a morphism of Hodge structure with respect to the standard Hodge structure on $sl_2(\mathbb{C})$ defined in §3 and pure Hodge structure

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_p \mathfrak{g}^{p,-p}$$

induced by F_o on $\mathfrak{g}_{\mathbb{C}}$. Accordingly, since $\mathfrak{g}_{\mathbb{C}}$ is the Lie algebra of a linear Lie group $\mathrm{G}_{\mathbb{C}}$, the equation

$$h^{-1}(y)\frac{d}{dy} = -\frac{1}{2}\Phi(\mathsf{h}) \tag{7.14}$$

therefore determines $h(y)$ up to left multiplication by $h_o \in \mathrm{H} = G_{\mathbb{R}}$.

Define

$$h(y) = g(y)y^{-H/2} \tag{7.15}$$

where $H = \Phi_0(\mathsf{h})$. Then, a standard weight argument shows that

$$g^{-1}(y)\frac{dg}{dy} = -\frac{1}{2}y^{-H/2}(\Phi(\mathsf{h}) - \Phi_0(\mathsf{h})y^{-1}) = \sum_{m\geq 2} B_m y^{-2}. \tag{7.16}$$

Consequently, $g(y)$ and $g^{-1}(y)$ have convergent series expansions about ∞ of the form

$$\begin{aligned} g(y) &= g(\infty)(1 + g_1 y^{-1} + g_2 y^{-2} + \cdots) \\ g^{-1}(y) &= (1 + f_1 y^{-1} + f_2 y^{-2} + \cdots)g^{-1}(\infty) \end{aligned} \tag{7.17}$$

where the coefficients g_k and f_k are universal non-commutative polynomials in the B_k with rational coefficients.

To connect these results with the SL_2-orbit theorem, we now assume that $\theta(z) = e^{zN}.F$ is a nilpotent orbit of pure Hodge structure and let

$$(F, {}^rW) = (e^{-i\delta}.\hat{F}, {}^rW) \tag{7.18}$$

be the splitting of the limiting mixed Hodge structure of $\theta(z)$ defined by Theorem (2.16). Define

$$F_o = \hat{\theta}(i) = e^{iN}.\hat{F}$$

where $\hat{\theta}(z) = e^{zN}.\hat{F}$ is the associated split orbit, and require Φ_0 to be the associated representation of $sl_2(\mathbb{R})$ defined by Theorem (3.13). Then,

$$h(y).F_o = g(y)y^{-H/2}.F_o = g(y)e^{iyN}.\hat{F}.$$

On the other hand, by Theorem (7.1), $h(y).F_o = e^{iy\tilde{N}}.\tilde{F}$ and hence

$$e^{iy\tilde{N}}.\tilde{F} = g(y)e^{iyN}.\hat{F}.$$

Therefore, in order to complete the proof of the SL_2 orbit theorem, it remains only to show that one can select data $(g(\infty), \{T_n\})$ such that $(\tilde{N}, \tilde{F}) = (N, F)$. Assuming that $g(\infty) \in \ker(N)$, this then boils down after a lengthy calculation to the requirement that

$$e^{i\delta} = g(\infty)\left(1 + \sum_{k>0} \frac{1}{k!}(-i)^k (\mathrm{ad}\, N_0)^k\, g_k\right).$$

At this point, the algebra/combinatorics of solving for $g(\infty)$ and $\{T_n\}$ becomes sufficiently involved that I shall leave the details to §8 and [**7**].

8. Nilpotent Orbits of Type (I)

In this section we prove Theorem (4.2) for admissible nilpotent orbits of type (I) by constructing a suitable series solution $\beta(y)$ of the Lax equation (7.2) using the outline of [**7**] developed in §7. To determine what form the series expansion of $\beta(y)$ should assume, consider the following two examples:

Example 8.1. Let $\pi : E \to \mathbb{C}$ denote the family of elliptic curves defined by the equation

$$v^2 = u(u-1)(u-s)$$

and $\tilde{\pi} : \tilde{E} \to \mathbb{C}$ denote the corresponding family of punctured curves obtained by deleting the points of E lying over $u = a$ for some fixed parameter $a \in \mathbb{C} - \{0, 1\}$. Then, the function

$$\beta(y) = \mathrm{Ad}\,(h^{-1}(y))N$$

attached by Theorem (6.11) to the nilpotent orbit of $R^1_{\tilde{\pi}*}(\mathbb{Q}) \otimes \mathcal{O}_{\mathbb{C}-\{0,1,a\}}$ at $s = 0$ is given by the formula

$$\beta(y) = \frac{N}{y} - \frac{\delta}{y^{3/2}}.$$

Example 8.2. Let $\hat{\theta}(z) = e^{zN}.\hat{F}$ be a split orbit of type (I) and $\mathfrak{U} = H(1) \otimes S(1)$ [cf. Theorem (3.14)]. Equip $\mathfrak{g}_{\mathbb{C}}$ with the associated sl_2-module structure defined by Theorem (3.16) and suppose that

$$\Psi : \mathfrak{U} \to \mathfrak{g}_{\mathbb{C}}$$

is a morphism of Hodge structure with respect to $F_o = \hat{\theta}(i)$ such that

$$\varsigma.\Psi(\tau) = \Psi(\varsigma.\tau)$$

for all $\varsigma \in sl_2(\mathbb{C})$ and $\tau \in \mathfrak{U}$. Then,

$$\theta(z) = e^{zN} e^{-i\Psi(f)}.\hat{F}$$

is an admissible nilpotent orbit of type (I) with split orbit $\hat{\theta}(z)$ and associated functions

$$\beta(y) = \frac{N}{y} + \frac{\Psi(f)}{y^{3/2}}, \qquad h(y) = (1 + \Psi(e)y^{-1})y^{-H/2}.$$

Based upon such examples, let us assume that the desired function $\beta(y)$ is horizontal with respect to F_o and has a convergent series expansion about ∞ of the form

$$\beta(y) = \sum_{n \geq 0} \beta_n y^{-1-n/2}. \tag{8.3}$$

Let $\Phi(y)$ be the corresponding function defined by equations (6.43)–(6.44) and $\Psi(y)$ be the linear map from $\mathfrak{U} = H(1) \otimes S(1)$ to $\mathfrak{g}_{\mathbb{C}}$ defined by the equation

$$\Psi(e + if) = 2i\beta^{0,-1}(y), \qquad \Psi(e - if) = -2i\beta^{-1,0}. \tag{8.4}$$

Then, a short calculation shows that equation (7.2) is equivalent to the pair of differential equations

$$-8\Phi'(y) = Q(\Phi, \Phi), \qquad -2\Psi'(y) = Q(\Phi, \Psi). \tag{8.5}$$

Thus, as in [**7**], the series expansion

$$\Phi(y) = \sum_{n \geq 0} \Phi_n y^{-1-n/2}$$

of Φ can be computed inductively starting from a collection of morphisms of Hodge structure

$$T_n : sl_2(\mathbb{C}) \to \mathfrak{g}(n) \tag{8.6}$$

such that $\Omega T_n = (n^2 - 2n)T_n$, where

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_r \mathfrak{g}(r) \tag{8.7}$$

denotes the decomposition of $\mathfrak{g}_{\mathbb{C}}$ into isotypical components with respect to the sl_2-module structure

$$x.y = [\Phi_0(x), y]$$

induced by Φ_0 on $\mathfrak{g}_{\mathbb{C}}$. Moreover, $\Phi_1 = 0$.

Similarly, the coefficients of the series expansion

$$\Psi = \sum_{n \geq 0} \Psi_n y^{-1-n/2} \tag{8.8}$$

satisfy the recursion relation

$$(n+2)\Psi_n = \sum_{j=0}^{n} Q(\Phi_j, \Psi_{n-j}). \tag{8.9}$$

Therefore, except for the contribution introduced by the term $Q(\Phi_0,\Psi_n)$, equation (8.9) allows us to inductively compute the coefficients of Ψ.

Let R be the endomorphism of $\mathrm{Hom}(\mathfrak{U},\mathfrak{g}_{\mathbb{C}})$ defined by $Q(\Phi_0,*)$ and recall that if U_r and U_s are irreducible sl_2-modules of highest weight r and s then

$$U_r \otimes U_s = \bigoplus_{|r-s|<t<r+s,\, t\equiv r+s \mod 2} U_t \tag{8.10}$$

where U_t is irreducible of highest weight t. In particular,

$$\mathrm{Hom}(\mathfrak{U},\mathfrak{g}(n)) = \mathrm{Hom}(\mathfrak{U},\mathfrak{g}(n))^+ \oplus \mathrm{Hom}(\mathfrak{U},\mathfrak{g}(n))^- \tag{8.11}$$

where $\mathrm{Hom}(\mathfrak{U},\mathfrak{g}_{\mathbb{C}}(n))^{\pm}$ is of highest weight n with respect to the left action of sl_2 on $\mathrm{Hom}(\mathfrak{U},\mathfrak{g}_{\mathbb{C}}) \cong \mathfrak{g}_{\mathbb{C}} \otimes (\mathfrak{U})^*$ and highest weight $n \pm 1$ with respect to the diagonal action.

Calculation 8.12. R acts semisimply on $\mathrm{Hom}(\mathfrak{U},\mathfrak{g}(n))$ as multiplication by $(n+2)$ on $\mathrm{Hom}(\mathfrak{U},\mathfrak{g}(n))^-$ and multiplication by $-n$ on $\mathrm{Hom}(\mathfrak{U},\mathfrak{g}(n))^+$.

Proof. Let $e=(1,0)$ and $f=(0,1)$ denote the standard basis of $\mathbb{C}^2$ and M be an irreducible submodule of $\mathfrak{g}_{\mathbb{C}}$ of highest weight n. Let $\{e^*, f^*\}$ be the corresponding dual basis of $(\mathbb{C}^2)^*$. Then, relative to the standard identification of M with $\mathrm{Sym}^n(\mathbb{C}^2)$,

$$M \otimes \mathfrak{U}^* \cong A \oplus B \tag{8.13}$$

where

$$\begin{aligned} A &= \mathrm{span}(a_0,\dots,a_{n+1}), & a_j &= (n-j+1)e^{n-j}f^j \otimes f^* \\ &&& \quad - je^{n-j+1}f^{j-1} \otimes e^* \\ B &= \mathrm{span}(b_0,\dots,b_{n-1}), & b_j &= e^{n-j-1}f^{j+1} \otimes f^* + e^{n-j}f^j \otimes e^* \end{aligned}$$

are irreducible submodules of highest weight $n+1$ and $n-1$ with respect to the diagonal action of $sl_2(\mathbb{C})$, and [cf. (3.15)]

$$\mathfrak{h}.(a_j) = (n+1-2j)a_j, \qquad \mathfrak{h}.(b_j) = (n-1-2j)b_j. \tag{8.14}$$

Accordingly, it suffices to compute $R(a_j)$ and $R(b_j)$. A short calculation shows that

$$Q(\sigma,\tau)(v) = 2[\sigma(\mathfrak{n}_0^+), \tau(\mathfrak{n}_0.v)] + 2[\sigma(\mathfrak{n}_0^-), \tau(\mathfrak{n}_0^+.v)] + [\sigma(\mathfrak{h}), \tau(\mathfrak{h}.v)].$$

Therefore,

$$\begin{aligned} R(a_j)(e) &= 2\mathfrak{n}_0^+.a_j(f) + \mathfrak{h}.a_j(e) \\ &= 2\mathfrak{n}_0^+.((n-j+1)e^{n-j}f^j) + \mathfrak{h}.(-je^{n-j+1}f^{j-1}) \\ &= 2(n-j+1)je^{n-j+1}f^{j-1} - j(n-2j+2)e^{n-j+1}f^{j-1} \\ &= j(2n-2j+2-n+2j-2)e^{n-j+1}f^{j-1} \\ &= jne^{n-j+1}f^{j-1} = -na_j(e). \end{aligned}$$

The remaining calculations of $R(a_j)(f)$, $R(b_j)(e)$ and $R(b_j)(f)$ are similar. q.e.d.

Corollary 8.15. $\Psi_0 = 0$, $\Psi_1 \in \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(1))^-$, $\Psi_2 \in \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(2))^-$.

Proof.
By equation (8.9), $R(\Psi_0) = 2\Psi_0$, and hence $\Psi_0 \in \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(0))^-$ by Calculation (8.12). However, $\mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(0))^- = 0$ since it is highest weight -1 with respect to the diagonal action of sl_2. Consequently, by virtue of the fact that $\Psi_0 = 0$ and $\Phi_1 = 0$, it then follows from equation (8.9) that $R(\Psi_1) = 3\Psi_1$ and $R(\Psi_2) = 4\Psi_2$. Therefore, by Calculation (8.12), $\Psi_1 \in \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(1))^-$ and $\Psi_2 \in \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(2))^-$. q.e.d.

To continue, given a semisimple endomorphism of A of a finite dimensional vector space V, let $[*]^A_\lambda$ denote projection from V onto the λ eigenspace of V. Then, by virtue of Calculation (8.12),

$$(8.16) \qquad (n-k)\Psi^-_{n,k} = \left[\sum_{0<j<n} Q(\Phi_j, \Psi_{n-j})\right]^R_{k+2}$$

$$(n+k+2)\Psi^+_{n,k} = \left[\sum_{0<j<n} Q(\Phi_j, \Psi_{n-j})\right]^R_{-k}$$

where $\Psi^\pm_{n,k}$ denotes the component of Ψ_n which takes values in $\mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(k))^\pm$. Therefore, subject to the compatibility condition

$$(8.17) \qquad \left[\sum_{0<j<n} Q(\Phi_j, \Psi_{n-j})\right]^R_{n+2} = 0$$

equation (8.9) allows one to compute Ψ_n modulo $\Psi^-_{n,n}$ from Φ and $\Psi_1, \dots, \Psi_{n-1}$.

To handle the compatibility condition (8.17), observe that by virtue of equation (8.10),

$$\mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}(n)) = \bigoplus_{\epsilon=-1}^{1} \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}(n))^\epsilon$$

where $\mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}(n))^\epsilon$ is highest weight n with respect to the left action of $sl_2(\mathbb{C})$ on $\mathfrak{g}_{\mathbb{C}}$ and highest weight $n+2\epsilon$ with respect to the diagonal action.

Lemma 8.18. *Let $C \in \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}(r))^{-1}$ and $B \in \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(s))^{-}$. Then,*

$$Q(C,B) \in \bigoplus_{|r-s|\le t\le r+s-2,\, t\equiv r+s \mod 2} \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(t)).$$

Proof. By equation (8.10) and the Jacobi identity,

$$[\mathfrak{g}(r), \mathfrak{g}(s)] \subseteq \bigoplus_{|r-s|\le t\le r+s,\, t= r+s \mod 2} \mathfrak{g}(t).$$

Therefore, it suffices to show that $Q(C, B)$ projects trivially onto $\mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(r+s))$. Direct calculation shows that every irreducible submodule of $\mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}(r))^{-1}$ is isomorphic to $\mathrm{span}(c_0, \dots, c_{r-2})$ where

$$c_k(\mathfrak{n}_0) = e^{r-k-2} f^{k+2}, \quad c_k(\mathfrak{h}) = 2e^{r-k-1} f^{k+1}, \quad c_k(\mathfrak{n}_0^+) = -e^{r-k} f^k.$$

Accordingly, by the semisimplicity of $sl_2(\mathbb{C})$, it is sufficient to show that

$$Q(c_k, b_j) = 0 \mod \bigoplus_{t\le r+s-2} \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(t)).$$

Consider $Q(c_k, b_j)(e)$:

$$\begin{aligned} (8.19)\quad Q(c_k, b_j)(e) &= 2[c_k(\mathfrak{n}_0^+), b_j(f)] + [c_k(\mathfrak{h}), b_j(e)] \\ &= -2[e^{r-k} f^k, e^{s-j-1} f^{j+1}] + 2[e^{r-k-1} f^{k+1}, e^{s-j} f^j] \\ &\in E_{r+s-2k-2j-2}(\mathfrak{h}). \end{aligned}$$

Suppose that $Q(c_k, b_j)(e)$ projects non-trivially onto $\mathfrak{g}(r+s)$. Then, by (8.19), $\mathfrak{n}_0^{r+s-j-k-1}.Q(c_k, b_j)(e) \neq 0$. But,

$$\begin{aligned} &\mathfrak{n}_0^{r+s-j-k-1}.[e^{r-k} f^k, e^{s-j-1} f^{j+1}] \\ &= \binom{r+s-j-k-1}{r-k} [\mathfrak{n}_0^{r-k}.e^{r-k} f^k, \mathfrak{n}_0^{s-j-1}.e^{s-j-1} f^{j+1}] \\ &= \binom{r+s-j-k-1}{r-k} [(r-k)! f^r, (s-j-1)! f^s] \\ &= (r+s-j-k-1)! [f^r, f^s]. \end{aligned}$$

Likewise, $\mathfrak{n}_0^{r+s-j-k-1}.[e^{r-k-1} f^{k+1}, e^{s-j} f^j] = (r+s-j-k-1)![f^r, f^s]$. Combining these two equations with (8.19), it then follows that $Q(c_k, b_j)(e)$ projects trivially onto $\mathfrak{g}(r+s)$. Similarly, one finds that $Q(c_k, b_j)(f)$ projects trivially onto $\mathfrak{g}(r+s)$, thereby proving the lemma. q.e.d.

Theorem 8.20. *For any choice of a collection of morphisms of Hodge structure*

$$S_n \in \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(n))^-, \qquad n > 0$$

there exists a unique, convergent $\mathfrak{h}$*-valued series solution* $\Psi = \sum_{n>0} \Psi_n y^{-1-n/2}$ *of equation* (8.5) *which is horizontal with respect to* F_o *such that*

(a) $\Psi_n \in \oplus_{r\leq n,\, r\equiv n \mod 2} \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(r))$*;*
(b) $\Psi^-_{n,n} = S_n$*;*
(c) $\Psi^+_{n,n} = 0$.

Proof. The desired function Ψ can now be constructed inductively using equation (8.16). Namely, by Corollary (8.15), we can assume by induction that Ψ_m satisfies conditions (a)–(c) for $m < n$. Therefore, by Lemma (8.18),

$$\sum_{0<j<n} Q(\Phi_j, \Psi_{n-j}) \in \bigoplus_{t<n,\, t\equiv n \mod 2} \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}(t)) \tag{8.21}$$

since

$$\Phi_k \in \bigoplus_{s\leq k,\, s\equiv k \mod 2} \mathrm{Hom}(sl_2, \mathfrak{g}(s))$$

by [**7**, 6.17]. Consequently, $\sum_{0<j<n} Q(\Phi_j, \Psi_{n-j})$ satisfies the compatibility condition (8.17), and hence we can solve for Ψ_n modulo $\Psi^-_{n,n}$ using equation (8.16). In particular, by equation (8.21) and (8.16), $\Psi^+_{n,n} = 0$. Likewise, $\Psi_{n,k} = 0$ for $n > k$. Thus, given Φ and $\Psi_1, \dots, \Psi_{n-1}$ there exists a unique solution Ψ_n to equations (8.9) which satisfies conditions (a)–(c).

Imposing the condition that $S_n = \Psi^-_{n,n}$ be a morphism of Hodge structure, it then follows from [**7**, 6.47] and equation (8.16) that Ψ_n is horizontal and takes values in $\mathfrak{h}$.

To prove that the formal series solution

$$\Psi(y) = \sum_{n\geq 0} \Psi_n y^{-1-n/2}$$

constructed above converges about $y = \infty$, recall that $\mathfrak{g}_{\mathbb{C}}$ is a subalgebra of $gl(V_{\mathbb{C}})$ and let $\| * \|$ be norm on $gl(V_{\mathbb{C}})$ such that $\|AB\| \leq \|A\|\|B\|$. Define

$$\|A\|_1 = 4(\|A(x^+)\| + \|A(x^-)\| + \|A(\mathfrak{z})\|) \qquad A \in \mathrm{Hom}(sl_2(\mathbb{C}), \mathfrak{g}_{\mathbb{C}})$$
$$\|B\|_2 = \|B(\nu_+)\| + \|B(\nu_-)\| \qquad B \in \mathrm{Hom}(\mathfrak{U}, \mathfrak{g}_{\mathbb{C}}).$$

Then, a short calculation shows that

$$\|Q(A, B)\|_2 \leq \|A\|_1 \|B\|_2.$$

Therefore, by equation (8.9),

$$(n+2)\|\Psi_n\|_2 \leq \|\Phi_0\|_1 \|\Psi_n\|_2 + \sum_{0<j<n} \|\Phi_j\|_1 \|\Psi_{n-j}\|_2$$

and hence

$$(n-1)\|\Psi_n\|_2 \leq \sum_{0<j<n} \|\Phi_j\|_1 \|\Psi_{n-j}\|_2 \tag{8.22}$$

upon rescaling $\| * \|$ so that $\|\Phi_0\|_1 = 3$.

To continue, we note that since $\mathfrak{g}_{\mathbb{C}}$ is finite dimensional, there exists an integer m such that $\mathfrak{g}(n) = 0$ for $n > m$. Consequently, $S_n = 0$ for $n > m$ and hence

$$\max_k \|S_k\|_2$$

is finite. Therefore, there exists a constant D such that[3]

$$\|\Psi_\ell\|_2 \leq D^\ell (\max_k \|S_k\|_2)^\ell \tag{8.23}$$

for $\ell \leq m$. Similarly, by [**7**, 6.24] there exists a constant C such that

$$\|\Phi_\ell\|_1 \leq C^\ell (\max_k \|T_k\|_1)^\ell.$$

Assume by induction that (8.23) holds for $\ell < n$, and enlarge D if necessary so that

$$D(\max_k \|S_k\|_2) \geq C(\max_k \|T_k\|_1).$$

Then, by equation (8.22),

$$\begin{aligned}(n-1)\|\Psi_n\|_2 &\leq \sum_{0<j<n} \|\Phi_j\|_1 \|\Psi_{n-j}\|_2 \\ &\leq \sum_{0<j<n} C^j (\max_k \|T_k\|_1)^j D^{n-j} (\max_k \|S_k\|_2)^{n-j} \\ &\leq \sum_{0<j<n} D^n (\max_k \|S_k\|_2)^n = (n-1) D^n (\max_k \|S_k\|_2)^n.\end{aligned}$$

Therefore,

$$\|\Psi_n\|_2 \leq D^n (\max_k \|S_k\|_2)^n$$

for all n, and hence the series $\sum_{n\geq 0} \Psi_n y^{-1-n/2}$ converges on some interval (a, ∞). q.e.d.

Invoking Theorem (7.1), we now obtain an H-valued function $h(y)$ such that

$$h^{-1}\frac{dh}{dy} = -\mathrm{L}\,\beta(y) = -\frac{1}{2}\Phi(\mathrm{h}) - \Psi(e). \tag{8.24}$$

Following [**7**], let $H = \Phi_0(\mathrm{h})$ and $g(y)$ be the H-valued function defined by the equation

$$h(y) = g(y) y^{-H/2}. \tag{8.25}$$

[3] In the degenerate case $\max_k \|S_k\|_2 = 0$ all $S_k = 0$ and hence $\Psi = 0$ by Theorem (8.20).

Then,

$$\left[g^{-1}\frac{dg}{dy}\right]_0^{\operatorname{ad} Y} = -\frac{1}{2}y^{-H/2}.(\Phi(\mathfrak{h}) - \Phi_0(\mathfrak{h})y^{-1}) \tag{8.26}$$

$$\left[g^{-1}\frac{dg}{dy}\right]_{-1}^{\operatorname{ad} Y} = -y^{-H/2}.\Psi(e)$$

where $Y = Y_{(F_o,W)}$.

Theorem 8.27. $g^{-1}(dg/dy) = \sum_{m\geq 2} B_m y^{-m}$.

Proof. Due to the short length of W,

$$g^{-1}\frac{dg}{dy} = \left[g^{-1}\frac{dg}{dy}\right]_0^{\operatorname{ad} Y} + \left[g^{-1}\frac{dg}{dy}\right]_{-1}^{\operatorname{ad} Y}.$$

Therefore, since Φ is isomorphic via the grading Y with the corresponding function defined by nilpotent orbits of pure Hodge structure induced by $\theta(z)$ on Gr^W, it then follows from [**7**, 6.30] that

$$\left[g^{-1}\frac{dg}{dy}\right]_0^{\operatorname{ad} Y} = \sum_{m\geq 2} [B_m]_0^{\operatorname{ad} Y} y^{-m}$$

where

$$[B_m]_0^{\operatorname{ad} Y} = -\frac{1}{2}\sum_{n\geq m} [\Phi_n(\mathfrak{h})]_{2(m-1)-n}^{\operatorname{ad} H}.$$

To establish that $[g^{-1}\frac{dg}{dy}]_{-1}^{Y}$ is also of this form, observe that by (8.26):

$$\begin{aligned}\left[g^{-1}\frac{dg}{dy}\right]_{-1}^{Y} &= -y^{-H/2}.\Psi(e) = -y^{-H/2}.\left(\sum_{n>0}\Psi_n(e)\, y^{-1-n/2}\right) \\ &= -y^{-H/2}.\left(\sum_{n>0}\sum_{r=0}^{n}[\Psi_n(e)]_{n-2r}^{H} y^{-1-n/2}\right) \\ &= -\sum_{n>0}\sum_{r=0}^{n}[\Psi_n(e)]_{n-2r}^{H}\, y^{-1-n+r}.\end{aligned} \tag{8.28}$$

However, by the description of the irreducible submodules B of $\operatorname{Hom}(\mathfrak{U}, \mathfrak{g}(n))^-$ presented in Calculation (8.12), $[\Psi_n(e)]_{-n}^{H} = 0$ and hence equation (8.28) reduces to

$$\left[g^{-1}\frac{dg}{dy}\right]_{-1}^{Y} = -\sum_{n>0}\sum_{r=0}^{n-1}[\Psi_n(e)]_{n-2r}^{H}\, y^{-1-n+r} = \sum_{m\geq 2}[B_m]_{-1}^{\operatorname{ad} Y} y^{-m}$$

where

$$[B_m]_{-1}^{\operatorname{ad} Y} = -\sum_{n\geq m-1}[\Psi_n(e)]_{2(m-1)-n}^{\operatorname{ad} H}. \tag{8.29}$$

q.e.d.

Corollary 8.30. *The functions $g(y)$ and $g^{-1}(y)$ have convergent Taylor expansions about $y = \infty$ of the form*

$$g(y) = g(\infty)(1 + g_1 y^{-1} + g_2 y^{-2} + \cdots),$$
$$g^{-1}(y) = (1 + f_1 y^{-1} + f_2 y^{-2} + \cdots)g^{-1}(\infty)$$

where $g(\infty)$ is an arbitrary element of H determined by the initial value of $h(y)$. Moreover, the coefficients g_n and f_n can be expressed as universal non-commutative polynomials in the B_k with rational coefficients, weighted homogeneous of degree n when B_k when B_k is assigned weight $k-1$. B_{n+1} occurs with coefficient $-1/n$ in g_n and with coefficient $1/n$ in the case of f_n.

Proof. See Lemma (6.32) in [**7**]. q.e.d.

Calculation 8.31. $\mathfrak{n}_0^k.B_k = 0$.

Proof. That $\mathfrak{n}_0^k.[B_k]_0^Y = 0$ is shown in [**7**, 6.32]. Moreover, by (8.29):

$$[B_k]_{-1}^Y = -\sum_{n\geq k-1} [\Psi_n(e)]_{2(k-1)-n}^H$$

and hence

$$\mathfrak{n}_0^k.[B_k]_{-1}^Y = -\sum_{n\geq k-1} \mathfrak{n}_0^k.[\Psi_n(e)]_{2(k-1)-n}^H = 0$$

since $\Psi_n(e)$ takes values in $\oplus_{r\leq n}\,\mathfrak{g}(r)$. q.e.d.

Corollary 8.32. $\mathfrak{n}_0^{k+1}.g_k = \mathfrak{n}_0^{k+1}.f_k = 0$.

Proof. By Corollary (8.30), g_k and f_k are homogeneous polynomials of degree k in $B_2, \dots, B_{k+1}$ with respect to the grading $\deg(B_\ell) = \ell - 1$. Therefore, by virtue of Calculation (8.31) and Leibniz rule, both $\mathfrak{n}_0^{k+1}.g_k$ and $\mathfrak{n}_0^{k+1}.f_k = 0$. q.e.d.

Theorem 8.33. *Let $\beta(y) = \Phi(\mathfrak{n}_0) + \Psi(f)$ denote the solution equation (7.2) constructed above, and $e^{z\tilde{N}}.\tilde{F}$ be the associated nilpotent orbit defined by Theorem (7.1). Then, $\tilde{N}$ coincides with $N_0 = \Phi_0(\mathfrak{n}_0)$ if and only if $g(\infty) \in \ker(\operatorname{ad} N_0)$.*

Proof. By definition,

$$\tilde{N} = h(y).\beta(y) = h(y).([\beta(y)]_0^Y + [\beta(y)]_{-1}^Y) = h(y).[\beta(y)]_0^Y + h(y).\psi(f).$$

Moreover, since $\Psi_0 = 0$,

$$\begin{aligned} &y^{-H/2}.\Psi(f) \\ &= y^{-H/2}.\left(\sum_{n>0}\sum_{r=0}^{n} [\Psi_n(f)]_{n-2r}^H\, y^{-1-n/2}\right) \\ &= \sum_{n>0}\sum_{r=0}^{n} [\Psi_n(f)]_{n-2r}^H\, y^{-1-n+r} = \{\cdots\}y^{-1} + \{\cdots\}y^{-2} + \cdots . \end{aligned}$$

Thus, making use of the calculations of [**7**], we have

$$\begin{aligned}\tilde{N} &= h(y).[\beta(y)]_0^Y + \{\cdots\}y^{-1} + \{\cdots\}y^{-2} + \cdots \\ &= g(y)y^{-H/2}.[\beta(y)]_0^Y + \{\cdots\}y^{-1} + \{\cdots\}y^{-2} + \cdots \\ &= g(y).(N_0 + \{\cdots\}y^{-1} + \{\cdots\}y^{-2} + \cdots) \\ &\quad + \{\cdots\}y^{-1} + \{\cdots\}y^{-2} + \cdots \\ &= g(\infty).N_0 + \{\cdots\}y^{-1} + \{\cdots\}y^{-2} + \cdots\end{aligned}$$

and hence $\tilde{N} = g(\infty).N_0$. q.e.d.

To connect previous constructions with Theorem (4.2), let us now suppose that $\theta(z) = e^{zN}.F$ is an admissible nilpotent orbit of type (I), and let

$$\hat{\theta}(z) = e^{zN}.\hat{F}$$

be the associated split orbit obtained by applying the splitting operation

$$(\hat{F}, {}^rW) = (e^{-i\delta}.F, {}^rW)$$

to the limiting mixed Hodge structure of θ. Define

$$F_o = \hat{\theta}(\sqrt{-1}) = e^{iN}.\hat{F} \tag{8.34}$$

and let (N_0, H, N_0^+) be the associated sl_2-triple obtained by application of Theorem (3.16) to $\hat{\theta}$. Set

$$\Phi_0(\mathfrak{n}_0) = N_0, \qquad \Phi(\mathfrak{h}) = H, \qquad \Phi_0(\mathfrak{n}_0^+) = N_0^+ \tag{8.35}$$

and recall that $N_0 = N$ due to the short length of W.

Theorem 8.36. *Let* $\beta(y) = \Phi(\mathfrak{n}_0) + \Psi(f)$ *denote the solution equation to* (7.2) *constructed above, and* $e^{z\tilde{N}}.\tilde{F}$ *be the associated nilpotent orbit obtained from Theorem* (7.1). *Assume that* F_o *and* Φ_0 *are given by equations* (8.34)–(8.35) *and* $g(\infty) \in \ker(\operatorname{ad} N_0)$. *Then,*

$$\tilde{F} = g(\infty)\left(1 + \sum_{k>0} \frac{1}{k!}(-i)^k(\operatorname{ad} N_0)^k\, g_k\right).\hat{F}.$$

Proof. By Theorem (7.1), $h(y).F_0 = e^{iyN_0}.\tilde{F}$. Therefore,

$$\tilde{F} = e^{-iyN_0}h(y).F_o = e^{-iyN_0}g(\infty)\left(1+\sum_{k>0} g_k y^{-k}\right) y^{-H/2}e^{iN_0}.\hat{F}$$

$$= e^{-iyN_0}g(\infty)\left(1+\sum_{k>0} g_k y^{-k}\right) e^{iyN_0}.\hat{F}$$

$$= g(\infty)e^{-iyN_0}\left(1+\sum_{k>0} g_k y^{-k}\right) e^{iyN_0}.\hat{F}$$

$$= g(\infty)\left(e^{-iy\,\mathrm{ad}\,N_0}\left(1+\sum_{k>0} g_k y^{-k}\right)\right).\hat{F}$$

$$= g(\infty)\left(1+\sum_{k>0,j\geq 0} \frac{(-i)^j}{j!}(adN_0)^j g_k y^{j-k}\right).\hat{F}.$$

Moreover, by Corollary (8.32), $(ad\,N_0)^j\, g_k = 0$ whenever $j > k$. Thus,

$$\tilde{F} = g(\infty)\left(1+\sum_{k>0}\sum_{j=0}^{k} \frac{(-i)^j}{j!}(adN_0)^j g_k y^{j-k}\right).\hat{F}_\infty$$

$$= g(\infty)\left(1+\sum_{k>0} \frac{1}{k!}(-i)^k(ad\,N_0)^k\, g_k\right).\hat{F}_\infty$$

$$+\{\cdots\}y^{-1}+\{\cdots\}y^{-2}+\cdots.$$

Accordingly, upon taking the limit as $y\to\infty$ in this last equation we obtain the stated formula for $\tilde{F}$. q.e.d.

Thus, in order to complete the proof of Theorem (4.2) for admissible nilpotent orbits of type (I), it is sufficient to show that we can select morphisms of Hodge structure

$$T_n \in \mathrm{Hom}(sl_2(\mathbb{C}),\mathfrak{g}(n))^{-1}, \qquad S_n \in \mathrm{Hom}(\mathfrak{U},\mathfrak{g}(n))^{-}$$

for $n > 0$ and element $\zeta = \log(g(\infty)) \in \mathfrak{h}\cap\ker(\mathrm{ad}\,N_0)\cap\Lambda^{-1,-1}_{(\hat{F},{}^rW)}$ such that

$$e^{i\delta} = e^{\zeta}\left(1+\sum_{k>0}\frac{1}{k!}(-i)^k(\mathrm{ad}\,N_0)^k\, g_k\right).$$

Theorem 8.37. *Let $\theta(z) = e^{zN}.F$ be an admissible nilpotent orbit of type* (I)*. Then, the solutions $\beta(y)$ of equation* (7.2) *which have the following three properties*

(1) *$\beta(y)$ is horizontal at $F_o = \hat{\theta}(i)$;*
(2) *$\beta(y) = \sum_{n\geq 0}\beta_n y^{-1-n/2}$;*
(3) *$\beta_0 = N_0$;*

are in 1-1 *correspondence with the elements* $\eta \in \mathfrak{h} \cap \ker(\operatorname{ad} N_0) \cap \Lambda^{-1,-1}_{(\hat{F},{}^rW)}$ *via the map*

$$\eta = \sum_{n>0} [\beta_n]^{\operatorname{ad} H}_{-n}.$$

Proof. If $\beta(y)$ satisfies the conditions stated above then so does $[\beta(y)]^{\operatorname{ad} Y}_0$. Therefore, by Lemma (6.41) in [**7**] the map

$$[\eta]^{\operatorname{ad} Y}_0 = \left[\sum_{n>0} [\beta_n]^{\operatorname{ad} H}_{-n}\right]^{\operatorname{ad} Y}_0 = \sum_{n>0} \left[[\beta_n]^{\operatorname{ad} Y}_0\right]^{\operatorname{ad} H}_{-n}$$

determines a bijective correspondence between the morphisms T_n and the elements of $\mathfrak{h} \cap \ker(\operatorname{ad} Y) \cap \ker(\operatorname{ad} N_0) \cap \Lambda^{-1,-1}_{(\hat{F},{}^rW)}$.

To recover the morphisms S_n from $[\eta]^{\operatorname{ad} Y}_{-1}$, observe that since (F_o, W) is split over $\mathbb{R}$,

$$\mathcal{H} = \bigoplus_{r+s=-1} \mathfrak{g}^{r,s}_{(F_o,W)}$$

is a pure Hodge structure of weight -1 with respect to which the representation of $sl_2(\mathbb{C})$ defined by $\operatorname{ad} \Phi_0$ is Hodge. Therefore, by Theorem (3.14) we can decompose $\mathcal{H}$ into a direct sum of irreducible submodules M, each of which is isomorphic to one of the following two standard types

(a) $H(d) \otimes S(n)$, $n = 2d - 1$ odd;
(b) $E^{p,q} \otimes S(n)$, $n + p + q = -1$, $p - q > 0$;

where $S(n) = \operatorname{Sym}^n(\mathbb{C}^2)$ is the standard representation of $sl_2(\mathbb{C})$ of highest weight n equipped with the Hodge structured obtained by declaring

$$\nu_r = (e + if)^r (e - if)^{n-r} \tag{8.38}$$

to be of type $(r, n - r)$, and $H(d) = \mathbb{C}\epsilon^{-d,-d}$ and $E(p,q) = \mathbb{C}\epsilon^{p,q} \oplus \mathbb{C}\epsilon^{q,p}$ are trivial representations of sl_2 equipped with the Hodge structure obtained by requiring $\epsilon^{r,s}$ to type (r, s) and $\overline{\epsilon^{r,s}} = \epsilon^{s,r}$.

Let S^M_n denote the projection of S_n onto such an irreducible module M. Then, a short calculation shows that

$$S^M_n(e + if) = \tau_M \epsilon^{-d,-d} \otimes \nu_d, \qquad S^M_n(e - if) = \tau_M \epsilon^{-d,-d} \otimes \nu_{d-1} \tag{8.39}$$

for some real number τ_M if M is of type (a). Similarly, if M is type (b) then

$$\begin{aligned} S^M_n(e + if) &= \tau_M \epsilon^{p,q} \otimes \nu_{-p} + \bar{\tau}_M \epsilon^{q,p} \otimes \nu_{-q}, \\ S^M_n(e - if) &= \tau_M \epsilon^{p,q} \otimes \nu_{-p-1} + \bar{\tau}_M \epsilon^{q,p} \otimes \nu_{-q-1} \end{aligned} \tag{8.40}$$

where $\tau_M \in \mathbb{C}$, $p, q < 0$ and $p + q + n = -1$.

In particular, if S^M_n is of type (8.40) then

$$2iS^M_n(f) = \tau_M \epsilon^{p,q} \otimes (\nu_{-p} - \nu_{-p-1}) + \bar{\tau}_M \epsilon^{q,p} \otimes (\nu_{-q} - \nu_{q-1}).$$

Moreover, for any index $0 \leq k \leq n$,

$$\begin{aligned} \nu_k - \nu_{k-1} &= (e+if)^k(e-if)^{n-k} - (e+if)^{k-1}(e-if)^{n-k+1} \\ &= i^k(-i)^{n-k}f^n - i^{k-1}(-i)^{n-k+1}f^n + e(\cdots) \\ &= (2i)i^{2k-n-1}f^n + e(\cdots). \end{aligned}$$

Accordingly, using the identity $p+q+n=-1$, it then follows that

$$[\beta_n^M]_{-n}^{\mathrm{ad}\,H} = [S_n^M(f)]_{-n}^{\mathrm{ad}\,H} = (-i)^\chi \tau_M \epsilon^{p,q} \otimes f^n + i^\chi \bar{\tau}_M \epsilon^{q,p} \otimes f^n \tag{8.41}$$

where $\chi = p - q$. Similarly, if S_n^M is of type (8.39) then

$$[\beta_n^M]_{-n}^{\mathrm{ad}\,H} = \tau_M \epsilon^{-d,-d} \otimes f^n. \tag{8.42}$$

Therefore, the sum

$$[\eta]_{-1}^{\mathrm{ad}\,Y} = \sum_M \eta^M = \sum_{n>0} \sum_M [\beta_n^M]_{-n}^{\mathrm{ad}\,H} \tag{8.43}$$

determines τ_M for all M.

To verify that the sum (8.43) takes values in $\Lambda^{-1,-1}_{(\hat{F},{}^rW)}$, suppose that S_n^M is of type (8.40) and observe that

$$e^{iN_0}.(\epsilon^{p,q} \otimes e^n) = \epsilon^{p,q} \otimes \nu_n \in \mathfrak{g}^{n+p,q}_{(F_o,W)}$$

and hence

$$\begin{aligned} \{e^{iN_0}.(\epsilon^{p,q} \otimes e^n)\}(F_o^r) &= e^{iN_0}(\epsilon^{p,q} \otimes e^n)e^{-iN_0}e^{iN_0}.\hat{F}^r \\ &= e^{iN_0}(\epsilon^{p,q} \otimes e^n).\hat{F}^r \subseteq e^{iN_0}.\hat{F}^{n+p+r}. \end{aligned}$$

Therefore,

$$(\epsilon^{p,q} \otimes e^n)(\hat{F}^r) \subseteq \hat{F}^{n+p+r}. \tag{8.44}$$

Furthermore, by Theorem (3.16),

$$H = {}^rY - Y$$

where ${}^rY$ is the grading of ${}^rW$ defined by the $I^{p,q}$'s of $(\hat{F},{}^rW)$ and Y is the grading of W defined by the $I^{p,q}$'s of (F_o,W). Consequently, the condition that $\epsilon^{p,q} \otimes e^n$ be of weight n with respect to H and weight -1 with respect to Y implies that

$$\epsilon^{p,q} \otimes e^n \in \bigoplus_t \mathfrak{g}^{t,n-1-t}_{(\hat{F},{}^rW)}.$$

Imposing the condition (8.44), it then follows that

$$\epsilon^{p,q} \otimes e^n \in \bigoplus_{t \geq n+p} \mathfrak{g}^{t,n-1-t}_{(\hat{F},{}^rW)}. \tag{8.45}$$

Likewise, switching the roles of p and q,

$$\epsilon^{q,p} \otimes e^n \in \bigoplus_{s \geq n+q} \mathfrak{g}^{s,n-1-s}_{(\hat{F},{}^rW)}. \tag{8.46}$$

Thus, since $\overline{\epsilon^{q,p}\otimes e^n} = \epsilon^{p,q}\otimes e^n$ and $(\hat{F},{}^rW)$ is split over $\mathbb{R}$, equations (8.45) and (8.46) imply that the Hodge components

$$(\epsilon^{p,q}\otimes e^n)^{t,n-1-t}$$

of $\epsilon^{p,q}\otimes e^n$ with respect to $(\hat{F},{}^rW)$ vanish unless

$$t = n-1-s, \qquad t \geq n+p, \qquad s \geq n+q. \tag{8.47}$$

Recalling that $p+q+n=-1$, it then follows from equation (8.47) that

$$(\epsilon^{p,q}\otimes e^n)^{t,n-1-t} = 0$$

unless $t = n+p$. Accordingly, since N_0 is a $(-1,-1)$-morphism of $(\hat{F},{}^rW)$,

$$\epsilon^{p,q}\otimes f^n = (N_0)^n.(\epsilon^{p,q}\otimes e^n) \in \mathfrak{g}^{p,q}_{(F_o,{}^rW)}. \tag{8.48}$$

Now, by equation (8.40), $p,q<0$. Therefore, by equation (8.41) and (8.48),

$$[S_n^M]^{\mathrm{ad}H}_{-n} \in \mathfrak{g}^{p,q}_{(\hat{F},{}^rW)} \oplus \mathfrak{g}^{q,p}_{(\hat{F},{}^rW)} \subseteq \Lambda^{-1,-1}_{(\hat{F},{}^rW)}.$$

Similarly, if S_n^M is of type (8.39) then

$$[S_n^M]^{\mathrm{ad}H}_{-n} \in \mathfrak{g}^{-d,-d}_{(\hat{F},{}^rW)} \subseteq \Lambda^{-1,-1}_{(\hat{F},{}^rW)}.$$

q.e.d.

Following [**7**], we now note that by virtue of Corollary (8.30)

$$1+\sum_{k>0}\frac{1}{k!}(-i)^k(\mathrm{ad}\,N_0)^k\,g_k = \exp\left(\sum_{k>0}Q_k(C_2,\dots,C_{k+1})\right) \tag{8.49}$$

where $C_{\ell+1} = \frac{(-i)^\ell}{\ell!}(\mathrm{ad}\,N_0)^\ell B_{\ell+1}$.

Calculation 8.51. Let $(1-x)^r(1+x)^s = \sum_t b^t_{r,s}x^t$. Then,

$$[C_{\ell+1}]^{\mathrm{ad}Y}_0 = i\sum_{p,q\geq 1,p+q\geq\ell+1} b^{\ell-1}_{p-1,q-1}([\eta]^{\mathrm{ad}Y}_0)^{-p,-q}$$

where $([\eta]^{\mathrm{ad}Y}_0)^{-p,-q}$ denotes the component of $[\eta]^{\mathrm{ad}Y}_0$ of type $(-p,-q)$ with respect to $(\hat{F},{}^rW)$.

Proof. See Lemma (6.60) in [**7**]. q.e.d.

Calculation 8.52.
$[C_{\ell+1}]^{\mathrm{ad}Y}_{-1} = i\sum_{p,q\geq 1,p+q\geq\ell+1} b^{\ell-1}_{p-1,q-1}([\eta]^{\mathrm{ad}Y}_{-1})^{-p,-q}$.

Proof. By equation (8.29),

$$
\begin{aligned}
[C_{\ell+1}]^{\operatorname{ad} Y}_{-1} &= -\frac{(-i)^\ell}{\ell!}(\operatorname{ad} N_0)^\ell \sum_{n\geq\ell}[\Psi_n(e)]^{\operatorname{ad} H}_{2\ell-n} \\
&= -\frac{(-i)^\ell}{\ell!}\sum_{n\geq\ell}(\operatorname{ad} N_0)^\ell[S_n(e)]^{\operatorname{ad} H}_{2\ell-n} \\
&= -\frac{(-i)^\ell}{\ell!}\sum_{n\geq\ell}\sum_{M}(\operatorname{ad} N_0)^\ell[S^M_n(e)]^{\operatorname{ad} H}_{2\ell-n}
\end{aligned}
\tag{8.53}
$$

where $S_n = \sum_M S^M_n$ denotes the decomposition of S_n into irreducible components of type (8.39) and (8.40).

Now, for any index $0\leq k\leq n$,

$$
\begin{aligned}
\nu_k &= (e+if)^k(e-if)^{n-k} = (i(f-ie))^k((-i)(f+ie))^{n-k} \\
&= i^{2k-n}(f-ie)^k(f+ie)^{n-k} = i^{2k-n}\sum_t i^t b^t_{k,n-k}e^t f^{n-t}.
\end{aligned}
\tag{8.54}
$$

Therefore, if S^M_n is of type (8.40) then

$$
\begin{aligned}
&[S^M_n(e)]^{\operatorname{ad} H}_{2\ell-n} \\
&= \frac{1}{2}\tau_M\epsilon^{p,q}\otimes[\nu_{-p}+\nu_{-p-1}]^{\operatorname{ad} H}_{2\ell-n} + \frac{1}{2}\bar{\tau}_M\epsilon^{q,p}\otimes[\nu_{-q}+\nu_{-q-1}]^{\operatorname{ad} H}_{2\ell-n} \\
&= \frac{1}{2}\tau_M\epsilon^{p,q}\otimes\left(i^{-2p-n+\ell}b^\ell_{-p,n+p} + i^{-2p-2+n+\ell}b^\ell_{-p-1,n+p+1}\right)e^\ell f^{n-\ell} \\
&\quad + \frac{1}{2}\bar{\tau}_M\epsilon^{q,p}\otimes\left(i^{-2q-n+\ell}b^\ell_{-q,n+q} + i^{-2q-2+n+\ell}b^\ell_{-q-1,n+q+1}\right)e^\ell f^{n-\ell} \\
&= \frac{1}{2}i^{1+\ell-\chi}\tau_M\epsilon^{p,q}\otimes\left(b^\ell_{-p,-q-1} - b^\ell_{-p-1,-q}\right)e^\ell f^{n-\ell} \\
&\quad + \frac{1}{2}i^{1+\ell+\chi}\bar{\tau}_M\epsilon^{q,p}\otimes\left(b^\ell_{-q,-p-1} - b^\ell_{-q-1,-p}\right)e^\ell f^{n-\ell}
\end{aligned}
$$

where $\chi = p-q$ [recall: p+q+n=-1]. To simplify the above expression, observe that

$$
\begin{aligned}
&\sum_t(b^t_{k,n-k} - b^t_{k-1,n-k+1})x^t \\
&= (1-x)^k(1+x)^{n-k} - (1-x)^{k-1}(1+x)^{n-k+1} \\
&= (1-x)^{k-1}(1+x)^{n-k}((1-x)-(1+x)) \\
&= (-2x)(1-x)^{k-1}(1+x)^{n-k} \\
&= (-2x)\sum_t b^t_{k-1,n-k}x^t
\end{aligned}
$$

and hence

$$b^{\ell}_{-p,-q-1} - b^{\ell}_{-p-1,-q} = -2b^{\ell-1}_{-p-1,-q-1}$$
$$b^{\ell}_{-q,-p-1} - b^{\ell}_{-q-1,-p} = -2b^{\ell-1}_{-q-1,-p-1}.$$

Accordingly,

$$[S^M_n(e)]^{\mathrm{ad}\,H}_{2\ell-n} = -b^{\ell-1}_{-p-1,-q-1}(i^{1+\ell-\chi}\tau_M\epsilon^{p,q}\otimes e^{\ell}f^{n-\ell}) - b^{\ell-1}_{-q-1,-p-1}(i^{1+\ell+\chi}\bar{\tau}_M\epsilon^{q,p}\otimes e^{\ell}f^{n-\ell}). \tag{8.55}$$

Inserting (8.55) into equation (8.53) it then follows by equation (8.48) that

$$\begin{aligned} C^M_{\ell+1} &= ib^{\ell-1}_{-p-1,-q-1}i^{-\chi}\tau_M\epsilon^{p,q}\otimes f^n + ib^{\ell-1}_{-p-1,-q-1}i^{\chi}\bar{\tau}_M\epsilon^{q,p}\otimes f^n \\ &= ib^{\ell-1}_{-p-1,-q-1}(\eta^M)^{p,q} + ib^{\ell-1}_{-q-1,-p-1}(\eta^M)^{q,p}. \end{aligned} \tag{8.56}$$

Similarly, if S^M_n is of type (8.39) then

$$C^M_{\ell+1} = ib^{\ell-1}_{d-1,d-1}(\eta^M)^{-d,-d}. \tag{8.57}$$

Thus, combining equations (8.56) and (8.57) and switching the signs of p and q, we obtain the formula:

$$[C_{\ell+1}]^{\mathrm{ad}\,Y}_{-1} = \sum_M C^M_{\ell+1} = i\sum_{p,q\geq 1,p+q\geq \ell+1} b^{\ell-1}_{p-1,q-1}([\eta]^{\mathrm{ad}\,Y}_{-1})^{-p,-q}.$$

q.e.d.

In particular, by virtue of Calculations (8.51) and (8.52),

$$C_{\ell+1} = i\sum_{p,q\geq 1,p+q\geq \ell+1} b^{\ell-1}_{p-1,q-1}\eta^{-p,-q}.$$

Therefore, since $C_{\ell+1}$ is of the same algebraic form as in Lemma (6.60) of [**7**], we can use this result verbatim to prove that given $\delta\in\mathfrak{h}\cap\ker(N)\cap\Lambda^{-1,-1}_{(\hat{F},{}^rW)}$ we can find unique elements $\zeta,\ \eta\in\mathfrak{h}\cap\ker(N)\cap\Lambda^{-1,-1}_{(\hat{F},{}^rW)}$ such that

$$e^{i\delta} = e^{\zeta}\left(1+\sum_{k>0}\frac{1}{k!}(-i)^k(\mathrm{ad}\,N_0)^k\,g_k\right).$$

By the above remarks, this completes the proof of Theorem (4.2) for admissible orbits of type (I).

9. Nilpotent Orbits of Type (II)

Suppose now that $\theta(z) = e^{zN}.F$ is an admissible nilpotent orbit of type (II) and let $\hat{\theta}(z) = e^{zN}.\hat{F}$ be the associated split orbit. Then, application of Theorem (3.16) to $\hat{\theta}(z)$ defines a corresponding splitting

$$N = N_0 + N_2 \tag{9.1}$$

of N such that $\hat{\theta}_0(z) = e^{zN_0}.\hat{F}$ is an SL_2-orbit. Consequently,

$$F_o = \hat{\theta}_0(i) \in \mathcal{M}_{\mathbb{R}}.$$

Furthermore, since $\theta(z)$ is of type (II), the Hodge decomposition of the associated function $\beta(y) = \mathrm{Ad}\,(h^{-1}(y))N$ defined by Theorem (6.11) is of the form

$$\beta(y) = \beta^{1,-1} + \beta^{0,0} + \beta^{-1,1} + \beta^{0,-1} + \beta^{-1,0} + \beta^{-1,-1}. \tag{9.2}$$

As in §7–8, the first five components of the right hand side of equation (9.2) are governed by the system of differential equations

$$-8\Phi' = Q(\Phi,\Phi), \qquad -2\Psi' = Q(\Phi,\Psi).$$

Therefore, as in §7–8, we can formally solve for these components starting from a collection of morphisms of Hodge structures

$$T_n : sl_2(\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}, \qquad S_n : \mathfrak{U} \to \mathfrak{g}_{\mathbb{C}}.$$

To solve for $\beta^{-1,-1}$, we now return to equation (6.41), which implies that

$$\frac{d}{dy}\beta^{-1,-1} = i[\beta^{0,0},\beta^{-1,-1}] + 2i[\beta^{0,-1},\beta^{-1,0}]. \tag{9.3}$$

Next, we recall that since $\theta(z)$ is of type (II) there exists an index k such that the Hodge decomposition of (F_o,W) is of the form

$$V_{\mathbb{C}} = I^{k,k} \oplus \left(\bigoplus_{p+q=2k-1} I^{p,q}\right) \oplus I^{k-1,k-1} \tag{9.4}$$

for some index k. Therefore, since Gr^W_{2k} and Gr^W_{2k-2} are of pure type (k,k) and $(k-1,k-1)$ it then follows from [**7**] that Φ acts trivially $I^{k,k}$ and $I^{k-1,k-1}$. Consequently, $[\beta^{0,0},\beta^{-1,-1}] = 0$ since $\beta^{-1,-1}$ maps $I^{k,k}$ to $I^{k-1,k-1}$ and annihilates the remaining summands appearing in (9.4). Thus, equation (9.3) simplifies to

$$\frac{d}{dy}\beta^{-1,-1} = 2i[\beta^{0,-1},\beta^{-1,0}]$$

and hence

$$\beta^{-1,-1} = \mu + 2i\int [\beta^{0,-1},\beta^{-1,0}]\,dy. \tag{9.5}$$

Remark. The assertion that Φ must act trivially on Gr^W_{2k} and Gr^W_{2k-2} is a simple consequence of the fact that Φ_0 must be a morphism of Hodge structure, and hence $\Phi_0(x^-)$, $\Phi_0(x^+)$ must be of type $(-1,1)$ and $(1,-1)$ respectively. Therefore, the purity of Gr^W_{2k} and Gr^W_{2k-2} implies that Φ_0 must act trivially. As such, the equation $-8\Phi' = Q(\Phi,\Phi)$ then implies that all of the higher coefficients of Φ must also act trivially Gr^W_{2k} and Gr^W_{2k-2}. In particular, N_0 and H commute with every element of $\Lambda^{-1,-1}_{(F_o,W)} = \mathrm{Lie}_{-2}(W)$.

To continue, we now observe that by (6.20) we know that if $\theta(z)$ was a split orbit then the associated function $h(y)$ defined by Theorem (6.11) would be given by the formula

$$h(y) = e^{iyN}e^{-iyN_0}y^{-H/2} = e^{iyN_{-2}}y^{-H/2}.$$

Accordingly, when $\theta(z)$ is not split we shall write

$$h(y) = g(y)e^{iyN_{-2}}y^{-H/2}. \tag{9.6}$$

Therefore, by equation (8.26),

$$g^{-1}(y)\frac{dg}{dy} = y^{-H/2}.\left(-\frac{1}{2}\Phi(\mathfrak{h}) + \frac{H}{2y} - \Psi(e)\right) + i\beta^{-1,-1} - iN_{-2}. \tag{9.7}$$

Setting $\mu = N_{-2}$ it then follows from equations (9.5) and (9.7) that

$$g^{-1}(y)\frac{dg}{dy} = y^{-H/2}.\left(-\frac{1}{2}\Phi(\mathfrak{h}) + \frac{H}{2y} - \Psi(e)\right) - 2\int [\beta^{0,-1}, \beta^{-1,0}]\, dy \tag{9.8}$$

where

$$-2\int [\beta^{0,-1}, \beta^{-1,0}]\, dy = y^{-2}\{\cdots\} + y^{-5/2}\{\cdots\} + \cdots \tag{9.9}$$

since $\beta^{0,-1}$ and $\beta^{-1,0}$ have leading order term $y^{-3/2}$. Combining equations (9.8) and (9.9) with Theorem (8.27) it then follows that

$$g^{-1}(y)\frac{dg}{dy} = \sum_{m\geq 2} B_m y^{-m}.$$

Thus, just as in Corollary (8.30),

$$g(y) = g(\infty)(1 + g_1 y^{-1} + g_2 y^{-2} + \cdots)$$
$$g^{-1}(y) = (1 + f_1 y^{-1} + f_2 y^{-2} + \cdots)g^{-1}(\infty)$$

where $g(\infty)$ is an arbitrary element of H and g_n and f_n can be expressed as universal non-commutative polynomials in the coefficients B_k.

Continuing the analogy with §8, it remains to show that we can select data $(g(\infty), \{T_n\}, \{S_n\})$ such that

$$h(y).F_o = e^{iyN}.F.$$

In particular, the proofs of Theorem (8.33) and (8.36) imply mutatis mutandis that $h(y).F_o = e^{iyN}.\tilde{F}$ where

$$\tilde{F} = g(\infty)\left(1 + \sum_{k>0}\frac{1}{k!}(-i)^k(\operatorname{ad} N_0)^k\, g_k\right).\hat{F} \tag{9.10}$$

provided $g(\infty) \in \ker(\operatorname{ad} N) = \ker(\operatorname{ad} N_0) \cap \ker(\operatorname{ad} N_{-2})$. Furthermore, just as in §8, for purely formal algebraic reasons (cf. [**7**])

$$1 + \sum_{k>0}\frac{1}{k!}(-i)^k(\operatorname{ad} N_0)^k\, g_k = \exp\left(\sum_{k>0} Q_k(C_2, \ldots, C_{k+1})\right) \tag{9.11}$$

where $C_{\ell+1} = \frac{(-i)^\ell}{\ell!}(\operatorname{ad} N_0)^\ell B_{\ell+1}$. Recycling the argument of Calculation (8.52), one then finds that

$$C_{\ell+1} = i \sum_{p,q\geq 1, p+q\geq \ell+1} b^{\ell-1}_{p-1,q-1}\eta^{-p,-q} \tag{9.12}$$

where $\eta = \sum_{n>0} [\beta_n]^{\operatorname{ad} H}_{-n}$ and $\beta(y) = N_{-2} + \sum_{n\geq 0} \beta_n y^{-1-n/2}$ is the series expansion of β.

To complete the proof of Theorem (4.2) for orbits of type (II), observe that since N_0 acts trivially on Gr^W_{2k} and Gr^W_{2k-2}, the corresponding limiting mixed Hodge structure on these graded pieces is also of type (k,k) and $(k-1,k-1)$. Therefore, if we decompose the splitting

$$(F, {}^rW) = (e^{i\delta}.\hat{F}, {}^rW)$$

of the limiting mixed Hodge structure of $\theta(z)$ as

$$\delta = \delta_0 + \delta_{-1} + \delta_{-2}$$

relative to the grading Y defined by application of Theorem (3.16) to $\hat{\theta}(z)$, then δ_0 acts trivially on $I^{k,k}$ and $I^{k-1,k-1}$. Consequently, δ_0 commutes with every element of $\mathrm{Lie}_{-2}(W)$, and hence

$$e^{i\delta} = e^{i\delta_{-2}} e^{i\delta_0 + i\delta_{-1}}.$$

Proceeding as in the last part of §8, we can therefore pick elements η and ζ' so that

$$e^{i\delta_0 + i\delta_{-1}} = e^{\zeta'} \exp\left(\sum_{k>0} Q_k(C_2, \ldots, C_{k+1})\right) \mod \exp(\mathrm{Lie}_{-2}(W)).$$

Accordingly, since H contains $\exp(Lie_{-2}(W))$, we can therefore pick elements η and ζ such that

$$e^{i\delta} = e^{\zeta}\left(1 + \sum_{k>0} \frac{1}{k!}(-i)^k (\operatorname{ad} N_0)^k g_k\right).$$

The remaining details regarding the uniqueness of η and ζ now follow as in §8.

References

[1] S. Arakelov, *Intersection theory of divisors on an arithmetic surface*, Izv. Akad. Nauk. SSSR Ser. Math. **38** (1974) 1167–1180, MR 0472815, Zbl 0355.14002.

[2] A. Beilinson, *Height pairing between algebraic cycles*, Current Trends in Arithmetical Algebraic Geometry, Contemp. Math. **67** (1987) 1–24, MR 0902590, Zbl 0624.14005.

[3] P. Brosnan & G. Pearlstein, *On the asymptotic behavior of the archimedean height*, preprint.

[4] J. Brylinski & S. Zucker, *An overview of recent advances in Hodge theory*, in 'Complex Manifolds' (Gamkrelidze R. ed.), Springer, 1990, 39–142, MR 1095090, Zbl 0793.14005.

[5] P. Candelas, X. de la Ossa, A. Font, S. Katz, & D. Morrison, *Mirror symmetry for two parameter models* I, Nuclear Phys. B**416** (1993) 481–562; II, Nuclear Phys. B**429** (1994) 626–674, MR 1274436, Zbl 0899.14017; MR 1302876, Zbl 0899.14018.

[6] E. Cattani & A. Kaplan, *Degenerating variations of Hodge structure*, Astérisque **179-180** (1986) 67–96, MR 1042802, Zbl 0705.14006.

[7] E. Cattani, A. Kaplan & W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. **123** (1986) 457–535, MR 0840721, Zbl 0617.14005.

[8] ______, *L^2 and intersection cohomologies for a polarizable variation of Hodge structure*, Invent. Math. **87** (1987) 217–252, MR 0870728, Zbl 0611.14006.

[9] A. Ceresole, R. d'Auria, & T. Regge, *Duality group for Calabi-Yau two moduli space*, Nuclear Phys. B**414** (1994) 517–537, MR 1264796, Zbl 1007.32500.

[10] H. Clemens, *Picard–Lefschetz Theorem for families of non-singular algebraic varieties*, TAMS **136** (1969) 93–108, MR 0233814, Zbl 0185.51302.

[11] P. Deligne, *Travaux de Griffiths*, Semin. Bourbaki 1969/70 **376** (1971) 213–237, Zbl 0208.48601.

[12] ______, *Théorie de Hodge* I, Actes, Congrés Intern. Math., Nice, 1971, 425–430; II, Publ. Math. IHES, **40** (1971) 5–58; III, Publ. Math. IHES **44** (1974) 5–77, MR 0441965, Zbl 0219.14006; MR 0498551, Zbl 0219.14007; MR 0498552, Zbl 0237.14003.

[13] ______, *La conjecture de Weil*, II Publ. Math. IHES **52** (1980) 137–252, MR 0601520, Zbl 0456.14014.

[14] ______, *Private communication*, 1995.

[15] F. El Zein, *Théorie de Hodge des cycles évanescents*, Ann. Sci. Ec. Norm. Sup. **19** (1986) 107–184, MR 0860812, Zbl 0538.14003.

[16] H. Gillet & C. Soulé, *Intersection sur les variétés d'Arakelov*, CR Acad. Sci. Paris Sér I Math. **299** (1984) 563–566, MR 0770447, Zbl 0607.14003.

[17] M. Green, *Griffiths' infinitesimal invariant and the Abel-Jacobi map*, Journal of Differential Geometry **29** (1989) 545–555, MR 0992330, Zbl 0692.14003.

[18] P. Griffiths, *Periods of Integrals on algebraic manifolds* I, Amer. J. Math. **90** (1968) 568–626; II, Amer. J. Math. **90** (1968) 805–865; III, Publ. Math. IHES **38** (1970) 125–180, MR 0229641, Zbl 0169.52303; MR 0233825, Zbl 0183.25501; MR 0282990, Zbl 0212.53503.

[19] R. Hain, *Biextensions and heights associated to curves of odd genus*, DMJ **61** (1990) 859–898, MR 1084463, Zbl 0737.14005.

[20] R. Hain & D. Reed, *On the Arakelov Geometry of Moduli Spaces of Curves*, Jour. Differential Geom. **67**(**2**) (2004) 195–228, MR 2153077.

[21] R. Hain & S. Zucker, *Unipotent variations of mixed Hodge structure*, Invent. Math. **88** (1987) 83–124, MR 0877008, Zbl 0622.14007.

[22] H. Hironaka, *Resolutions of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. **79** (1964) 109–326, MR 0199184, Zbl 0122.38603.

[23] N. Hitchin, *On the construction of monopoles*, Comm. Math. Phys. **89** (1983) 145–190, MR 0709461, Zbl 0517.58014.

[24] Kaplan A. *Notes on the moduli spaces of Hodge structures*, preprint, 1995.

[25] A. Kaplan & G. Pearlstein, *Singularities of variations of mixed Hodge structure*, Asian J. Math. **7** (2003) 307–336, MR 2129326.

[26] M. Kashiwara, *A study of variations of mixed Hodge structure*, Puble. RIMS **22** (1986) 991-1024, MR 0866665, Zbl 0621.14007.

[27] D. Lear, *Extensions of normal functions and the asymptotics of the height pairing*, Ph.D. Thesis, University of Washington, 1990.

[28] B. Lian, A. Todorov & S.-T. Yau, *Maximal Unipotent Monodromy for Complete Intersection CY Manifolds*, Amer. J. Math. **127** (2005) 1–50, MR 2115660, Zbl 1067.14035.

[29] G. Pearlstein, *The geometry of the Deligne–Hodge decomposition*, Ph.D. thesis, University of Massachusetts, Amherst, 1999.

[30] ———, *Variations of mixed Hodge structure, Higgs fields and quantum cohomology*, Manuscripta Math. **102** (2000) 269–310, MR 1777521, Zbl 0973.32008.

[31] ———, *Degenerations of mixed Hodge structure*, DMJ **110** (2001) 217–251, MR 1865240.

[32] M. Saito, *Mixed Hodge modules and admissible variations*, CRAS Paris **309** (1989) 351–356, MR 1054250, Zbl 0765.14006.

[33] ———, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990) 221–333, MR 1047415, Zbl 0727.14004.

[34] W. Schmid, *Variation of Hodge structure: The singularities of the period mapping*, Invent. Math. **22** (1973) 211–319, MR 0382272, Zbl 0278.14003.

[35] C. Schwarz, *Relative monodromy weight filtrations*, Math. Zeit. **236** (2001) 11–21, MR 1812447, Zbl 1005.14003.

[36] J. Steenbrink, *Limits of Hodge structure*, Invent. Math. **31** (1975) 229–257, MR 0429885, Zbl 0303.14002.

[37] J. Steenbrink & S. Zucker, *Variation of mixed Hodge structure* I, Invent. Math. **80** (1985) 489–542, MR 0791673, Zbl 0626.14007.

[38] S. Usui, *Variation of mixed Hodge structure arising from a family of logarithmic deformations*, Ann. Sci. École Norm. Sup. **16** (1983) 91–107; II. *Classifying space*, DMJ **51** (1984) 851–875; *Supplement to Part* II [with Saito M.and Shimizu Y.], Duke Math. Jour. *52* (1985) 529–534, MR 0719764, Zbl 0516.14006; MR 0771384, Zbl 0558.14005; MR 0792187, Zbl 0593.14007.

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