

## MINIMAL LAGRANGIAN SUBMANIFOLDS IN INDEFINITE COMPLEX SPACE

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ABSTRACT. Consider the complex linear space endowed with the canonical pseudo-Hermitian form of arbitrary signature. This yields both a pseudo-Riemannian and a symplectic structure. We prove that those submanifolds which are both Lagrangian and minimal with respect to these structures minimize the volume in their Lagrangian homology class. We also describe several families of minimal Lagrangian submanifolds. In particular, we characterize the minimal Lagrangian surfaces in pseudo-Euclidean complex plane endowed with its natural neutral metric and the equivariant minimal Lagrangian submanifolds of indefinite complex space with arbitrary signature.

### Introduction

It has been discovered in the seminal paper of Harvey and Lawson [HL1] (see also [Ha]) that a minimal Lagrangian submanifold of complex Euclidean space is calibrated and therefore minimizes the volume in its homology class. This remarkable fact no longer holds true in an arbitrary Kähler manifold, but it does so in a certain class of Kähler manifolds, namely the *Calabi–Yau manifolds*. The study of minimal Lagrangian submanifolds (usually called *Special Lagrangian submanifolds*) in Calabi–Yau manifolds has attracted much attention, in particular because of its close relationship with mirror symmetry, a important issue in theoretical physics ([SYZ]).

Most of the theory of submanifolds in Riemannian geometry may be extended to the realm of pseudo-Riemannian geometry, and recently there have been growing interest on this topic. In particular, Mealy extended in [Me] (see also [HL2]) the concept of calibration in pseudo-Riemannian manifolds. On

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the other hand, Dong addressed in [Do] the local study of minimal Lagrangian submanifolds in complex linear space  $\mathbb{C}^n$  endowed with the pseudo-Hermitian form defined by:

$$\langle\langle \cdot, \cdot \rangle\rangle_p := - \sum_{j=1}^p dz_j d\bar{z}_j + \sum_{j=p+1}^n dz_j d\bar{z}_j,$$

where  $0 \leq p \leq n$ . If  $p = 0$  or  $n$ , we fall back in the classical, Riemannian setting of [HL1]. One of Dong’s main observations is that, although the geometry of a minimal Lagrangian in  $(\mathbb{C}^n, \langle\langle \cdot, \cdot \rangle\rangle_p), p \neq 0, n$  is somehow analogous to that of the Riemannian case  $p = 0$ , they are always unstable (in the classical sense), so in particular they can not be homology minimizing.

The main result of this paper is that although the original calibration of Harvey and Lawson does not calibrate minimal Lagrangian submanifolds (as pointed by Dong), it does calibrate them in their Lagrangian homology class (Main Theorem, Section 2). In the remainder of the paper, we describe some families of minimal Lagrangian submanifolds. In particular, we show that a minimal Lagrangian surface of  $(\mathbb{C}^2, \langle\langle \cdot, \cdot \rangle\rangle_1)$  must be the Cartesian product of two curves contained in two mutually orthogonal, null, non-Lagrangian planes (Theorem 2, Section 3.1). We also characterize  $SO(p, n - p)$ -equivariant minimal Lagrangian submanifolds of  $(\mathbb{C}^n, \langle\langle \cdot, \cdot \rangle\rangle_p)$  (Section 3.2). This family generalizes the *Lagrangian catenoid*, which was first described by Harvey and Lawson and studied in more detail in [CU]. Finally, inspired by a construction due to Joyce ([Jo]), we produce a larger family of minimal Lagrangian submanifolds obtained from evolving quadrics (Section 3.3).

### 1. Preliminaries

Consider the complex linear space  $\mathbb{C}^n$  of arbitrary dimension  $n$ , endowed with its canonical complex structure  $J$  and, for  $0 \leq p \leq n$ , the pseudo-Hermitian form of arbitrary signature  $(p, n - p)$  defined by:

$$\langle\langle \cdot, \cdot \rangle\rangle_p := - \sum_{j=1}^p dz_j d\bar{z}_j + \sum_{j=p+1}^n dz_j d\bar{z}_j.$$

The real and imaginary parts of  $\langle\langle \cdot, \cdot \rangle\rangle_p$

$$\langle \cdot, \cdot \rangle_{2p} = \text{Re}\langle\langle \cdot, \cdot \rangle\rangle_p \quad \text{and} \quad \omega_p = - \text{Im}\langle\langle \cdot, \cdot \rangle\rangle_p$$

yield two different structures: while the bilinear form  $\langle \cdot, \cdot \rangle_{2p}$  is a pseudo-Riemannian metric with signature  $(2p, 2(n - p))$ , the closed 2-form  $\omega_p$  is, up to a (real) linear change of coordinates, the canonical symplectic form of  $\mathbb{C}^n \simeq T^*\mathbb{R}^n$  regarded as the cotangent bundle of  $\mathbb{R}^n$ .

A smooth, immersed submanifold  $\mathcal{S}$  of  $(\mathbb{C}^n, \langle\langle \cdot, \cdot \rangle\rangle_p)$  is said to be *non-degenerate* if the induced metric on  $\mathcal{S}$  is itself non-degenerate. Moreover, a non-degenerate submanifold is said to be *minimal* if it is a critical point

of the volume with respect to compactly supported variations. On the other hand, an  $n$ -dimensional submanifold is said to be *Lagrangian* if  $\omega_p$  vanishes on it. The equation  $\omega_p = \langle J, \cdot \rangle_{2p}$  shows that a non-degenerate submanifold is Lagrangian if and only if its tangent and normal bundles  $T\mathcal{L}$  and  $N\mathcal{L}$  are isometrically exchanged by the complex structure  $J$ . Since  $T\mathcal{L} \oplus N\mathcal{L} = T\mathbb{C}^n$ , the following fact holds:

LEMMA 1. *The induced metric on a non-degenerate Lagrangian submanifold of  $(\mathbb{C}^n, \langle \cdot, \cdot \rangle_p)$  has signature  $(p, n - p)$ .*

We furthermore introduce the holomorphic volume form  $\Omega := dz_1 \wedge \cdots \wedge dz_n$ , which turns out to be useful for the description of the geometry of a Lagrangian submanifold.

DEFINITION 1. The *Lagrangian angle*  $\beta$  of a non-degenerate, Lagrangian, oriented, submanifold  $\mathcal{L}$  is the map  $\beta : \mathcal{L} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  defined by

$$\beta := \arg \Omega(X_1, \dots, X_n),$$

where  $(X_1, \dots, X_n)$  is a tangent moving frame along  $\mathcal{L}$  (it is easy to check that the definition of  $\beta$  does not depend on the choice of the moving frame, see [An]).

The importance of the Lagrangian angle map is due to the following formula, which was first derived by Chen and Morvan in the definite case (see [CM]) and extended to the indefinite case in ([Do]):

THEOREM 1. *Let  $\mathcal{L}$  be a non-degenerate, Lagrangian submanifold of  $(\mathbb{C}^n, \langle \cdot, \cdot \rangle_p)$  with Lagrangian angle  $\beta$  and mean curvature vector  $\vec{H}$ . Then the following formula holds*

$$n\vec{H} = J\nabla\beta,$$

where  $\nabla$  denotes the gradient operator with respect to the induced metric.

COROLLARY 1. *A Lagrangian submanifold  $\mathcal{L}$  of  $(\mathbb{C}^n, \langle \cdot, \cdot \rangle_p)$  is minimal if and only if it has constant Lagrangian angle.*

## 2. Minimizing properties

We recall here the simple but powerful concept developed in [HL1]: let  $(\mathcal{M}, g)$  be a Riemannian manifold. A *calibration* is a closed  $n$ -form  $\Theta$  of  $\mathcal{M}$  which is bounded from above by the  $n$ th dimensional volume form induced from  $g$ , that is, for any  $n$ -vector  $X_1 \wedge \cdots \wedge X_n$ , we have

$$\begin{aligned} \Theta(X_1, \dots, X_n) &\leq \sqrt{|\det_{\mathbb{R}}[g(X_j, X_k)]_{1 \leq j, k \leq n}|} \\ &:= d \text{Vol}(X_1, \dots, X_n). \end{aligned}$$

A  $n$ -dimensional submanifold  $\mathcal{S}$  of  $\mathcal{M}$  is said to be *calibrated* by  $\Theta$  if the restriction of  $\Theta$  to  $\mathcal{S}$  is equal to the  $n$ -volume, that is, equality is attained in

the expression above when  $(X_1, \dots, X_n)$  is a tangent moving frame along  $\mathcal{S}$ . By Stokes theorem, it follows that if  $\mathcal{S}'$  is any submanifold belonging to the homology class of a calibrated submanifold  $\mathcal{S}$ , we have

$$\begin{aligned} \text{Vol}(\mathcal{S}) &= \int_{\mathcal{S}} d\text{Vol} = \int_{\mathcal{S}} \Theta \\ &= \int_{\mathcal{S}'} \Theta \leq \int_{\mathcal{S}'} d\text{Vol} = \text{Vol}(\mathcal{S}'). \end{aligned}$$

Therefore a calibrated submanifold minimizes the volume in its homology class, hence it is in particular minimal and stable.

Among the few known examples of calibrations is the 1-parameter family of  $n$ -forms of  $\mathbb{C}^n$ ,

$$\Theta_0 := \text{Re}(e^{-i\beta_0}\Omega), \quad \beta_0 \in \mathbb{R}/2\pi\mathbb{Z},$$

discovered in [HL1]. The calibrated submanifolds of  $\Theta_0$  are precisely those Lagrangian submanifold of  $(\mathbb{C}^n, \langle\langle \cdot, \cdot \rangle\rangle_0)$  with constant Lagrangian angle  $\beta_0$ .

On the other hand, it was proved in [Do] (see also [An]) that a minimal submanifold of  $(\mathbb{C}^n, \langle\langle \cdot, \cdot \rangle\rangle_p)$  whose tangent or normal bundle is indefinite is unstable. By Lemma 1, it follows that there is no hope to find a calibration for minimal Lagrangian submanifolds of  $(\mathbb{C}^n, \langle\langle \cdot, \cdot \rangle\rangle_p)$  in the usual sense when  $p \neq 0, n$ .

Nevertheless, the following result holds.

**MAIN THEOREM.** *Let  $\mathcal{L}$  be a minimal Lagrangian submanifold of  $(\mathbb{C}^n, \langle\langle \cdot, \cdot \rangle\rangle_p)$ . Then  $\mathcal{L}$  minimizes the volume in its Lagrangian homology class.*

*Proof.* Let  $\beta_0$  be the constant Lagrangian angle of  $\mathcal{L}$ . We claim that if  $X_1, \dots, X_n$  are  $n$  vectors spanning a non-degenerate Lagrangian subspace, then  $\Theta_0(X_1, \dots, X_n) \leq d\text{Vol}(X_1, \dots, X_n)$ , with equality if and only if  $\beta(X_1, \dots, X_n) = \beta_0$ . To see this, observe that given a vector  $X$  of  $\mathbb{C}^n$ , we have

$$X = \sum_{j=1}^n \varepsilon_j \langle\langle X, e_j \rangle\rangle_p e_j,$$

where  $(e_1, \dots, e_n)$  is the canonical Hermitian basis of  $(\mathbb{C}^n, \langle\langle \cdot, \cdot \rangle\rangle_p)$  and  $\varepsilon_j := \langle\langle e_j, e_j \rangle\rangle_p = \pm 1$ . Setting  $M := [\langle\langle X_j, e_k \rangle\rangle_p]_{1 \leq j, k \leq n}$ , it follows that

$$|\Omega(X_1, \dots, X_n)| = |\det_{\mathbb{C}} [\varepsilon_k \langle\langle X_j, e_k \rangle\rangle_p]_{1 \leq j, k \leq n}| = |\det_{\mathbb{C}} M|.$$

On the other hand, by the Lagrangian assumption, we have

$$\begin{aligned} \langle X_j, X_k \rangle_{2p} &= \langle\langle X_j, X_k \rangle\rangle_p \\ &= \sum_{l=1}^n \langle\langle X_j, e_l \rangle\rangle_p \langle\langle X_k, e_l \rangle\rangle_p. \end{aligned}$$

Therefore,

$$\begin{aligned} d\text{Vol}(X_1, \dots, X_n) &= \left| \det_{\mathbb{R}}([\langle X_j, X_k \rangle_{2p}]_{1 \leq j, k \leq n}) \right|^{1/2} \\ &= \left| \det_{\mathbb{R}}(M \cdot M^*) \right|^{1/2} \\ &= \left| \det_{\mathbb{C}}(M \cdot M^*) \right|^{1/2} \\ &= \left| \det_{\mathbb{C}} M \right|, \end{aligned}$$

where  $M^*$  denotes the complex transpose of  $M$ . It follows that

$$\Theta_0(X_1, \dots, X_n) = |\Omega(X_1, \dots, X_n)| = d\text{Vol}(X_1, \dots, X_n),$$

and of course equality holds if and only if

$$\beta(X_1, \dots, X_n) = \arg(\Omega(X_1, \dots, X_n)) = \beta_0.$$

To conclude the proof, we proceed exactly as in the case of a classical calibration: given a Lagrangian submanifold  $\mathcal{L}'$  in the homology class of  $\mathcal{L}$ , we have

$$\text{Vol}(\mathcal{L}) = \int_{\mathcal{L}} d\text{Vol} = \int_{\mathcal{L}} \Theta_0 = \int_{\mathcal{L}'} \Theta_0 \leq \int_{\mathcal{L}'} d\text{Vol} = \text{Vol}(\mathcal{L}'). \quad \square$$

### 3. Examples of minimal Lagrangian surfaces in complex space

**3.1. Minimal Lagrangian surfaces in complex Lorentzian plane.** In this section, we characterize minimal Lagrangian surfaces of  $\mathbb{C}^2$  endowed with the ‘‘Lorentzian Hermitian metric’’

$$\langle \cdot, \cdot \rangle_1 := -dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 = \langle \cdot, \cdot \rangle_2 - i\omega_1.$$

**THEOREM 2.** *Let  $\mathcal{L}$  be a minimal Lagrangian surface of  $(\mathbb{C}^2, \langle \cdot, \cdot \rangle_1)$ . Then  $\mathcal{L}$  is the product  $\gamma_1 \times J\gamma_2 \subset P \oplus JP$ , where  $\gamma_1$  and  $\gamma_2$  are two planar curves contained in a non-Lagrangian (and therefore non-complex) null plane  $P$ .*

**LEMMA 2.** *Let  $P$  be a plane of  $(\mathbb{C}^2, \langle \cdot, \cdot \rangle_1)$ . Then the induced metric on  $P$  is totally null (i.e.,  $\langle \cdot, \cdot \rangle_2|_P = 0$ ) if and only if  $JP = P^{\omega_1}$ , where  $P^{\omega_1}$  denotes the symplectic orthogonal of  $P$ .*

*Proof.* Suppose first that  $P$  is totally null and let  $X$  be a vector of  $P$ . For all vector  $Y$  in  $P$ , we have

$$0 = \langle X, Y \rangle_2 = -\omega_1(JX, Y),$$

so  $JX \in P^{\omega_1}$ . Since it holds  $\forall X \in P$ , we deduce that  $JP \subset P^{\omega_1}$ , and the two-form  $\omega_1$  being non-degenerate,  $P^{\omega_1}$  is a two-dimensional subspace. Hence,  $JP = P^{\omega_1}$ .

Conversely, if  $JP = P^{\omega_1}$ , then, for all vector  $X$  in  $P$ , we have  $|X|_2^2 = -\omega_1(JX, X) = 0$ . By the polarization formula  $2\langle X, Y \rangle_2 = |X + Y|_2^2 - |X|_2^2 - |Y|_2^2$ , it implies that  $P$  is totally null. □

REMARK 1. This lemma proves in particular that a plane may be both complex and Lagrangian. This fact may sound strange to the reader familiar with Kähler geometry, where complex and Lagrangian planes are two distinct classes. More precisely, if a plane enjoys any two of the three properties: {totally null, Lagrangian, complex}, then the third one holds as well.

*Proof of Theorem 2.* Let  $f : \mathcal{L} \rightarrow \mathbb{C}^2$  be a local parametrization of a minimal Lagrangian surface of  $(\mathbb{C}^2, \langle\langle \cdot, \cdot \rangle\rangle_1)$ . By Lemma 1, the induced metric on  $\mathcal{L}$  is Lorentzian, so it enjoys null coordinates  $(u, v)$  (see [We]). We claim that  $\vec{H} = \frac{2f_{uv}}{\langle f_u, f_v \rangle_2} : ^1$  on the one hand a straightforward computation (see [An]) shows that  $\vec{H} = \frac{2f_{uv}^\perp}{\langle f_u, f_v \rangle_2}$ , where  $(\cdot)^\perp$  denotes the projection onto the normal space; on the other hand, differentiating the assumptions  $|f_u|_2^2 = |f_v|_2^2 = 0$ , we get that  $f_{uv}$  is normal to  $\mathcal{L}$ . It follows that the immersion  $f$  is minimal if and only if  $f_{uv}$  vanishes. Hence,  $f$  must take the form

$$f(u, v) = \gamma_1(u) + \tilde{\gamma}_2(v),$$

where  $\gamma_1, \tilde{\gamma}_2$  are two curves of  $\mathbb{C}^2$ . Moreover, the assumption that  $(u, v)$  are null coordinates translates into the fact that the two curves have null (i.e., lightlike) velocity vector, and the non-degeneracy assumption is

$$\langle \gamma'_1(u), \tilde{\gamma}'_2(v) \rangle_2 \neq 0, \quad \forall (u, v) \in I_1 \times I_2.$$

On the other hand, the Lagrangian assumption is:

$$\omega_1(\gamma'_1(u), \tilde{\gamma}'_2(v)) = 0, \quad \forall (u, v) \in I_1 \times I_2.$$

The remainder of the proof relies on the analysis of the dimension of the two linear spaces  $P_1 := \text{Span}\{\gamma'_1(u), u \in I_1\}$  and  $P_2 := \text{Span}\{\tilde{\gamma}'_2(v), v \in I_2\}$ . We first observe that  $\dim P_1, \dim P_2 \geq 1$  and that the case  $\dim P_1 = \dim P_2 = 1$  corresponds to the trivial case of  $\mathcal{L}$  being planar. Since the rôles of  $\gamma_1$  and  $\tilde{\gamma}_2$  are symmetric, we may assume without loss of generality that  $\dim P_1 \neq 1$ .

Next, the Lagrangian assumption is equivalent to  $P_2 \subset P_1^{\omega_1}$  and  $P_1 \subset P_2^{\omega_1}$ , so  $\dim P_2 \leq \dim P_1^{\omega_1}$  and  $\dim P_1 \leq \dim P_2^{\omega_1}$ . By the non-degeneracy of  $\omega_1$ , it follows that  $\dim P_1 \leq \dim P_2^{\omega_1} = 4 - \dim P_2 \leq 3$ . We claim that in fact  $\dim P_1 = 2$ . To see this, assume by contradiction that  $\dim P_1 = 3$ . It follows that  $\dim P_2 \leq \dim P_1^{\omega_1} = 1$ , so the curve  $\tilde{\gamma}_2$  is a straight line, which may be parametrized as follows:  $\tilde{\gamma}_2(v) = e_0 v$ , where  $e_0$  is a null vector of  $\mathbb{C}^2$ . Then  $\gamma'_1$  is contained in the intersection of the light cone  $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2|\}$  with the hyperplane  $\{e_0\}^{\omega_1}$ . An easy computation, using the fact that  $e_0$  is null, shows that

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2|\} \cap \{e_0\}^{\omega_1} = \Pi_1 \cup \Pi_2,$$

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<sup>1</sup> This is the Lorentzian equivalent of the well known fact that the mean curvature of a surface is given by the Laplacian of its coordinates.

where  $\Pi_1$  and  $\Pi_2$  are two null planes. Moreover, one of these planes, say  $\Pi_2$ , is contained in the metric orthogonal of  $e_0$ . By the non-degeneracy assumption

$$\langle \gamma'_1(u), \tilde{\gamma}'_2(v) \rangle_2 = v \langle \gamma'_1(u), e_0 \rangle_2 \neq 0,$$

we deduce that  $\gamma'_1 \in \Pi_1$ , which implies that  $\dim P_1 \leq 2$ , a contradiction.

To conclude, observe that, using Lemma 2,  $\tilde{\gamma}_2 \in P_2 \subset P_1^{\omega_1} = JP_1$ . Hence, we just need to set  $P := P_1$  and  $\gamma_2 := -J\tilde{\gamma}_2$ , to get that  $\gamma_1, \gamma_2 \subset P$ , so that  $\mathcal{L}$  takes the required expression.  $\square$

**3.2. Equivariant Lagrangian submanifolds in  $\mathbb{C}^n$ .** In this subsection, we give a characterization of those minimal Lagrangian submanifolds of  $(\mathbb{C}^n, \langle \cdot, \cdot \rangle_p)$  which are equivariant with respect to the canonical action of the group  $SO(p, n - p)$  defined as follows: for  $z = x + iy \in \mathbb{C}^n$  and  $M \in SO(p, n - p)$  we simply set  $Mz := Mx + iMy$ . Of course we have  $\langle \langle Mz, Mz' \rangle_p \rangle_p = \langle \langle z, z' \rangle_p \rangle_p$ , so  $SO(p, n - p)$  can be identified with a subgroup of

$$U(n - p, p) := \{ M \in Gl(\mathbb{C}^n) \mid \langle \langle MX, MY \rangle_p \rangle_p = \langle \langle X, Y \rangle_p \rangle_p \}.$$

Observe that the orbits of the action  $SO(p, n - p)$  on  $\mathbb{R}^n$  are the quadrics

$$\mathbb{X}_{p,c}^{n-1} := \{ x \in \mathbb{R}^n \mid \langle x, x \rangle_p = c \}.$$

**THEOREM 3.** *Let  $\mathcal{L}$  be an  $SO(p, n - p)$ -equivariant Lagrangian submanifold of  $\mathbb{C}^n$ . Then it is locally congruent to the image of an immersion of the form*

$$\begin{aligned} f : I \times \mathbb{X}_{p,\varepsilon}^{n-1} &\rightarrow \mathbb{C}^n, \\ (s, x) &\mapsto \gamma(s)x, \end{aligned}$$

where  $\varepsilon = 1$  or  $-1$  and  $\gamma : I \rightarrow \mathbb{C}^*$  is a planar curve. Moreover, the Lagrangian angle of  $\mathcal{L}$  is given by

$$\beta = \arg(\gamma' \gamma^{n-1}).$$

**REMARK 2.** In the definite, two-dimensional case ( $p = 0, n = 2$ ), the  $SO(2)$ -action mentioned in the theorem above is not the only possible one, and there do exist Lagrangian surfaces of  $\mathbb{C}^2$  equivariant by *another*  $SO(2)$ -action. For example, let  $\gamma(s) = (\gamma_1(s), \gamma_2(s))$  be a regular curve of the sphere  $\mathbb{S}^3$  such that  $\langle \gamma', J\gamma \rangle_0 \neq 0$ . Then the map  $f(s, t) = (\gamma_1(s)e^{it}, \gamma_2(s)e^{it})$  is a Lagrangian immersion which is equivariant by the action  $M(z_1, z_2) = (Mz_1, Mz_2), M \in SO(2)$ . These surfaces have been studied in [Pi], where they are called *Hopf surfaces*.

*Proof of Theorem 3. First case:  $n = 2$ .* Recall that the metric of  $\mathbb{C}^2$  is  $\langle \cdot, \cdot \rangle_p = \varepsilon_1 dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$  with  $\varepsilon_1 = 1$  or  $-1$ . Introducing

$$M_{\varepsilon_1} := \begin{pmatrix} 0 & -\varepsilon_1 \\ 1 & 0 \end{pmatrix},$$

we have

$$SO(2) = \{ e^{M_1 t} \mid t \in \mathbb{R} \} \quad \text{and} \quad SO(1, 1) = \{ e^{M_{-1} t} \mid t \in \mathbb{R} \}.$$

A surface of  $\mathbb{C}^2$  which is  $SO(2)$  or  $SO(1, 1)$ -equivariant may be locally parametrized by an immersion of the form

$$f(s, t) = e^{M_{\varepsilon_1} t}(z_1(s), z_2(s)).$$

We first compute the first derivatives of the immersion:

$$\begin{aligned} f_s &= e^{M_{\varepsilon_1} t}(z'_1, z'_2), \\ f_t &= e^{M_{\varepsilon_1} t} M_{\varepsilon_1}(z_1, z_2) = e^{M_{\varepsilon_1} t}(-\varepsilon_1 z_2, z_1). \end{aligned}$$

Therefore, the Lagrangian condition yields:

$$\begin{aligned} 0 &= \omega_p(f_s, f_t) = \omega_p((z'_1, z'_2), (-\varepsilon_1 z_2, z_1)) \\ &= -\operatorname{Im}(z'_1 \bar{z}_2) + \operatorname{Im}(z'_2 \bar{z}_1) \\ &= \frac{d}{ds} \operatorname{Im}(z_2 \bar{z}_1). \end{aligned}$$

Hence,  $z_1 \bar{z}_2$  must be constant. Observe that there is no loss of generality in assuming that  $\operatorname{Im}(z_1 \bar{z}_2)$  vanishes: otherwise, we introduce

$$\tilde{f} := \begin{pmatrix} 1 & 0 \\ 0 & e^{i \arg z_2(0)} \end{pmatrix} f,$$

which is congruent to  $f$ . Thus,  $z_1$  and  $z_2$  have the same argument. Next, introduce polar coordinates  $z_1 = r_1 e^{i\varphi}$  and  $z_2 = r_2 e^{i\varphi}$  and consider separately the definite and indefinite cases:

*The definite case  $p = 0$ .* The second coordinate of  $f$  is

$$z_2(s) \cos t + z_1(s) \sin t = (r_2(s) \cos t + r_1(s) \sin t) e^{i\varphi(s)}.$$

Clearly,  $\forall s \in I$ , there exists  $t(s) \in \mathbb{R}$  such that  $r_2(s) \sin t(s) + r_1(s) \cos t(s) = 0$ , hence the second coordinate of  $f$  vanishes at  $(s, t(s))$ . Setting  $\gamma(s) := z_1(s) \cos t(s) - z_2(s) \sin t(s)$ , that is,  $\gamma(s)$  is the first coordinate of  $f$  at  $(s, t(s))$ , we see that  $f(s, t) = e^{M_1(t-s(t))}(\gamma(s), 0)$ . Hence, the immersion  $\tilde{f}(s, t) := e^{M_1 t}(\gamma(s), 0) = (\gamma \cos t, \gamma \sin t)$  parameterizes the same surface as  $f$ , and we get the required parameterization for the surface  $\mathcal{L}$ .

*The indefinite case  $p = 1$ .* We first observe that  $r_1 \neq r_2$  since otherwise the immersion would be degenerate. If  $r_1 > r_2$ , there exists  $t(s)$  such that  $r_2(s) \cosh t(s) + r_1(s) \sinh t(s) = 0$ , hence the second coordinate of  $f$  vanishes at  $(s, t(s))$ . Analogously to the definite case, we set  $\gamma(s) = r_1(s) \cosh t(s) + r_2(s) \sinh t(s)$ , and as before, we check that  $\tilde{f}(s, t) := (\gamma(s) \cosh t, \gamma(s) \sinh t)$  parametrizes the same surface as  $f$ . The argument is similar if  $r_1 < r_2$ : we find  $t(s)$  in order to make the first coordinate vanish and find  $\tilde{f}(s, t) = (\gamma(s) \sinh t, \gamma(s) \cosh t)$ .

*Second case:  $n \geq 3$ .* First, set three different indexes  $j, k$  and  $l$  and consider the two matrices  $M_{jl}$  and  $M_{kl}$  defined by

$$M_{jl} e_j = e_l, \quad M_{jl} e_l = \varepsilon_j \varepsilon_l e_j \quad \text{and} \quad M_{jl} e_m = 0 \quad \text{for } m \neq j, l,$$

and

$$M_{kl}e_k = e_l, \quad M_{kl}e_l = \varepsilon_k \varepsilon_l e_k \quad \text{and} \quad M_{kl}e_m = 0 \quad \text{for } m \neq k, l.$$

The reader may check that  $M_{jl}$  and  $M_{kl}$  are skew with respect to  $\langle \cdot, \cdot \rangle_p$ . Hence, given a point  $z$  in  $\mathcal{L}$ , the two curves  $s \mapsto e^{M_{jl}s}$  and  $s \mapsto e^{M_{kl}s}$  belong to  $SO(p, n - p)$ . By the equivariance assumption, it follows that the curves  $s \mapsto e^{M_{jl}s}z$  and  $s \mapsto e^{M_{kl}s}z$  belong to  $\mathcal{L}$ , so the two vectors  $M_{jl}z$  and  $M_{kl}z$  are tangent to  $\mathcal{L}$  at  $z$ . Moreover, the Lagrangian assumption yields

$$0 = \omega_p(M_{jl}z, M_{kl}z) = \operatorname{Re} z_j \operatorname{Im} z_k - \operatorname{Re} z_k \operatorname{Im} z_j.$$

Since this holds for any pair of indexes  $(j, k)$ , it follows that  $\operatorname{Re} z$  and  $\operatorname{Im} z$  are collinear. Therefore, there exist  $\varphi \in \mathbb{R}$  and  $y \in \mathbb{R}^n$  such that  $z = e^{i\varphi}y$ . Let  $r > 0$  and  $x \in \mathbb{X}_{p,\varepsilon}^{n-1}$  such that  $y = rx$ , and set  $\gamma := re^{i\varphi}$ . By the equivariance assumption, the  $(n - 1)$ -dimensional quadric  $\gamma\mathbb{X}_{p,\varepsilon}^{n-1}$  of  $\mathbb{C}^n$  is contained in  $\mathcal{L}$ . Finally, since  $\mathcal{L}$  is  $n$ -dimensional, it must be locally foliated by a one-parameter family of quadrics  $\gamma(s)\mathbb{X}_{p,\varepsilon}^{n-1}$ , which proves the first part of the theorem (characterization of equivariant Lagrangian submanifolds).

We now prove the second part of the theorem: let  $f$  be an immersion as described in the statement of the theorem,  $x$  a point of  $\mathbb{X}_{p,\varepsilon}^{n-1}$  and  $(e_1, \dots, e_{n-1})$  an oriented orthonormal basis of  $T_x\mathbb{X}_{p,\varepsilon}^{n-1}$ . Setting

$$X_j := \gamma e_j \quad \text{and} \quad X_n := \gamma'x,$$

it is easy to check that  $(X_1, \dots, X_n)$  is a basis of  $T_x\mathcal{L}$ . Then, we calculate

$$\begin{aligned} \omega_p(X_j, X_k) &= \langle JX_j, X_k \rangle_{2p} = \langle i\gamma, \gamma \rangle \langle e_j, e_k \rangle_p = 0, \\ \omega_p(X_j, X_n) &= \langle JX_j, X_n \rangle_{2p} = \langle i\gamma, \gamma' \rangle \langle e_j, x \rangle_p = 0, \end{aligned}$$

which shows that  $\mathcal{L}$  is Lagrangian. Finally, we get the Lagrangian angle of  $\mathcal{L}$  as follows:

$$\begin{aligned} e^{i\beta} &= \Omega(X_1, \dots, X_n) \\ &= \Omega(\gamma e_1, \dots, \gamma e_{n-1}, \gamma'x) \\ &= \gamma' \gamma^{n-1} \Omega(e_1, \dots, e_{n-1}, x) = \gamma' \gamma^{n-1}. \end{aligned} \quad \square$$

From Theorem 3, it is straightforward to describe equivariant minimal Lagrangian submanifolds:  $\beta$  vanishes if and only if  $\operatorname{Im} \gamma' \gamma^{n-1} = 0$ , which we easily integrate to get  $\operatorname{Im} \gamma^n = c$  for some real constant  $c$ . If  $c$  vanishes, the curve  $\gamma$  is made up of  $n$  straight lines passing through the origin, and the corresponding Lagrangian submanifold is nothing but the union of  $n$  linear spaces of  $\mathbb{C}^n$ . If  $c$  does not vanish, the curve is a made up of  $2n$  pieces, each of one contained in an angular sector  $\{\varphi_0 < \arg \gamma < \varphi_0 + \frac{\pi}{n}\}$ . If  $n = 2$ , they are hyperbolae. Summing up, we have obtained the following characterization of equivariant, minimal Lagrangian submanifolds:

**COROLLARY 2.** *Let  $\mathcal{L}$  be a connected, minimal Lagrangian submanifold of  $(\mathbb{C}^n, \langle \cdot, \cdot \rangle_p)$  which is  $SO(p, n - p)$ -equivariant. Then  $\mathcal{L}$  is congruent to an open subset of either an affine Lagrangian  $n$ -plane, or of the Lagrangian catenoid*

$$\{\gamma \cdot x \in \mathbb{C}^n \mid x \in \mathbb{X}_{p,\varepsilon}^{n-1}, \gamma \in \mathbb{C}, \text{Im } \gamma^n = c\},$$

where  $c$  is a non-vanishing real constant.

**3.3. Lagrangian submanifolds from evolving quadrics.** This section describes a class of Lagrangian submanifolds which generalize the former ones and follows ideas from [Jo] (see also [LW], [JLT]). Consider a real, invertible  $n \times n$  matrix  $M$  which is self-adjoint with respect to  $\langle \cdot, \cdot \rangle_p$ , i.e.  $\langle Mx, y \rangle_p = \langle x, My \rangle_p, \forall x, y \in \mathbb{R}^n$ .

**THEOREM 4.** *Let  $c \in \mathbb{R}$  such that the quadric*

$$\mathcal{S} := \{x \in \mathbb{R}^n \mid \langle x, Mx \rangle_p = c\}$$

*is a non-degenerate hypersurface of  $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_p)$  and  $r(s)$  a positive function on an interval  $I$  of  $\mathbb{R}$ . Then the immersion*

$$\begin{aligned} f : I \times \mathcal{S} &\rightarrow \mathbb{C}^n, \\ (s, x) &\mapsto r(s)e^{iMs}x \end{aligned}$$

*is a Lagrangian and its Lagrangian angle is given by*

$$\beta = \text{tr } Ms + \arg\left(c \frac{r'}{r} + i|Mx|_p^2\right) + \pi/2.$$

*Proof.* Let  $(e_1, \dots, e_{n-1})$  be an orthonormal basis of  $T_x\mathcal{S} = (Mx)^\perp$ , that we complete by  $e_n$  in such a way that  $(e_1, \dots, e_n)$  is an oriented, orthonormal basis of  $\mathbb{R}^n$ . Hence,  $e_n$  is collinear to  $Mx$  and, setting  $\varepsilon_n := |e_n|_p^2$  we have

$$Mx = \varepsilon_n \langle Mx, e_n \rangle_p e_n.$$

We obtain a basis of tangent vectors to  $f(I \times \mathcal{S})$  at a point  $z = re^{iMs}x$ , setting

$$Z_j = e^{iMs}e_j \quad \text{and} \quad Z_n = (r' + riM)e^{iMs}x.$$

Using the fact that  $e^{iMs} \in U(p, n - p)$ , it is easily checked that  $\omega_p(Z_j, Z_n)$  and  $\omega_p(Z_j, Z_k)$  vanish, hence the immersion  $f$  is Lagrangian. To complete the proof, we compute

$$\begin{aligned} \Omega(Z_1, \dots, Z_n) &= \Omega(e^{iMs}e_1, \dots, e^{iMs}e_{n-1}, (r' + irM)e^{iMs}x) \\ &= i \det_{\mathbb{C}}[e^{iMs}] \det_{\mathbb{C}}(e_1, \dots, e_{n-1}, (r' + irM)x) \\ &= i \det_{\mathbb{C}}[e^{iMs}] (r'\varepsilon_n \langle x, e_n \rangle_p + ir\varepsilon_n \langle Mx, e_n \rangle_p). \end{aligned}$$

Using the fact that

$$\langle x, e_n \rangle_p = \frac{\langle x, Mx \rangle_p}{\varepsilon_n \langle Mx, e_n \rangle_p} = \frac{c}{\varepsilon_n \langle Mx, e_n \rangle_p},$$

we get

$$\Omega(Z_1, \dots, Z_n) = ie^{i \operatorname{tr} Ms} \frac{r}{\langle Mx, e_n \rangle_p} \left( c \frac{r'}{r} + i \varepsilon_n \langle Mx, e_n \rangle_p^2 \right).$$

We deduce, using the fact that  $\varepsilon_n \langle Mx, e_n \rangle_p^2 = |Mx|_p^2$ ,

$$\begin{aligned} \beta &= \arg(\Omega(Z_1, \dots, Z_n)) \\ &= \frac{\pi}{2} + \operatorname{tr} Ms + \arg\left(c \frac{r'}{r} + i |Mx|_p^2\right), \end{aligned}$$

which is the required formula. □

EXAMPLE 1. Assume that  $M = Id$  and  $c = 1$ . Then  $f$  becomes

$$\begin{aligned} f : I \times \mathbb{X}_{p,\varepsilon}^{n-1} &\rightarrow \mathbb{C}^n, \\ (s, x) &\mapsto r(s)e^{is}x. \end{aligned}$$

In particular, the image of the immersion is a  $SO(p, n - p)$ -equivariant submanifold as in Section 3.2.

COROLLARY 3. *The Lagrangian immersion  $f$  introduced in Theorem 4 above is minimal if and only if one of the three statements holds:*

- (i)  $\operatorname{tr} M = 0$  and the function  $r$  is constant;
- (ii)  $\operatorname{tr} M = 0$  and the constant  $c$  vanishes;
- (iii) the image of  $f$  is a part of the Lagrangian catenoid described in the previous section.

*Proof.* The Lagrangian angle  $\beta$  must be constant, so the term  $\arg(c \frac{r'}{r} + i |Mx|_p^2)$  must be independent of  $x$ . This happens if and only if either  $r'$  or  $c$  vanishes, or both  $|Mx|_p^2$  and  $\frac{r'}{r}$  are constant. If  $r'$  or  $c$  vanish, the first term  $\operatorname{tr} Ms$  of  $\beta$  must be constant as well, hence we must have  $\operatorname{tr} M = 0$ . These are the first two cases of the corollary. Suppose now  $|Mx|_p^2$  is constant on  $\mathcal{S}$ , that is,

$$\forall x \in \mathbb{R}^n \text{ such that } \langle Mx, x \rangle_p = c, \quad |Mx|_p^2 = c'.$$

Since  $M$  is invertible, it is equivalent to

$$\forall y \in \mathbb{R}^n \text{ such that } \langle y, M^{-1}y \rangle_p = c, \quad |y|_p^2 = c'.$$

It follows that the quadric  $\{\langle y, M^{-1}y \rangle_p = c\}$  is contained in the quadric  $\mathbb{X}_{p,c'}^{n-1}$ , hence  $M^{-1}$  is a multiple of the identity and so is  $M$ . Hence, the immersion is equivariant and we are in the situation described in Example 1 above. The result follows from Corollary 2. □

EXAMPLE 2. Set  $n = 2$ ,  $p = 1$  and  $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Since  $\operatorname{tr} M = 0$  the immersion  $f(s, x) = r(s)e^{iMs}x$  is minimal if  $c$  vanishes or if  $r$  is constant. The case of vanishing  $c$  is trivial: the quadric  $\mathcal{S}$  reduces to the union of the two

straight lines  $\{x_1 = 0\}$  and  $\{x_2 = 0\}$ , and the image of  $f$  is the union the two complex planes  $\{z_1 = 0\}$  and  $\{z_2 = 0\}$ .

In the case of non-vanishing  $c$ , constant  $r$ , the set

$$\mathcal{S} = \{x \in \mathbb{R}^2 \mid \langle x, Mx \rangle_1 = 2x_1x_2 = c\}$$

is an hyperbola which may be parametrized by  $t \mapsto (e^t, \frac{2}{c}e^{-t})$ . On the other hand

$$e^{iMs} = \begin{pmatrix} \cosh s & -i \sinh s \\ i \sinh s & \cosh s \end{pmatrix}.$$

So, setting  $r = 1$ , we are left with the immersion

$$f(s, t) = \left( e^t \cosh s - i \frac{2}{c} e^{-t} \sinh s, \frac{2}{c} e^{-t} \cosh s + i e^t \sinh s \right).$$

Observing that  $(s, t)$  are conformal coordinates, we obtain null coordinates setting  $u := s + t$  and  $v := s - t$ . It follows that the immersion takes the form  $f(u, v) = \gamma_1(u) + J\gamma_2(v)$ , where

$$\gamma_1(u) := \frac{1}{2} \left( e^u + \frac{2}{c} i e^{-u}, e^{-u} + \frac{2}{c} i e^u \right)$$

and

$$\gamma_2(v) := \frac{-1}{2} \left( e^v + i \frac{2}{c} e^{-v}, e^v + i \frac{2}{c} e^{-v} \right)$$

are two hyperbolae in the null plane  $P = \{x_1 - y_2 = 0, x_2 - y_1 = 0\}$ . We therefore recover a special case of Theorem 2. We observe that local descriptions of minimal Lagrangian surfaces in  $(\mathbb{C}^2, \langle \cdot, \cdot \rangle_1)$  have been given independently in [AGR], [Do], [Ch].

EXAMPLE 3. In the definite case, since the metric  $\langle \cdot, \cdot \rangle_0$  is positive, there exists an orthonormal basis of eigenvectors of  $M$ . So we may assume without loss of generality that  $M = \text{diag}(\lambda_1, \dots, \lambda_n)$ , where the  $\lambda_j$ s are real constants. It follows that a point  $z$  of  $\mathcal{L}$  takes the form

$$(x_1 e^{i\lambda_1 s}, \dots, x_n e^{i\lambda_n s}),$$

where  $\sum_{j=1}^n \lambda_j x_j^2 = c$ . We observe furthermore that  $\mathcal{L}$  is a properly immersed submanifold if and only if all the coefficients  $\lambda$  are rationally related. In this case, we may assume without loss of generality that they are integer numbers. This case is studied [LW]. Observe moreover that  $c$  cannot vanish (otherwise  $\mathcal{S}$  reduces to the origin), so by Theorem 3,  $\mathcal{L}$  is minimal if and only if  $\text{tr } M = 0$ . Example 2 above proves that the situation is richer in the indefinite case, since there may not exist an orthonormal basis of eigenvectors.

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