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KURANISHI SPACES OF MEROMORPHIC CONNECTIONS

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ABSTRACT. We construct the Kuranishi spaces, or in other words, the versal deformations, for the following classes of connections with fixed divisor of poles D: all such connections, as well as for its subclasses of integrable, integrable logarithmic and integrable logarithmic connections with a parabolic structure over D. The tangent and obstruction spaces of deformation theory are defined as the hypercohomology of an appropriate complex of sheaves, and the Kuranishi space is a fiber of the formal obstruction map.

0. Introduction

We construct the Kuranishi space, or in other words, the versal deformation, of connections belonging to each one of the following classes:

meromorphic connections with fixed divisor of poles D;
integrable meromorphic connections with fixed divisor of poles D;
integrable logarithmic connections with fixed divisor of poles D;
integrable logarithmic connections on curves with parabolic structure at singular points.

The interest in versal deformations is twofold. First, a versal deformation is a kind of a local moduli space which exists in a much wider range of situations than the moduli spaces in the proper sense do. Second, versal deformations are usually easier to write down than the moduli spaces, and one can use the versal deformation to determine the germ of the moduli space up to analytic, formal or étale equivalence.

Historically, versal deformations were introduced for the first time in late 50s in the work of Kodaira and Spencer ([KS-1], [KS-2]), and Kuranishi ([Ku-1], [Ku-2]). In the beginning, this theory was only concerned with deformations of compact complex manifolds and was viewed as a replacement for

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Riemann's insight of moduli of compact complex curves in higher dimensions. But since then, the theory has been significantly formalized and extended to a much wider range of domains: singularities [Ar], [Schl-2], [AGZV], vector bundles and sheaves [Rim-1], [Rim-2], [Artam-1], [Artam-2], singular complex spaces [Gro], [Illu-1], [Illu-2], [Pa-1], [Pa-2], and morphisms of varieties or complex spaces [Fl], [Bi], [Ran-1], [Ran-2].

Recently, many people believe that a deformation theory over a field of characteristic 0 should be taken over by a differential graded Lie algebra (denoted DGLA). This principle deriving from researches regarding homotopy theory, quantization, mirror symmetry, etc. (see, for instance, [Kon]). One prototype example to this principle is the deformation theory of compact complex manifold via Maurer–Cartan equation on the vector field valued $(0,1)$ forms. This is the Newlander–Nirenberg theorem (or rather Kuranishi's proof of the existence of the Kuranishi space). If we restrict to infinitesimal deformations, we can describe the situation as a bijection between

$$\frac{\{\text{Maurer–Cartan solutions in } \mathrm{KS}^1_X \otimes m_A\}}{\text{gauge equivalence}} \simeq \frac{\{\text{deformations of } X \text{ on } A\}}{\text{isomorphisms}},$$

where A is a local Artinian $\mathbb{C}$-algebra and $\mathrm{KS}^\bullet_X = (A^{0;\bullet}_X(\Theta_X), \partial, [-,-])$ the Kodaira–Spencer algebra on X. This isomorphism is functorial in A. The left-hand side is the deformation functor associated to the Kodaira–Spencer DGLA $\mathrm{KS}^\bullet_X$, denoted by $\mathrm{Def}_{\mathrm{KS}_X}$, and the right-hand side is the usual deformation functor Def_X of X.

All the constructions are enclosed in the paradigm of the Kuranishi space associated to a "good" deformation theory. A "good" deformation theory for some type of object X consists in determining a triple (T^1_X, T^2_X, f), where T^1_X is the tangent space to deformations of X, T^2_X is the obstruction space, $f : \hat{T^1_X} \to \hat{T^2_X}$ a formal map without linear terms, called the Kuranishi map ($\hat{}$ denotes the formal completion at zero). Then the formal scheme $f^{-1}(0)$ is the Kuranishi space, or a formal germ of the versal deformation of X.

We provide the triples (T^1_X, T^2_X, f) for the above four classes of connections. In all the 4 cases, $T^i_X = \mathbb{H}^i(\mathcal{C}^\bullet)$, the hypercohomology of an appropriate complex of sheaves, and the initial component f_2 of f is the Yoneda square map. For instance, in the case $X = (\mathcal{E}, \nabla)$ is a meromorphic connection with fixed divisor of poles D, the complex $\mathcal{C}^\bullet$ is a two-term one and is

$$\mathcal{C}^\bullet = \big[\mathcal{E}\mathrm{nd}(\mathcal{E}) \xrightarrow{\nabla} \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)\big].$$

A similar situation occurs in the deformation theory of Higgs bundles or Hitchin pairs [B-R], where $T^1_X = \mathbb{H}^1(\mathcal{C}^\bullet)$ with complex

$$\mathcal{C}^\bullet = \big[\mathcal{E}\mathrm{nd}(\mathcal{E}) \xrightarrow{\mathrm{ad}\,\varphi} \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)\big]$$

defined by the Higgs field $\varphi:\mathcal{E}\to\mathcal{E}\otimes\Omega^1(D)$; contrary to our case, $\operatorname{ad}\varphi$ is $\mathcal{O}_X$-linear.

Let X be a complete scheme of finite type over k or a compact complex space (then $k=\mathbb{C}$). The existence of a versal deformation and the theoretical approach to its construction are known for coherent sheaves on X. The construction of the Kuranishi space (= versal deformation) for coherent sheaves is done in using the injective resolutions. We are studying vector bundles $\mathcal{E}$ with an additional structure (a connection ∇), and in this case the deformation theory of both $\mathcal{E}$ and $(\mathcal{E},\nabla)$ can be stated in terms of the Čech cohomology of a sufficiently fine open covering of X. This approach is easier than the one via injective resolutions. We start by the construction of the Kuranishi space of vector bundles serving as a model for that of the pairs $(\mathcal{E},\nabla)$. This is done in Section 1, where it is also explained how the versal deformations can be used to construct analytic moduli spaces of simple vector bundles. In Section 2, we introduce connections with fixed divisor of poles and show that their isomorphism classes of first order deformations are classified by the hypercohomology $\mathbb{H}^1(\mathcal{C}^\bullet)$ of some two-term complex of sheaves. In Section 3, we show that the first obstruction to lifting the first order deformation is given by the Yoneda square and construct the Kuranishi space. We also define several versions of the Atiyah class. In Section 4, we describe the construction of the Kuranishi space for integrable and integrable logarithmic connections. The last Section 5 treats the Kuranishi space of parabolic connections.

0.1. Deformation theory. In this section, we follow [Ma], and [H-L] to remind the framework of the deformation theory.

Let **Art** be the category of local Artinian $\mathbb{C}$-algebra A such that $A/m_A\simeq\mathbb{C}$, where m_A is the maximal ideal of A. We mean by a functor of Artinian rings [Schl-1] a covariant functor

$D:\mathbf{Art}\to\mathbf{Set}$ such that $D(\mathbb{C})$ is the one-point set. The tangent space T_D to a functor of Artinian rings D is defined by $T_D=D(\mathbb{C}[\varepsilon])$, where $\mathbb{C}[\varepsilon]$ is the ring of dual numbers $\mathbb{C}[x]/(x^2)$.

Let A, B, C be local artinian $\mathbb{C}$-algebras and $\eta: D(B\times_A C)\to D(B)\times_{D(A)} D(C)$ be the natural map. We call a functor of Artinian rings D a deformation functor if it satisfies (i) if $B\to A$ is onto, so is η, and (ii) if $A=\mathbb{C}$, η is bijective [Ma], Definition 2.5. Note that these conditions are closely related to Schlessinger's criterion of existence of a hull (see Remark to Definition 2.7 in [F-M]).

An obstruction theory of a functor of Artinian rings D is a pair $(U,\operatorname{ob}(-))$, consisting of a finite dimensional $\mathbb{C}$-vector space U, the obstruction space, and a map $\operatorname{ob}(\alpha): D(A')\to U\otimes a$, the obstruction map such that for any small extension

$$\alpha: 0\to a\to A\to A'\to 0,$$

with kernel a such that $m_A a=0$, the following conditions are satisfied:

1. If $x' \in D(A')$ lifts to $D(A)$, then $\mathrm{ob}(\alpha)(x') = 0$.
2. For any morphism φ of small extensions

$$\begin{array}{lccccccccc} \alpha_1: & 0 & \longrightarrow & a_1 & \longrightarrow & A_1 & \longrightarrow & A'_1 & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \varphi_a} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} & & \\ \alpha_2: & 0 & \longrightarrow & a_2 & \longrightarrow & A_2 & \longrightarrow & A'_2 & \longrightarrow & 0, \end{array}$$

we have the compatibility $\mathrm{ob}(\alpha_2)(\varphi'_*(x')) = (\mathrm{id}_U \otimes \varphi_a)(\mathrm{ob}(\alpha_1)(x'))$, for every $x' \in D(A'_1)$. Moreover, if $\mathrm{ob}(\alpha)(x') = 0$ implies the existence of a lifting of x' to $D(A)$, the obstruction is called complete.

In the sequel, we always assume that k is an algebraically closed field or $k = \mathbb{C}$. For instance, if X is a smooth projective variety over k, and let F be a coherent $\mathcal{O}_X$-module which is simple. If $A \in \mathbf{Art}\,/\boldsymbol{k}$, let $D_F(A)$ be the set of isomorphim classes of pairs (F_A, φ) where F_A is a flat family of coherent sheaves on X parameterized by $\mathrm{Spec}(A)$ and $\varphi : F_A \otimes_A k \to F$ is an isomorphism of $\mathcal{O}_X$-modules. Following [H-L], the map $D_F(\alpha) : D_F(A) \to D_F(A')$ has for fibers affine spaces with affine group $\mathrm{Ext}^1(F,F) \otimes_k a$, and the image of $D_F(\alpha)$ lies in the kernel of the obstruction map $\mathrm{ob}(\alpha) : D_F(A') \to \mathrm{Ext}^2(F,F) \otimes_k a$.

Proposition 0.1 (See [Ma], Proposition 2.17). *Let D_1 and D_2 be deformation functors and $\varphi : D_1 \to D_2$ a morphism of functors, (V_1, ob_{D_1}) and (V_2, ob_{D_2}) obstruction theories for D_1 and D_2, respectively. Assume that*

(i) *φ induces a surjection (resp. bijection) on the tangent spaces $T_{D_1} \to T_{D_2}$.*

(ii) *There is an injective linear map between obstruction spaces $\Phi : V_1 \to V_2$ such that $\mathrm{ob}_{D_2} \circ \varphi = \Phi \circ \mathrm{ob}_{D_1}$.*

(iii) *The obstruction theory (V_1, ob_{D_1}) is complete.*

Then, the morphism φ is smooth (resp étale).

1. Construction of the Kuranishi space in the case of vector bundles over any base

Let X be a complete scheme of finite type over k or a complex space (then $k = \mathbb{C}$), $\mathfrak{U} = (U_\alpha)$ be an open covering of X, e_α a trivialization of $\mathcal{E}_{|U_\alpha}$. The transition functions $g_{\alpha\beta}$ relate the trivializations by the formula $e_\beta = e_\alpha g_{\alpha\beta}$ over $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and satisfy the following relations

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}, \qquad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1. \tag{1}$$

In other words, $(g_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathrm{GL}(r, \mathcal{O}_X))$ is a skew-symmetric multiplicative 1-cocycle.

1.1. Construction of the Kuranishi space in the case of simple vector bundles over any base.

Definition 1.1. A vector bundle $\mathcal{E}$ on X is simple if and only if $H^0(X, \mathcal{E}\mathrm{nd}(\mathcal{E})) = k\,\mathrm{id}$.

In the case of a simple vector bundle, the versal deformation is in fact universal and this is a local version of the moduli space.

Proposition 1.2. *Let $\mathcal{E}$ be a simple vector bundle on a scheme X of finite type on k or a complex space (in which case $k=\mathbb{C}$). Then there exists an analytic space $M(\mathcal{E})$ with a reference point $*$ and a vector bundle E on $X \times M(\mathcal{E})$ which satisfy the following properties:*

(1) $E|_{X\times *} \simeq \mathcal{E}$.

(2) *If T is an analytic space with a reference point $*$ and E' a vector bundle on $X \times T$ such that $E'|_{X\times *} \simeq \mathcal{E}$, then there is a holomorphic mapping $\Phi : T \to M(\mathcal{E})$ such that $\Phi(*) = *$ and $E' \simeq (1 \times \Phi)^*(E)$.*

(3) *The above mapping Φ is unique as a germ of a holomorphic mapping from $(T, *)$ to $(M(\mathcal{E}), *)$. $(M(\mathcal{E}), *)$ and E are called the Kuranishi space and the Kuranishi family of $\mathcal{E}$, respectively.*

Proof. See [Mu-1]. □

We define SV_X as the set of isomorphism classes of simple vector bundles on X. Using Proposition 1.2, we can endow it with an analytic structure so that SV_X has a universal family only locally in the étale or classical topology. Then there exists a sufficiently small open set U of SV_X in the classical or étale topology and a vector bundle E on $X \times U$ satisfiying the following property: For any analytic space S, there exists a functorial bijection between the sets $\{\text{morphisms } S \to U\} \to \{\text{vector bundles } E \text{ on } X \times S \text{ such that } \forall s \in S, E_s \text{ is simple and its class belongs to } U\}/\sim$ given by $\varphi \mapsto (1 \times \varphi)^*(E)$.

Proposition 1.3. *Let $X, \mathcal{E}$ be as in Proposition 1.2. Every obstruction to the smoothness of SV_X at $[\mathcal{E}]$ lies in $\ker(H^2(\mathrm{Tr}) : H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E})) \to H^2(X, \mathcal{O}_X))$. In particular, SV_X is smooth at $[\mathcal{E}]$ if $H^2(\mathrm{Tr})$ is injective.*

Proof. See [Mu-1]. □

Note, however, that SV_X, even if it is smooth, is not a nice concept of moduli space: it is non-separated in many examples.

We now treat the case of vector bundles over any base.

1.2. First order deformations. Deform the transition functions: $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \varepsilon g_{\alpha\beta,1}$, where $g_{\alpha\beta,1} \in \Gamma(U_{\alpha\beta}, M_r(\mathcal{O}_X))$ and $\varepsilon^2 = 0$. We have $g_{\alpha\beta,1} =$

$\frac{\mathrm{d}\tilde{g}_{\alpha\beta}}{\mathrm{d}\varepsilon}$. Differentiating (1), we obtain:

$$g_{\beta\alpha,1} = \frac{\mathrm{d}\tilde{g}_{\alpha\beta}^{-1}}{\mathrm{d}\varepsilon} = -g_{\alpha\beta}^{-1} g_{\alpha\beta,1} g_{\alpha\beta}^{-1}, \tag{2}$$
$$g_{\alpha\beta,1} g_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma,1} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha,1} = 0,$$

and by (2), $g_{\gamma\alpha,1} = -g_{\alpha\gamma}^{-1} g_{\alpha\gamma,1} g_{\alpha\gamma}^{-1}$. Plugging this into the previous formula, we get

$$g_{\alpha\beta,1} g_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma,1} g_{\gamma\alpha} = g_{\alpha\beta} g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\gamma,1} g_{\alpha\gamma}^{-1}.$$

Multiply by $g_{\alpha\gamma}$ on the right:

$$g_{\alpha\beta,1} g_{\beta\gamma} + g_{\alpha\beta} g_{\beta\gamma,1} = g_{\alpha\gamma,1}. \tag{3}$$

We want to represent this in the form $a_{\alpha\beta} + a_{\beta\gamma} = a_{\alpha\gamma}$ for an appropriate additive 1-cocycle $a = (a_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$, associated with $(g_{\alpha\beta,1})$ and skew-symmetric: $a_{\alpha\beta} = -a_{\beta\alpha}$. Define $a_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ by its matrix: $g_{\alpha\beta}^{-1} g_{\alpha\beta,1}$ in the basis e_β and $g_{\alpha\beta,1} g_{\alpha\beta}^{-1}$ in the basis e_α. Then (2) gives $g_{\alpha\beta} g_{\beta\alpha,1} + g_{\alpha\beta,1} g_{\alpha\beta}^{-1} = 0$, written in terms of matrices with respect to the basis e_α, and (3) amounts to $a_{\alpha\beta} + a_{\beta\gamma} = a_{\alpha\gamma}$. Thus the first order deformations of $\mathcal{E}$ are classified by the 1-cocycles $a = (a_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$. Such a deformation is trivial if the vector bundle $\tilde{\mathcal{E}}$ defined over $X \times \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$ by the 1-cocycle $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \varepsilon g_{\alpha\beta,1}$ is isomorphic to $\mathrm{pr}_1^*(\mathcal{E})$, where $\mathrm{pr}_1 : X \times \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2) \to X$ is the natural projection. This means that there exists a change of basis $e_\alpha \mapsto \tilde{e}_\alpha = e_\alpha(1 + \varepsilon h_\alpha)$ which transforms $\tilde{g}_{\alpha\beta}$ into $g_{\alpha\beta}$. We compute $\tilde{e}_\beta = e_\beta(1+\varepsilon h_\beta) = e_\alpha g_{\alpha\beta}(1+\varepsilon h_\beta) = \tilde{e}_\alpha(1-\varepsilon h_\alpha) g_{\alpha\beta}(1+\varepsilon h_\beta)$ and we want that this coincides with $\tilde{e}_\beta = \tilde{e}_\alpha \tilde{g}_{\alpha\beta}$. That is: $g_{\alpha\beta} + \varepsilon g_{\alpha\beta,1} = (1 - \varepsilon h_\alpha) g_{\alpha\beta}(1 + \varepsilon h_\beta)$, or $g_{\alpha\beta,1} = -h_\alpha g_{\alpha\beta} + g_{\alpha\beta} h_\beta$. Interpreting h_α as the matrix of $b_\alpha \in \Gamma(U_\alpha, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ with respect to the basis e_α, we obtain $a_{\alpha,\beta} = -b_\alpha + b_\beta$ which is written in the basis e_α in the form $g_{\alpha\beta,1} g_{\alpha\beta}^{-1} = -h_\alpha + g_{\alpha\beta} h_\beta g_{\alpha\beta}^{-1}$. Thus the equivalence classes of first order deformations of $\mathcal{E}$ over $V = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$ are classified by

$$\check{H}^1\big(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) = \frac{\{\text{1-cocycles } (a_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))\}}{\{\text{coboundaries } a_{\alpha\beta} = b_\beta - b_\alpha, \text{ where } (b_\alpha) \in \check{C}^0(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))\}}.$$

1.3. First obstruction. We denote $V_k = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon)^{k+1}$. We will investigate the following question: which of the deformations of $\mathcal{E}$ over V_1 lift to V_2?

Let $G_{\alpha\beta} = g_{\alpha\beta,0} + \varepsilon g_{\alpha\beta,1} + \varepsilon^2 g_{\alpha\beta,2}$ be a deformation of the cocycle $g_{\alpha\beta} = g_{\alpha\beta,0}$ over V_2.

We want to prove, in other words that $G_{\alpha\beta}$ gives a valid 2nd-order deformation if and only if it satisfies the cocycle condition.

Assume that $G_{\alpha\beta} \mod \varepsilon^2$ is a 1-cocycle, then (2) and (3) are verified, and compute the coefficient $K_{\alpha\beta\gamma,2}$ of ε^2 in $G_{\alpha\beta} G_{\beta\gamma} G_{\gamma\alpha}$, which will be denoted

$K_{\alpha\beta\gamma,2}$:

$$K_{\alpha\beta\gamma,2} = g_{\alpha\beta,0}g_{\beta\gamma,1}g_{\gamma\alpha,1} + g_{\alpha\beta,1}g_{\beta\gamma,0}g_{\gamma\alpha,1} + g_{\alpha\beta,1}g_{\beta\gamma,1}g_{\gamma\alpha,0} + g_{\alpha\beta,2}g_{\beta\gamma,0}g_{\gamma\alpha,0} + g_{\alpha\beta,0}g_{\beta\gamma,2}g_{\gamma\alpha,0} + g_{\alpha\beta,0}g_{\beta\gamma,0}g_{\gamma\alpha,2}. \tag{4}$$

Similar to the above, introduce the sections $a_{\alpha\beta,i}$ $(i=1,2)$, of the endomorphism sheaf $\mathcal{E}\mathrm{nd}(\mathcal{E}_{|(U_{\alpha\beta})})$ having $g_{\alpha\beta,i}g_{\alpha\beta}^{-1}$ for their matrices in the bases e_α. Then, as above, $g_{\alpha\beta,2}g_{\beta\gamma,0}g_{\gamma\alpha,0} + g_{\alpha\beta,0}g_{\beta\gamma,2}g_{\gamma\alpha,0} + g_{\alpha\beta,0}g_{\beta\gamma,0}g_{\gamma\alpha,2}$ is the matrix of $a_{\alpha\beta,2} + a_{\beta\gamma,2} + a_{\gamma\alpha,2}$ in the basis e_α, and $g_{\alpha\beta,0}g_{\beta\gamma,1}g_{\gamma\alpha,1} + g_{\alpha\beta,1}g_{\beta\gamma,0} \times g_{\gamma\alpha,1} + g_{\alpha\beta,1}g_{\beta\gamma,1}g_{\gamma\alpha,0}$ is the matrix of

$$a_{\beta\gamma,1}a_{\gamma\alpha,1} + a_{\alpha\beta,1}a_{\gamma\alpha,1} + a_{\alpha\beta,1}a_{\beta\gamma,1} \tag{5}$$

in the basis e_α. Let a_1 denote the cocycle $(a_{\alpha\beta,1})$ and $[a_1]$ its class in $\check{H}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$. Then $a_{\beta\gamma,1}a_{\gamma\alpha,1} = c_{\beta\gamma\alpha}$ represents the Yoneda product $[a_1] \circ [a_1] = [c] \in \check{H}^2(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$; see for instance 10.1.1. of [H-L] for the definition of the Yoneda product

$$\check{H}^i\big(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) \times \check{H}^j\big(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) \to \check{H}^{i+j}\big(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})\big).$$

The whole expression (5) is the skew-symmetrization $\hat{c}_{\alpha\beta\gamma}$ of $c_{\beta\gamma\alpha}$, hence it represents the same cohomology class $[c]$. Let also a_2 denote the Čech cochain $(a_{\alpha\beta,2})$. We can rewrite $K_2 = (K_{\alpha\beta\gamma,2})$ in the form

$$K_2 = \hat{c} + \check{d}a_2. \tag{6}$$

We now see that we can find a_2 in such a way that $(G_{\alpha\beta})$ is a cocycle over V_2 if and only if $\hat{c}$ is $\check{d}$-exact. We have proved the following proposition.

Proposition 1.4. *Let X be a complete scheme of finite type over k or a complex space (and then $k=\mathbb{C}$), $\mathcal{E}$ a vector bundle on X, $[a] \in H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))$. Then the first order deformation of $\mathcal{E}$ over V_1 defined by $[a]$ lifts to a deformation over V_2 if and only if the Yoneda square $[a] \circ [a]$ is zero in $H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))$.*

Definition 1.5. The map

$$\begin{aligned} H^1\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) &\to H^2\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E})\big), \\ \big([a]\big) &\mapsto [a] \circ [a] \end{aligned} \tag{7}$$

will be called first obstruction, and denoted $\mathrm{ob}^{(2)}$.

Thus $\mathrm{ob}^{(2)}$ is the map of taking the Yoneda square. We will now construct a universal first order deformation of $\mathcal{E}$ on X. Let $W = H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))$, $t_1, \dots, t_N$ a coordinate system on W, $W_k = \operatorname{Spec} k[t_1, \dots, t_N]/(t_1, \dots, t_N)^{k+1}$ the k-th infinitesimal neighborhood of the origin in W. The universal first order deformation $\mathcal{E}_1$ of $\mathcal{E}$ over W_1 can be described as follows.

Choose an open covering of X as above, so that $\mathcal{E}$ is defined by a 1-cocycle $(g_{\alpha\beta})$. We deform $\mathcal{E}$ by specifying a family $G_{\alpha\beta}(t_1, \dots, t_N)$ of 1-cocyles over $X \times W_1$. Pick up N cocycles $a_i = (a^{(i)}_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ whose cohomology classes $[a_1], \dots, [a_N]$ form a basis of W dual to the coordinates $t_1, \dots, t_N$. Then we set $g^{(i)}_{\alpha\beta} = a^{(i)}_{\alpha\beta} g_{\alpha\beta}$, where $a^{(i)}_{\alpha\beta}$ is represented by its matrix in the basis e_α and write $G_{\alpha\beta}(t_1, \dots, t_N) = g_{\alpha\beta} + \sum_{i=1}^N g^{(i)}_{\alpha\beta} t_i$. Then $G_{\alpha\beta}$ is a 1-cocycle and defines a vector bundle $\mathcal{E}_1$ over $X \times W_1$ called a universal first order deformation of $\mathcal{E}$. The whole universal deformation over W_1 cannot be lifted to a deformation on W_2. Proposition 1.4 implies the following.

Proposition 1.6. *There is a maximal subscheme $K_2 \subset W_2$ with the property that $\mathcal{E}_1$ extends as a vector bundle from $X \times W_1$ to $X \times K_2$. This maximal subscheme K_2 is the (second infinitesimal neighborhood of the origin in the cone) defined by the equation* $\mathrm{ob}^{(2)}(z) = 0$ *in W_2.*

We will now prove the following theorem, providing a construction of the formal Kuranishi space.

Theorem 1.7. *Let $X, \mathcal{E}$ be as above, $W = H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))$, $(\delta_1, \dots, \delta_N)$ a basis of W and $(t_1, \dots, t_N)$ the dual coordinates on W. Let $W_k = \operatorname{Spec} k[t_1, \dots, t_N]/(t_1, \dots, t_N)^{k+1}$ be the k-th infinitesimal neighborhood of the origin in W, $\mathcal{E}_1$ a universal first order deformation of $\mathcal{E}$ over $X \times W_1$ as above. Then there exists a formal power series*

$$f(t_1, \dots, t_N) = \sum_{k=2}^{\infty} f_k(t_1, \dots, t_N) \in H^2\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E})\big)[[t_1, \dots, t_N]],$$

where f_k is homogeneous of degree k, with the following property. Let I be the ideal of $k[[t_1, \dots, t_N]]$ generated by the image of the map $f^* : H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))^* \to k[[t_1, \dots, t_N]]$, adjoint to f. Then for any $k \geq 2$, the universal first deformation $\mathcal{E}_1$ of $\mathcal{E}$ over $X \times W_1$ extends to a vector bundle $\mathcal{E}_k$ on $X \times K_k$, where K_k is a closed subscheme of W_k defined by the ideal $I \otimes k[[t_1, \dots, t_N]]/(t_1, \dots, t_N)^{k+1}$.

Definition 1.8. The inverse limit $\mathbb{K} = \varprojlim K_k$ is called the formal Kuranishi space of $\mathcal{E}$, and $\boldsymbol{\mathcal{E}} = \varprojlim \mathcal{E}_k$ the formal universal bundle over $\mathbb{K}$.

Proof of Theorem 1.7. Let $\mathfrak{U} = (U_k)$ be an open covering, sufficiently fine so that $\mathcal{E}_{|U_\alpha}$ is trivialized by a basis e_α, and the groups $H^i(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ are computed by the Čech complex $(\check{C}^\bullet(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})), \check{d})$. Let $\check{Z}^i(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})), \check{B}^i(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ denote the subspaces of cocycles and coboundaries in $\check{C}^i(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ respectively. Let us fix some cross-sections $\sigma_i : H^i(X, \mathcal{E}\mathrm{nd}(\mathcal{E})) \to \check{Z}^i(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ and $\tau : \check{B}^2(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})) \to \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ of the natural maps in the opposite direction. Let $a_i = (a^{(i)}_{\alpha\beta}) = \sigma_1(\delta_i)$, and denote, as above, by $(g_{\alpha\beta})$ the 1-cocycle defining $\mathcal{E}$, so that $e_\beta = e_\alpha g_{\alpha\beta}$. We will construct by induction on $k \geq 0$ the homogeneous forms of degree k in $t_1, \ldots, t_N$

$$\begin{aligned} G_{\alpha\beta,k}(t_1,\ldots,t_N) &\in \Gamma\big(U_{\alpha\beta}, M_r(\mathcal{O}_X)\big) \otimes k[t_1,\ldots,t_N],\\ F_{\alpha\beta\gamma,k}(t_1,\ldots,t_N) &\in \Gamma\big(U_{\alpha\beta\gamma}, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) \otimes k[t_1,\ldots,t_N],\\ f_k(t_1,\ldots,t_N) &\in H^2\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) \otimes k[t_1,\ldots,t_N] \end{aligned} \tag{8}$$

with the following properties:

(i) $G_{\alpha\beta,0} = g_{\alpha\beta}, G_{\alpha\beta,1} = \sum_{i=1}^N a_{\alpha\beta}^{(i)} g_{\alpha\beta} t_i$, where $a_{\alpha\beta}^{(i)}$ are represented by their matrices in the basis e_α.

(ii) $f_k = 0, F_{\alpha\beta\gamma,k} = 0$ for $k = 0, 1$.

(iii) For each k ≥ 1, let $f^{(k)} = \sum_{i\leq k} f_i$, and let $I^{(k+1)}$ be the ideal generated by $(t_1,\ldots,t_N)^{k+2}$ and the image of the adjoint map $f^{(k)*} : H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))^* \to k[t_1,\ldots,t_N]$. Then $(F_{\alpha\beta\gamma,k+1})$ is a cocycle modulo $I^{(k+1)}$ and f_{k+1} is a lift to $H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes k[t_1,\ldots,t_N]$ of the cohomology class $[(F_{\alpha\beta\gamma,k+1} \mod I^{(k+1)})] \in H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes k[t_1,\ldots,t_N]/I^{(k+1)}$.

(iv) For any $k \geq 1$, set $G_{\alpha\beta}^{(k)} = \sum_{i\leq k} G_{\alpha\beta,i}$. Then $G_{\alpha\beta}^{(k)} G_{\beta\gamma}^{(k)} G_{\gamma\alpha}^{(k)} \equiv (1 + F_{\alpha\beta\gamma,k+1}) \mod I^{(k+1)}$. Properties (i), (ii) determine $G_{\alpha\beta,k}, F_{\alpha\beta\gamma,k}$ for $k \leq 1$.

The proof of Proposition 1.4 allows us to see that (iii), (iv) are verified for $k = 1$ with

$$F_{\alpha\beta\gamma,2} = \sum_{i,j=1}^N \big(a_{\beta\gamma}^{(i)} a_{\gamma\alpha}^{(j)} + a_{\alpha\beta}^{(i)} a_{\gamma\alpha}^{(j)} + a_{\alpha\beta}^{(i)} a_{\beta\gamma}^{(j)}\big) t_i t_j$$

and to determine $G_{\alpha\beta,2}$ we proceed as follows. Let $f_2 = [(F_{\alpha\beta\gamma,2})]$, and $I^{(2)}$ be the ideal of K_2, that is the ideal generated by $(t_1,\ldots,t_N)^3$ and the image of the adjoint map $f^{(2)*} : H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))^* \to k_2[t_1,\ldots,t_N] = \mathrm{Sym}^2(W^*)$ (the degree-2 homogeneous part of $k[t_1,\ldots,t_N]$). Then the reduction $\mod I^{(2)}$ of $F_2 = (F_{\alpha\beta\gamma,2})$ is an element $\bar{F}_2 = (F_{\alpha\beta\gamma,2}) \mod I^{(2)} \in \check{B}^2(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes (\mathrm{Sym}^2(W^*)/I^{(2)} \cap \mathrm{Sym}^2(W^*))$. We define a skew-symmetric 1-cochain $a_2 = a_{\alpha\beta,2} \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes \mathrm{Sym}^2(W^*)$ as an arbitrary lift of $(\tau \otimes \mathrm{id})(\bar{F}_2) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes (\mathrm{Sym}^2(W^*)/I^{(2)} \cap \mathrm{Sym}^2(W^*))$ under the quotient map. Next we define $G_{\alpha\beta,2}$ by $G_{\alpha\beta,2} = a_{\alpha\beta,2} g_{\alpha\beta}$, where the matrix of $a_{\alpha\beta,2}$ is taken in the basis e_α.

Likewise, assuming that $G_{\alpha\beta}^{(k-1)}$, $F_{\alpha\beta}^{(k)}$ are already fixed, we can choose $F_{\alpha\beta\gamma,k+1}$ and $G_{\alpha\beta,k}$ as follows. By the induction hypothesis, we have $G_{\alpha\beta}^{(k-1)} \times G_{\beta\gamma}^{(k-1)} G_{\gamma\alpha}^{(k-1)} \equiv (1 + F_{\alpha\beta\gamma,k}) \mod I^{(k)}$. Then $(F_{\alpha\beta\gamma,k})$ is a cocycle modulo $I^{(k)}$, and is a coboundary modulo $I^{(k+1)} : \bar{F}_k = (F_{\alpha\beta\gamma,k} \mod I^{(k+1)}) \in \check{B}^2(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes (\mathrm{Sym}^k(W^*)/I^{(k+1)} \cap \mathrm{Sym}^k(W^*))$. We define $G_{\alpha\beta,k} = a_{\alpha\beta,k} \times g_{\alpha\beta}$ with $(a_{\alpha\beta,k}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes \mathrm{Sym}^k(W^*)$ an arbitrary skew-symmetric lift to $\mathrm{Sym}^k(W^*)$ of $(\tau \otimes \mathrm{id})(\bar{F}_k)$. Then $G_{\alpha\beta}^{(k)} G_{\beta\gamma}^{(k)} G_{\gamma\alpha}^{(k)} \equiv 1 \mod (I^{(k+1)} + (t_1, \ldots, t_N)^{(k+1)})$, and we can define $F_{\alpha\beta\gamma,k+1}$ as the degree-$(k+1)$ homogeneous component of $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha}$. To end this inductive construction of the sequences $G_{\alpha\beta,k}$, $F_{\alpha\beta\gamma,k+1}$, we need only to prove that $F_{k+1} = (F_{\alpha\beta\gamma,k+1})$ is a 2-cocycle modulo $I^{(k+1)}$ with values in $\mathcal{E}\mathrm{nd}(\mathcal{E})$. □

The latter is proved in Lemma 1.9 below.

LEMMA 1.9. *The* 2*-cochain* $(F_{\alpha\beta\gamma,k+1})$, *constructed in the proof of Theorem 1.7 as the degree-*$(k+1)$ *homogeneous component of* $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha}$, *is a* 2*-cocycle modulo* $I^{(k+1)}$ *with values in* $\mathcal{E}\mathrm{nd}(\mathcal{E})$.

Proof. The hypotheses, under which we have to prove the assertion of Lemma 1.9, are the following: $G^{(k)}_{\alpha\beta} = \sum_{i=0}^{k} G_{\alpha\beta,i} \in \Gamma(U_{\alpha\beta}, M_r(\mathcal{O}_X)) \otimes k[t_1, \dots, t_N]$ are the matrix polynomials of degree $\leq k$ in $t_1, \dots, t_N$ and there is an ideal $J \subset (t_1, \dots, t_N)^2$ such that $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\alpha} \equiv 1 \mod J$ and $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha} \equiv 1 \mod (J + (t_1, \dots, t_N)^{k+1})$. The ideal J in Theorem 1.7 is $I^{(k+1)}$. The collection $(F_{\alpha\beta\gamma,k})$ is considered not as a 2-cochain in $M_r(\mathcal{O}_X)$, but as a 2-cochain in $\mathcal{E}\mathrm{nd}(\mathcal{E})$, $\mathcal{E}$ being defined by the multiplicative cocycle $(g_{\alpha\beta}) = G_{\alpha\beta,0} \in \check{Z}^1(\mathfrak{U}, \mathrm{GL}_r(\mathcal{O}_X))$. Thus, $F_{\alpha\beta\gamma} = F_{\alpha\beta\gamma,k+1}$ is a certain section of $\mathcal{E}\mathrm{nd}(\mathcal{E})$ over $U_{\alpha\beta\gamma}$ given by its matrix in the basis e_α of $\mathcal{E}_{|U_{\alpha\beta\gamma}}$. We want to show that

$$F_{\alpha\beta\gamma} - F_{\alpha\beta\delta} + F_{\alpha\gamma\delta} - F_{\beta\gamma\delta} \equiv 0 \mod J. \tag{9}$$

We will replace it by a slightly different identity

$$F_{\alpha\beta\gamma} + F_{\alpha\gamma\delta} + F_{\alpha\delta\beta} + F_{\beta\delta\gamma} \equiv 0 \mod J, \tag{10}$$

which is the same as (9) as soon as we know that $(F_{\alpha\beta\gamma})$ is skew symmetric. We have:

$$\begin{aligned}
&F_{\alpha\beta\gamma} = [G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha}]_{k+1}, \qquad F_{\alpha\gamma\delta} = [G_{\alpha\gamma}G_{\gamma\delta}G_{\delta\alpha}]_{k+1},\\
&F_{\alpha\delta\beta} = [G_{\alpha\delta}G_{\delta\beta}G_{\beta\alpha}]_{k+1},\\
&F_{\beta\delta\gamma} = G_{\alpha\beta,0}\big([G_{\beta\delta}G_{\delta\gamma}G_{\gamma\beta}]_{k+1}\big)G^{-1}_{\alpha\beta,0} = [G_{\alpha\beta}G_{\beta\delta}G_{\delta\gamma}G_{\gamma\beta}G_{\beta\alpha}]_{k+1},
\end{aligned}$$

where we omitted the superscript k in $G^{(k)}_{\alpha\beta}$, $[\dots]_{k+1}$ stands for the homogeneous component of degree $k+1$ in $t_1, \dots, t_N$, and all the four terms are given by their matrices in the basis e_α. Now

$$\begin{aligned}
&F_{\alpha\beta\gamma} + F_{\alpha\gamma\delta} + F_{\alpha\delta\beta} + F_{\beta\delta\gamma}\\
&\quad = [G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} + G_{\alpha\gamma}G_{\gamma\delta}G_{\delta\alpha} + G_{\alpha\delta}G_{\delta\beta}G_{\beta\alpha} + G_{\alpha\beta}G_{\beta\delta}G_{\delta\gamma}G_{\gamma\beta}G_{\beta\alpha}]_{k+1}\\
&\quad \equiv [G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} \times G_{\alpha\gamma}G_{\gamma\delta}G_{\delta\alpha} \times G_{\alpha\delta}G_{\delta\beta}G_{\beta\alpha} \times G_{\alpha\beta}G_{\beta\delta}G_{\delta\gamma}G_{\gamma\beta}G_{\beta\alpha}]_{k+1}\\
&\quad \equiv 0 \mod J.
\end{aligned}$$

The skew symmetry of $(F_{\alpha\beta\gamma})$ is a particular case of (10) when $\delta = \gamma$. □

2. Connections

Let $X, \mathcal{E}$ be as above. A rational (or meromorphic in the case when X is a complex space) connection on $\mathcal{E}$ is a k-linear morphism of sheaves $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X(D)$ satisfying the Leibniz rule:

$$\forall p \in X, \forall f \in \mathcal{O}_p, \forall s \in \mathcal{E}_p, \quad \nabla(fs) = f\nabla s + s \otimes \mathrm{d}f.$$

We assume that D is an effective Cartier divisor and call D the divisor of poles of ∇. We can extend ∇ in a natural way to

$$\mathcal{E} \otimes \Omega^\bullet(*D) = \varinjlim_n \bigoplus_{i \geq 0} \mathcal{E} \otimes \Omega^i(nD)$$

as a k-linear map $\nabla : \mathcal{E} \otimes \Omega^i(*D) \to \mathcal{E} \otimes \Omega^{i+1}(*D)$ satisfying the Leibniz rule $\nabla(s \otimes \omega) = \nabla s \wedge \omega + s \otimes \mathrm{d}\omega$. The connection is integrable if $\nabla^2 = 0$. In this case, ∇ defines the generalized de Rham complex

$$0 \to \mathcal{E}(*D) \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1(*D) \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^2(*D) \xrightarrow{\nabla} \cdots. \tag{11}$$

If X is smooth at all the points of $X \setminus D$, then this complex is exact over $X \setminus D$ in all degrees different from 0 by the Poincaré lemma. Under the same assumption, the subsheaf $\mathcal{E}^h$ of sections s of $\mathcal{E}|_{X \setminus D}$ satisfying $\nabla(s) = 0$ is a local system of rank r, that is a vector bundle with constant transition functions, and $\mathcal{E}|_{X\setminus D} = \mathcal{E}^h \otimes \mathcal{O}_{X \setminus D}$; the sections of $\mathcal{E}^h$ are called horizontal sections of $(\mathcal{E}, \nabla)$. The complex defined above, when restricted to $X \setminus D$, is a resolution of $\mathcal{E}^h$.

A connection ∇ on $\mathcal{E}$ induces natural connections on $\mathcal{E}^*, \mathcal{E}\mathrm{nd}(\mathcal{E}), (\mathcal{E}^*)^{\otimes m} \otimes \mathcal{E}^{\otimes n}$, and more generally, on any Schur functor of $\mathcal{E}$ or $\mathcal{E}^*$. We will use in the sequel the induced connection $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}$ on $\mathcal{E}\mathrm{nd}(\mathcal{E})$. Taking a local section φ of $\mathcal{E}\mathrm{nd}(\mathcal{E})$, we can think of φ as a sheaf homomorphism $\mathcal{E} \to \mathcal{E}$ over an open set $U \subset X$, and $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}$ is defined by

$$\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(\varphi) = \nabla \circ \varphi - \varphi \circ \nabla,$$
$$\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})} : \mathcal{E}\mathrm{nd}(\mathcal{E}) \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D).$$

If ∇ is integrable, then $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}$ is also integrable, and $\mathcal{E}\mathrm{nd}(\mathcal{E})^h = \mathcal{E}\mathrm{nd}(\mathcal{E}^h)$.

Let now $\mathfrak{U} = (U_\alpha)$ be a sufficiently fine open covering of X, e_α a trivialization of $\mathcal{E}$ over U_α, $(g_{\alpha\beta})$ the transition functions of $\mathcal{E}$ with respect to the trivilizations (e_α). The connection matrices $A_\alpha \in \Gamma(U_\alpha, M_r(\mathcal{O}_X) \otimes \Omega^1(D))$ of ∇ are defined by $\nabla(e_\alpha) = e_\alpha A_\alpha$. The transition rule for the matrices A_α is

$$A_\beta = g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} \tag{12}$$

over $U_{\alpha\beta}$. This equation can be given a cohomological interpretation. To this end, introduce the cochains $\mathcal{A} = (\mathcal{A}_\alpha) \in \check{C}^0(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))$, $\mathcal{G} = (\mathcal{G}_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1)$ by saying that the matrix of $\mathcal{A}_\alpha$ (resp. $\mathcal{G}_{\alpha\beta}$) in the basis e_α is A_α (resp. $\mathrm{d}g_{\alpha\beta} g_{\alpha\beta}^{-1}$). Then $\mathcal{G}$ is a cocycle.

DEFINITION 2.1. The cohomology class $[\mathcal{G}]$ of $\mathcal{G}$ in $H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1)$ does not depend on the choice of trivializations (e_α) and is called the Atiyah class of $\mathcal{E}$. We will denote this class by $\mathrm{At}(\mathcal{E})$ and its image in $H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))$, in $H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(*D))$ by $\mathrm{At}^D(\mathcal{E})$, (resp. $\mathrm{At}^{*D}(\mathcal{E})$).

Now we can write (12) in the form

$$\mathcal{G} = \check{d}\mathcal{A},$$

and we get the following assertion.

PROPOSITION 2.2. *Let $X, \mathcal{E}$ be as above, D an effective Cartier divisor in X. Then $\mathcal{E}$ admits a connection with divisor of poles D if and only if* $\mathrm{At}^D(\mathcal{E})$ *vanishes in $H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))$.*

Informally speaking, this property is expressed by saying that the Atiyah class is the obstruction to the existence of a connection on a vector bundle. For future use, we also provide the integrability condition of ∇ in terms of the local data A_α:

$$\mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha = 0. \tag{13}$$

2.1. First order deformations of connections with fixed divisor of poles D. Let $(\mathcal{E}, \nabla)$ be defined as above and $V_1 = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$. We represent the deformed pair $(\tilde{\mathcal{E}}, \tilde{\nabla})$ over V_1 by the local data

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \varepsilon g_{\alpha\beta,1}, \qquad \tilde{A_\alpha} = A_\alpha + \varepsilon A_{\alpha,1}.$$

We have already studied the compatibility conditions which guarantee that $\tilde{g}_{\alpha\beta}$ ia a cocycle; they can be stated by saying that the cochain $a = (a_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$, defined over $U_{\alpha\beta}$ by the matrix $g_{\alpha\beta,1}g_{\alpha\beta}^{-1}$ in the basis e_α, is a cocycle. Now, we fix this cocycle and search for a cochain $(\mathcal{A}_{\alpha,1})$ compatible with a. Expanding (12) to order 1, we obtain:

$$A_{\beta,1} = g_{\beta\alpha,1}\mathrm{d}g_{\alpha\beta} + g_{\beta\alpha}\mathrm{d}g_{\alpha\beta,1} + g_{\beta\alpha,1}A_\alpha g_{\alpha\beta} + g_{\beta\alpha}A_{\alpha,1}g_{\alpha\beta} + g_{\beta\alpha}A_\alpha g_{\alpha\beta,1}. \tag{14}$$

LEMMA 2.3. *Define the* 0*-cochain $\mathcal{A}_1 = (\mathcal{A}_{\alpha,1})$ in $\mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1_X(D)$ whose matrix over U_α is $A_{\alpha,1}$ in the basis e_α. Then* (14) *implies*:

$$(\check{d}\mathcal{A}_1)_{\alpha\beta} = \mathcal{A}_{\beta,1} - \mathcal{A}_{\alpha,1} = \mathrm{d}a_{\alpha\beta} + [A_\alpha, a_{\alpha\beta}]. \tag{15}$$

Proof. Conjugate (14) by $g_{\alpha\beta}$:

$$\begin{aligned} g_{\alpha\beta}A_{\beta,1}g_{\alpha\beta}^{-1} = g_{\beta\alpha}^{-1}g_{\beta\alpha,1}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} + \mathrm{d}g_{\alpha\beta,1}g_{\alpha\beta}^{-1} \\ + g_{\alpha\beta}g_{\beta\alpha,1}A_\alpha + A_{\alpha,1} + A_\alpha g_{\alpha\beta,1}g_{\alpha\beta}^{-1}. \end{aligned} \tag{16}$$

Then $g_{\alpha\beta}A_{\beta,1}g_{\alpha\beta}^{-1}$, $A_{\alpha,1}$ are the matrices of $\mathcal{A}_{\beta,1}, \mathcal{A}_{\alpha,1}$ respectively in the basis e_α; we will also interpret all the remaining terms of (16) as matrices of some sections of $\mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)$. We have

$$g_{\beta\alpha}^{-1}g_{\beta\alpha,1} = a_{\beta\alpha} = -a_{\alpha\beta}; \qquad g_{\alpha\beta,1}g_{\beta\alpha}^{-1} = a_{\alpha\beta}, \tag{17}$$

so that

$$g_{\alpha\beta}g_{\beta\alpha,1}A_\alpha + A_\alpha g_{\alpha\beta,1}g_{\alpha\beta}^{-1} = [A_\alpha, a_{\alpha\beta}]. \tag{18}$$

Next, $g_{\alpha\beta,1} = a_{\alpha\beta}g_{\alpha\beta}$, so that

$$\mathrm{d}g_{\alpha\beta,1} = \mathrm{d}a_{\alpha\beta}g_{\alpha\beta} + a_{\alpha\beta}\mathrm{d}g_{\alpha\beta}. \tag{19}$$

Further, by (17),

$$g_{\beta\alpha}^{-1}g_{\beta\alpha,1}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} = -a_{\alpha\beta}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1}. \tag{20}$$

Combining (19), (20), we obtain

$$\begin{aligned} &g_{\beta\alpha}^{-1}g_{\beta\alpha,1}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} + \mathrm{d}g_{\alpha\beta,1}g_{\alpha\beta}^{-1} \\ &\quad = -a_{\alpha\beta}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} + \mathrm{d}a_{\alpha\beta} + a_{\alpha\beta}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} = \mathrm{d}a_{\alpha\beta}. \end{aligned} \tag{21}$$

Substituting (18), (21) into (16), we obtain (15). □

Corollary 2.4. *The pair* $(\tilde{g}_{\alpha\beta}), (\tilde{\mathcal{A}}_\alpha)$ *defines a first order deformation of* $(\mathcal{E}, \nabla)$ *if and only if the cochains* $a = (a_{\alpha\beta}) = (g_{\alpha\beta,1}g_{\alpha\beta}^{-1}), \mathcal{A}_{\alpha,1} = A_{\alpha,1}$ *(both given in the basis* e_α*) satisfy the relations* $\check{d}(a_{\alpha\beta}) = 0, \check{d}(\mathcal{A}_{\alpha,1}) = (\mathrm{d}a_{\alpha\beta} + [A_\alpha, a_{\alpha\beta}])$.

We will interpret the latter result in terms of the induced connection on $\mathcal{E}\mathrm{nd}\,(\mathcal{E})$. As we saw, given a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$ on $\mathcal{E}$, we can define a connection $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})} : \mathcal{E}\mathrm{nd}(\mathcal{E}) \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)$ by $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(\varphi) = \nabla \circ \varphi - \varphi \circ \nabla$. If we represent φ by its matrix M_α in the basis e_α, then $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(\varphi) = \mathrm{d}M_\alpha + [A_\alpha, M_\alpha]$. Now, we can reformulate Corollary 2.4 as follows.

Proposition 2.5. *The first order deformations of* $(\mathcal{E}, \nabla)$ *with fixed divisor of poles* D *are classified by the pairs* $(a, \mathcal{A}_1) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})) \times \check{C}^0(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))$ *such that*

$$\check{d}(a) = 0, \qquad \check{d}(\mathcal{A}_1) = \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(a). \tag{22}$$

Now, let us assume in addition that the initial connection is integrable. Then the condition that the deformed connection $(\tilde{\mathcal{E}}, \tilde{\nabla})$, given by the data $(a, \mathcal{A}_1)$ as in Proposition 2.5 , remains integrable, can be written in the form:

$$\mathrm{d}A_{\alpha,1} = -A_{\alpha,1} \wedge A_\alpha - A_\alpha \wedge A_{\alpha,1}, \tag{23}$$

or in an invariant form, $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(A_1) = 0$. We remark that here we consider $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}$ extended to $\mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^\bullet(*D)$ in the same way as was explained for $\nabla = \nabla_{\mathcal{E}}$.

Proposition 2.6. *The first order deformations of integrable connections* $(\mathcal{E}, \nabla)$ *with fixed divisor of poles* D *are classified by the pairs* $(a, \mathcal{A}_1)$ *as above satisfying three relations*

$$\check{d}(a) = 0, \qquad \check{d}(\mathcal{A}_1) = \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(a), \qquad \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(\mathcal{A}_1) = 0. \tag{24}$$

2.2. Hypercohomology. Let $K^\bullet = (K^p, d_K)$ be a complex of sheaves over X, and $\mathfrak{U} = (U_\alpha)$ a sufficiently fine open covering of X. The Čech complex of $K^\bullet$ is the double complex

$$\big(\check{C}^p\big(\mathfrak{U}, K^q\big), \check{d}, (-1)^p d_K\big). \tag{25}$$

The hypercohomology group $\mathbb{H}^i(X, K^\bullet)$ is by definition the i-th cohomology of the simple complex $(L^\bullet, D)$ associated to (25):

$$L^n = \bigoplus_{p+q=n} \check{C}^p\big(\mathfrak{U}, K^q\big), \qquad D_{|_{\check{C}^p(\mathfrak{U}, K^q)}} = \check{d} + (-1)^p d_K,$$

$$\mathbb{H}^i\big(X, K^\bullet\big) := H^i\big(L^\bullet, D\big).$$

A hypercohomology class $c \in \mathbb{H}^i(X, K^\bullet)$ is represented by a cocycle $c \in L^i$, $c = (\ldots, c^{p-1,q+1}, c^{p,q}, c^{p+1,q-1}, \ldots)$, where $p + q = i$, and the cocycle condition is $(\ldots, \check{d}c^{p-1,q+1} + (-1)^p d_K c^{p,q} = 0, \check{d}c^{p,q} + (-1)^{p+1} d_K c^{p+1,q-1} = 0, \ldots)$. A cocycle $(c^{p,q})_{p+q=n}$ is a coboundary if there exists a cochain $(b^{p,q})_{p+q=n-1}$ such that

$$c^{p,q} = \check{d}b^{p-1,q} + (-1)^p d_K b^{p,q-1}.$$

We denote the i-cocycles $\check{Z}^i(\mathfrak{U}, K^\bullet)$ and the i-coboundaries $\check{B}^i(\mathfrak{U}, K^\bullet)$, so that

$$\mathbb{H}^i\big(X, K^\bullet\big) = \check{Z}^i\big(\mathfrak{U}, K^\bullet\big) / \check{B}^i\big(\mathfrak{U}, K^\bullet\big).$$

Let now come back to the setting of Proposition 2.5. Define the two-term complex of sheaves

$$\mathcal{C}^\bullet = \big[\mathcal{C}^0 \to \mathcal{C}^1\big], \tag{26}$$

where $\mathcal{C}^0 = \mathcal{E}\mathrm{nd}(\mathcal{E})$, $\mathcal{C}^1 = \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)$, and differential $d_{\mathcal{C}} = \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}$. Then the equations (22) express the fact that $(a, \mathcal{A}_1) \in \check{Z}^1(\mathfrak{U}, \mathcal{C}^\bullet)$. Changing the bases e_α over $V_1 = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ by the rule $\tilde{e}_\alpha = e_\alpha(1 + \varepsilon h_\alpha)$, where $h = (h_\alpha) \in \check{C}^0(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})) = \check{C}^0(\mathfrak{U}, \mathcal{C}^0)$, we obtain the transformation rule of the cocycle $(a, \mathcal{A}_1)$ in the following form: $(a, \mathcal{A}_1) \to (a + \check{d}h, \mathcal{A}_1 + d_{\mathcal{C}}h)$, so that isomorphic first order deformations differ by a 1-coboundary. We deduce the following theorem.

THEOREM 2.7. *Let X be a complete scheme of finite type over k or a complex space* (*then $k = \mathbb{C}$*). *Let $\mathcal{E}$ be a vector bundle on X and ∇ a rational* (*or meromorphic*) *connection on $\mathcal{E}$ with divisor of poles D. Then the isomorphism classes of first order deformations of $(\mathcal{E}, \nabla)$ with fixed divisor of poles are classified by $\mathbb{H}^1(X, \mathcal{C}^\bullet)$.*

In order to characterize the first order deformations of integrable connections, we introduce two other complexes:

$$\mathcal{R}^\bullet = \big[\mathcal{E}\mathrm{nd}(\mathcal{E}) \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D) \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^2(*D) \to \ldots\big]$$

with differential $d_{\mathcal{R}} = \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}$, and

$$\mathcal{F}^\bullet = \left[\mathcal{F}^0 \xrightarrow{d_{\mathcal{F}}} \mathcal{F}^1\right], \tag{27}$$

where $\mathcal{F}^0 = \mathcal{E}\mathrm{nd}(\mathcal{E})$, $d_{\mathcal{F}} = \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}$, and $\mathcal{F}^1 = \ker(\mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)) \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^2(*D))$. It is easy to see that these complexes have the same 1-cocycles and 1-coboundaries, so that

$$\mathbb{H}^1\big(X, \mathcal{F}^\bullet\big) = \mathbb{H}^1\big(X, \mathcal{R}^\bullet\big).$$

The formulas (20) express the fact that the pair $(a, \mathcal{A}_1)$ is a 1-cocycle in either one of the complexes $\mathcal{F}^\bullet, \mathcal{R}^\bullet$.

THEOREM 2.8. *Let X be a scheme of finite type over k or a complex space (then $k = \mathbb{C}$). Let $\mathcal{E}$ a vector bundle on X and ∇ a rational (or meromorphic) integrable connection on $\mathcal{E}$ with fixed divisor of poles D. Then the isomorphism classes of first order deformations of $(\mathcal{E}, \nabla)$ in the class of integrable connections with fixed divisor of poles D are classified by*

$$\mathbb{H}^1\big(X, \mathcal{F}^\bullet\big) = \mathbb{H}^1\big(X, \mathcal{R}^\bullet\big).$$

3. Obstructions

3.1. First obstruction. Let $X, \mathcal{E}, \nabla, (a, \mathcal{A}_1)$ be as in Theorem 2.7, and let $(\mathcal{E}_1, \nabla_1)$ be the first order deformation of $(\mathcal{E}, \nabla)$ over V_1 associated to $(a, \mathcal{A}_1)$. We want to determine the obstruction to extend $(\mathcal{E}_1, \nabla_1)$ to $(\mathcal{E}_2, \nabla_2)$ over $V_2 = \operatorname{Spec} k[\varepsilon]/(\varepsilon^3)$. As before, we only consider deformations with fixed divisor of poles D. We search for the extended data

$$G_{\alpha\beta} = \big(1 + \varepsilon a_{\alpha\beta} + \varepsilon^2 a_{\alpha\beta,2}\big) g_{\alpha\beta} = g_{\alpha\beta} + \varepsilon g_{\alpha\beta,1} + \varepsilon^2 g_{\alpha\beta,2},$$
$$\tilde{A_\alpha} = A_\alpha + \varepsilon A_{\alpha,1} + \varepsilon^2 A_{\alpha,2}, \qquad \mathcal{A}_{\alpha,1} = A_{\alpha,1},$$

with respect to the basis e_α. We assume that they satisfy the cocycle condition modulo ε^2. Then the cocycle condition modulo ε^3 has two counterparts: the one expressing the extendability of $\mathcal{E}_1$, which we have already treated in Section 2, and the other expressing the extendability of the connection. The latter has the following form:

$$\begin{aligned} A_{\beta,2} = {} & g_{\beta\alpha,2} \mathrm{d} g_{\alpha\beta} + g_{\beta\alpha,1} \mathrm{d} g_{\alpha\beta,1} + g_{\beta\alpha} \mathrm{d} g_{\alpha\beta,2} \\ & + g_{\beta\alpha,2} A_\alpha g_{\alpha\beta} + g_{\beta\alpha} A_{\alpha,2} g_{\alpha\beta} + g_{\beta\alpha} A_\alpha g_{\alpha\beta,2} \\ & + g_{\beta\alpha,1} A_{\alpha,1} g_{\alpha\beta} + g_{\beta\alpha,1} A_\alpha g_{\alpha\beta,1} + g_{\beta\alpha} A_{\alpha,1} g_{\alpha\beta,1}. \end{aligned} \tag{28}$$

Introduce the cochain $\mathcal{A}_2 \in \check{C}^0(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))$ given over U_α by the matrix $A_{\alpha,2}$ in the basis e_α. By transformations similar to those used in the proof of (10), and in using formulas (22) and $a_{\beta\alpha,2} - (a_{\alpha\beta,1})^2 + a_{\alpha\beta,2} = 0$, we reduce (28) to the following equation:

$$\begin{aligned} & \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(a_{\alpha\beta,2}) - \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(a_{\alpha\beta,1}) a_{\alpha\beta,1} - [a_{\alpha\beta,1}, \mathcal{A}_{\beta,1}] \\ & \quad = \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(a_{\alpha\beta,2}) + \mathcal{A}_{\alpha,1} a_{\alpha\beta,1} - a_{\alpha\beta,1} \mathcal{A}_{\beta,1} = \mathcal{A}_{\beta,2} - \mathcal{A}_{\alpha,2}. \end{aligned} \tag{29}$$

Let us denote

$$k_{\alpha\beta} = \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(a_{\alpha\beta,2}) + \mathcal{A}_{\alpha,1}a_{\alpha\beta,1} - a_{\alpha\beta,1}\mathcal{A}_{\beta,1}. \tag{30}$$

We consider $k = (k_{\alpha\beta})$ as a cochain in $\check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))$.

LEMMA 3.1. *k is a skew-symmetric cocycle.*

Proof. A straightforward calculation using the relations

$$a_{\alpha\beta,2} + a_{\beta\gamma,2} + a_{\gamma\alpha,2} = -a_{\alpha\beta,1}a_{\beta\gamma,1} - a_{\beta\gamma,1}a_{\gamma\alpha,1} - a_{\alpha\beta,1}a_{\gamma\alpha,1} \tag{31}$$

and $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(XY) = \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(X)Y + Y\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(X)$, for any local sections X, Y of $\mathcal{E}\mathrm{nd}(\mathcal{E})$. □

PROPOSITION 3.2. *Let $(a, \mathcal{A}_1) \in \check{Z}^1(\mathfrak{U}, \mathcal{C}^\bullet)$, and let $(\mathcal{E}_1, \nabla_1)$ be the deformation of $(\mathcal{E}, \nabla)$ over V_1 defined by $(a, \mathcal{A}_1)$. Then $(\mathcal{E}_1, \nabla_1)$ extends to a deformation $(\mathcal{E}_2, \nabla_2)$ over V_2 if and only if the following two conditions are verified*:

(i) *The Yoneda square $[a_1] \circ [a_1] \in H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ vanishes.*

(ii) *Provided* (i) *holds, let $a_2 = (a_{\alpha\beta,2}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ be a solution of* (31), *and let $k = (k_{\alpha\beta})$ be the cocycle* (30) *determined by this choice of a_2. Then $[k] \in H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))$ vanishes.*

The expression $\mathcal{A}_{\alpha,1}a_{\alpha\beta,1} - a_{\alpha\beta,1}\mathcal{A}_{\beta,1}$ entering (30) is a component $c^{1,1}$ of the Čech cocycle $(c^{1,1}, c^{2,0}) \in \check{Z}^2(\mathfrak{U}, \mathcal{C}^\bullet)$ representing the Yoneda square $[a_1, \mathcal{A}_1] \circ [a_1, \mathcal{A}_1]$. The other component is $c^{2,0}_{\alpha\beta\gamma} = a_{\alpha\beta,1}a_{\beta\gamma,1} + a_{\beta\gamma,1}a_{\gamma\alpha,1} + a_{\alpha\beta,1}a_{\gamma\alpha,1}$. Hence, we have the following proposition.

PROPOSITION 3.3. *Under the assumptions of Proposition 3.2, $(\mathcal{E}_1, \nabla_1)$ extends to $(\mathcal{E}_2, \nabla_2)$ over V_2 with fixed divisor of poles D if and only if the Yoneda square $[a_1, \mathcal{A}_1] \circ [a_1, \mathcal{A}_1]$ vanishes in $\mathbb{H}^2(X, \mathcal{C}^\bullet)$.*

3.2. Infinitesimal deformations of the Atiyah class. We fix a vector bundle $\mathcal{E}$ on X given by a cocycle $g_{\alpha\beta}$. Recall that we defined the Atiyah class of $\mathcal{E}$ as the cohomology class of the cocycle $\mathcal{G}_{\alpha\beta} = \mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1}$ (here $\mathcal{G}_{\alpha\beta}$ is considered as a section of $\mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)$ given by the matrix $\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1}$ in the basis e_α).

If $\mathcal{E}_i$ is an extension of $\mathcal{E}$ (as a vector bundle) to $X \times V_i$, where $V_i = \operatorname{Spec} k[\varepsilon]/(\varepsilon^{i+1})$, then we can define the Atiyah class $\mathrm{At}(\mathcal{E}_i) \in H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}_i) \otimes \Omega^1)$ by the cocycle $\mathcal{G}_{i,\alpha\beta} = \mathrm{d}g_{i,\alpha\beta}g_{i,\alpha\beta}^{-1}$, where $(g_{i,\alpha\beta})$ is a cocycle defining $\mathcal{E}_i$, $g_{i,\alpha\beta} \in \Gamma(U_{\alpha\beta}, M_r(\mathcal{O}_X) \otimes k[\varepsilon]/(\varepsilon^{i+1}))$. The following assertion is obvious.

LEMMA 3.4. *Assume that $\mathcal{E}$ admits a connection ∇ with fixed divisor of poles D. Then ∇ extends to a connection ∇_i on $\mathcal{E}_i$ with fixed divisor of poles D if and only if the image $\mathrm{At}^D(\mathcal{E}_i)$ of $\mathrm{At}(\mathcal{E}_i)$ in $H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}_i) \otimes \Omega^1(D))$ is zero.*

COROLLARY 3.5. *Let $j > 0$, and assume $\mathcal{E}$ extends to a vector bundle $\mathcal{E}_j$ over $X \times V_j$. For any $i \geq 0, i \leq j$, denote by $\mathcal{E}_i$ the restriction of $\mathcal{E}_j$ to $X \times V_i$. The following assertions hold*:

(i) *if ∇_j is a connection with fixed divisor D of poles on $\mathcal{E}_j$, then $\nabla_i = \nabla_j|_{\mathcal{E}_i}$ is such a connection on $\mathcal{E}_i$. Thus $\mathrm{At}^D(\mathcal{E}_j) = 0 \Rightarrow \mathrm{At}^D(\mathcal{E}_i) = 0$ $(i \leq j)$.*

(ii) *Let $\mathrm{At}^D(\mathcal{E}_j) = 0$. Introduce the natural restriction map*

$$\mathrm{res}_{ji} : H^0\big(\mathcal{E}\mathrm{nd}(\mathcal{E}_j) \otimes \Omega^1(D)\big) \to H^0\big(\mathcal{E}\mathrm{nd}(\mathcal{E}_i) \otimes \Omega^1(D)\big);$$
$$\varphi \mapsto \varphi \otimes k[\varepsilon]/\big(\varepsilon^{i+1}\big).$$

Then any connection with fixed divisor of poles D on $\mathcal{E}_i$ extends to such a connection on $\mathcal{E}_j$ if and only if res_{ji} is surjective.

Proof. (i) is obvious. To prove (ii), we use the following observation: for two connections ∇_j, ∇'_j on $\mathcal{E}_j$ with fixed divisor D of poles, the difference $\nabla_j - \nabla'_j$ is an element of $H^0(\mathcal{E}\mathrm{nd}(\mathcal{E}_j) \otimes \Omega^1(D))$ and $(\nabla_j - \nabla'_j)|_{\mathcal{E}_i} = \mathrm{res}_{ji}(\nabla_j - \nabla'_j) \in H^0(\mathcal{E}\mathrm{nd}(\mathcal{E}_i) \otimes \Omega^1(D))$. □

In this corollary, it is possible that both $\mathcal{E}_i, \mathcal{E}_j$ admit connections with fixed divisor of poles D, but not every connection with the same D on $\mathcal{E}_i$ extends to such a connection on $\mathcal{E}_j$. To produce an example, set $D = 0, i = 0, j = 1, X$ an elliptic curve, $\mathcal{E} = \mathcal{O}_X^{\oplus 2}$. Define $\mathcal{E}_1$ as a nontrivial extension of vector bundles

$$0 \to \mathcal{O}_{X \times V_1} \xrightarrow{\mu} \mathcal{E}_1 \xrightarrow{\nu} \mathcal{O}_{X \times V_1} \to 0. \tag{32}$$

Such extensions are classified by $\mathrm{Ext}^1(\mathcal{O}_{X \times V_1}, \mathcal{O}_{X \times V_1}) = H^1(\mathcal{O}_{X \times V_1}) \simeq k[\varepsilon]/(\varepsilon^2)$, and we choose an extension class in the form $\varepsilon[f]$, so that the extension is trivial modulo ε^2. We can describe $[f]$ and the associated extension explicitly as follows. Let $\mathfrak{U} = \{U_{+-}\}$ be an open covering of X, and $f \in \Gamma(U_\pm, \mathcal{O}_X)$ a function whose cohomology class $[f]$ generates $H^1(X, \mathcal{O}_X)$. Let $e_\pm = (e_{\pm 1}, e_{\pm 2})$ be a basis of $\mathcal{E}|_{U_{+-}}$, and define the transition matrix over U_{+-} by

$$\begin{pmatrix} 1 & \varepsilon f \\ 0 & 1 \end{pmatrix}. \tag{33}$$

Define the maps μ, ν in (32) by $\mu : 1 \mapsto e_{\pm 1}$, $\nu : (e_{\pm 1}, e_{\pm 2}) \mapsto (0, 1)$. To be more explicit, we will give X by the Legendre equation

$$y^2 = x(x-1)(x-t) \quad \big(t \in k \setminus \{0, 1\}\big),$$

and define an open covering $\mathfrak{U}$ of X by $U_+ = X \setminus \{\infty\}, U_- = X \setminus \{0\}$. Then we can choose $f = \frac{y}{x}$ as a function having two simple poles at 0 and ∞ and no other singularities. The Residue theorem implies that it is impossible to represent f as the difference of two functions, one regular on U_+ and the other on U_-, so the cohomology class of f considered as a Čech cocycle of the covering $\mathfrak{U}$ with coefficients in $\mathcal{O}_X$ is nonzero. We now verify that $\mathrm{At}(\mathcal{E}_1)=0$. It is represented by the cocycle

$$\mathrm{d}g_{+-}g_{+-}^{-1}=\begin{pmatrix}0 & \varepsilon df\\ 0 & 0\end{pmatrix}, \tag{34}$$

and

$$df=\mathrm{d}\left(\frac{y}{x}\right)=\frac{\mathrm{d}y}{x}-y\frac{\mathrm{d}x}{x^2}=\omega_+-\omega_-,$$

where

$$\omega_+=2\frac{\mathrm{d}y}{x}-y\frac{\mathrm{d}x}{x^2},\qquad \omega_-=\frac{\mathrm{d}y}{x},$$

ω_+ (resp. w_-) being regular on U_+ (resp. U_-). Hence,

$$\mathrm{d}g_{+-}g_{+-}^{-1}=\begin{pmatrix}0 & \varepsilon\omega_+\\ 0 & 0\end{pmatrix}-\begin{pmatrix}0 & \varepsilon\omega_-\\ 0 & 0\end{pmatrix} \tag{35}$$

is a Čech coboundary, and $\mathrm{At}(\mathcal{E}_1)=0$. Thus, $\mathcal{E}_1$ has a regular connection.

Now, we will show that the map res_{10} defined in the last corollary is not surjective, so not every regular connection on $\mathcal{E}$ extends to a regular connection on $\mathcal{E}_1$. We remark that in our case Ω^1_X is trivial, $D=0$, so res_{10} is just the restriction map $\mathrm{res}_{10}: H^0(\mathcal{E}\mathrm{nd}(\mathcal{E}_1))\to H^0(\mathcal{E}\mathrm{nd}(\mathcal{E}_0))$. Consider $\mathcal{E}_1$ as an extension of another kind:

$$0\to\varepsilon\mathcal{E}\to\mathcal{E}_1\to\mathcal{E}\to 0,$$

where $\varepsilon\mathcal{E}\simeq\mathcal{O}_X^{\oplus 2}$ and $\mathcal{E}\simeq\mathcal{E}_1/\varepsilon\mathcal{E}\simeq\mathcal{O}_X^{\oplus 2}$. Apply to it $\mathcal{H}\mathrm{om}(\mathcal{E}_1,\cdot)$ (the Hom-sheaf as $\mathcal{O}_{X\times V_1}$-modules):

$$0\to\mathcal{H}\mathrm{om}(\mathcal{E}_1,\mathcal{E})\to\mathcal{E}\mathrm{nd}\,(\mathcal{E}_1)\to\mathcal{H}\mathrm{om}(\mathcal{E}_1,\mathcal{E})\to 0.$$

As $\mathcal{E}\simeq\mathcal{O}_X^{\oplus 2}$, the first and the third terms of the last triple are described as follows:

$$\mathcal{H}\mathrm{om}(\mathcal{E}_1,\mathcal{E})\simeq\mathcal{E}\mathrm{nd}\,(\mathcal{E})=M_2(\mathcal{O}_X).$$

Take an element in $H^0(\mathcal{H}\mathrm{om}(\mathcal{E}_1,\mathcal{E}))\simeq M_2(k)$ given by the matrix

$$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{36}$$

(as above, $\mathcal{E}_1,\mathcal{E}$ are trivialized by the bases $e_\pm=(e_{\pm1},e_{\pm2})$). We will see that it is not in the image of the restriction map $\mathrm{res}_{1,0}$.

Indeed, assume there is a lift of

$$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{37}$$

to $H^0(\mathcal{E}\mathrm{nd}(\mathcal{E}_1))$. Then it is given in the basis e_+ by a matrix of the form

$$A_+=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}+\varepsilon B, \tag{38}$$

$B \in M_2(k[U_+])$. Transforming it to the basis e_-, we obtain the matrix

$$A_- = \begin{pmatrix} 0 & -\varepsilon f \\ 0 & 1 \end{pmatrix} + \varepsilon B, \tag{39}$$

which has to be regular in U_-. Thus $\varepsilon f = \varepsilon b_{12} - a_{-12}$, where b_{12} is regular in U_+ and a_{-12} is regular in U_-. This contradicts the fact that f is not a Čech coboundary in $\check{C}(\mathfrak{U}, \mathcal{O}_X)$, and this ends the proof.

3.3. Kuranishi space for deformations of connections.

THEOREM 3.6. *Let X be a complete scheme of finite type over k or a complex space (in which case $k = \mathbb{C}$), $\mathcal{C}^\bullet$ the 2-term complex of sheaves on X defined by (26), $W = \mathbb{H}^1(X, \mathcal{C}^\bullet)$,$(\delta_1, \dots, \delta_N)$ a basis of W and $(t_1, \dots, t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of 0 in W, and $(\mathcal{E}_1, \nabla_1)$ the universal first order deformation over $X \times W_1$ of a connection $(\mathcal{E}, \nabla)$ on X with fixed divisor of poles D. Then there exists a formal power series*

$$f(t_1, \dots, t_N) = \sum_{k=2}^{\infty} f_k(t_1, \dots, t_N) \in \mathbb{H}^2\big(X, \mathcal{C}^\bullet\big)[[t_1, \dots, t_N]],$$

where f_k is homogeneous of degree k ($k \geq 2$), with the following property. Let I be the ideal of $k[[t_1, \dots, t_N]]$ generated by the image of the map $f^* : \mathbb{H}^2(X, \mathcal{C}^\bullet)^* \to k[[t_1, \dots, t_N]]$, adjoint to f. Then for any $k \geq 2$, the pair $(\mathcal{E}_1, \nabla_1)$ extends to a connection $(\mathcal{E}_k, \nabla_k)$ on $X \times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I \otimes k[[t_1, \dots, t_N]]/(t_1, \dots, t_N)^{k+1}$.

Proof. We will start by fixing a particular choice of coordinates $(t_1, \dots, t_N)$, coming from the spectral sequence $E_1^{p,q} = H^q(\mathcal{C}^p) \Rightarrow \mathbb{H}^{p+q}(\mathcal{C}^\bullet)$. The latter is supported on two vertical strings $p = 0$ and $p = 1$ (see Figure 1).

Thus the spectral sequence degenerates in the second term E_2, and we have the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0\big(X, \mathcal{C}^\bullet\big) \longrightarrow H^0\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) \xrightarrow{d_1} H^0\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^1(D)\big) \\ &\longrightarrow \mathbb{H}^1\big(X, \mathcal{C}^\bullet\big) \longrightarrow H^1\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) \xrightarrow{d_1} H^1\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^1(D)\big) \\ &\longrightarrow \mathbb{H}^2\big(X, \mathcal{C}^\bullet\big) \longrightarrow H^2\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) \xrightarrow{d_1} H^2\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^1(D)\big) \to \dots. \end{aligned}$$

We deduce the exact triple

$$0 \to W' \to W \to W'' \to 0,$$

with

$$W' = \frac{H^0(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^1(D))}{\mathrm{im}\, d_1}, \qquad W = \mathbb{H}^1\big(X, \mathcal{C}^\bullet\big),$$
$$W'' = \ker(H^1\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E})\big) \to H^1\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^1(D)\big).$$

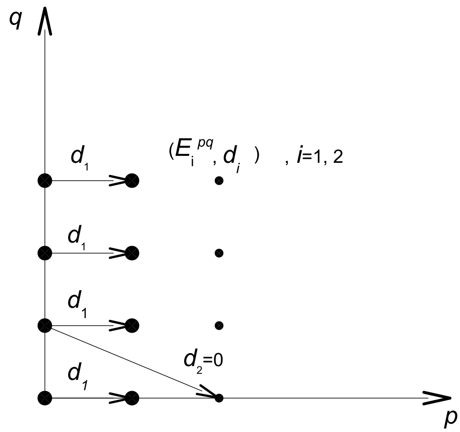

FIGURE 1. The spectral sequence is supported on 2 vertical strings $p=0, p=1$.

Let $N'=\dim W'$, $N''=\dim W''$; choose $t_1,\dots,t_N$ in such a way that $s_1 = t_{N'+1},\dots,s_{N''}=t_{N'+N''}(N=N'+N'')$, are coordinates on W'' and $t_1,\dots,t_{N'}$ restrict to W' as coordinates on W'. We will construct by induction on $k\geq 0$ the homogeneous forms

$$
\begin{aligned}
G_{\alpha\beta,k}(s_1,\dots,s_{N''}) &\in \Gamma\big(U_{\alpha\beta},\mathcal{E}\mathrm{nd}(\mathcal{E})\big)\otimes k[s_1,\dots,s_{N''}],\\
F_{\alpha\beta\gamma,k}(s_1,\dots,s_{N''}) &\in \Gamma\big(U_{\alpha\beta\gamma},\mathcal{E}\mathrm{nd}(\mathcal{E})\big)\otimes k[s_1,\dots,s_{N''}],\\
\bar{f}_k(s_1,\dots,s_{N''}) &\in H^2\big(X,\mathcal{E}\mathrm{nd}(\mathcal{E})\big)\otimes k[s_1,\dots,s_{N''}],\\
A_{\alpha,k}(t_1\dots,t_N) &\in \Gamma\big(U_\alpha,\mathcal{E}\mathrm{nd}(\mathcal{E})\otimes\Omega^1_X(D)\big)\otimes k[t_1,\dots,t_N],\\
\kappa_k(t_1\dots,t_N) &\in H^1\big(X,\mathcal{E}\mathrm{nd}(\mathcal{E})\otimes\Omega^1_X(D)\big)\otimes k[t_1,\dots,t_N],\\
K_{\alpha\beta,k}(t_1\dots,t_N) &\in \Gamma\big(U_{\alpha\beta},\mathcal{E}\mathrm{nd}(\mathcal{E})\otimes\Omega^1_X(D)\big)\otimes k[t_1,\dots,t_N]
\end{aligned}
\tag{40}
$$

with the following properties:

(i) $G_{\alpha\beta,0}=g_{\alpha\beta}$, and $A_{\alpha,0}$ define $\mathcal{E}$ and resp. ∇ with respect to the local trivializations e_α of $\mathcal{E}$ on U_α.

(ii) $\bar{f}_k=0, F_{\alpha\beta\gamma,k}=0$ for $k=0,1$, and $K_{\alpha\beta,0}=0$.

(iii) For each $k\geq 1$, let $\bar{f}^{(k)}=\sum_{i\leq k}\bar{f}_i$, and let $\bar{I}^{(k+1)}$ be the ideal generated by $(s_1,\dots,s_{N''})^{k+2}$ and the image of the adjoint map $\bar{f}^{(k)*}: H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))^*\to k[s_1,\dots,s_{N''}]$. Then $(F_{\alpha\beta\gamma,k+1})$ is a cocycle modulo $\bar{I}^{(k+1)}$ and $\bar{f}_{k+1}$ is a lift to $W''\otimes k[s_1,\dots,s_{N''}]$ of the cohomology class

$$\big[\big(F_{\alpha\beta\gamma,k+1}\mod \bar{I}^{(k+1)}\big)\big]\in W''\otimes k\big[s_1,\dots,s''_N\big]/\bar{I}^{(k+1)}.$$

(iv) For any $k\geq 1$, set $G^{(k)}_{\alpha\beta}=\sum_{i\leq k}G_{\alpha\beta,i}$. Then

$$G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha}\equiv(1+F_{\alpha\beta\gamma,k+1})\mod \bar{I}^{(k+1)}. \tag{41}$$

(v) For each $k \geq 1$, set $\kappa^{(k)} = \sum_{i\leq k} \kappa_i$, and let $J^{(k+1)}$ be the ideal generated by $(t_1,\ldots,t_N)^{k+2}$ and by the image of the adjoint map $\kappa^{(k)*} : H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))^* \to k[t_1,\ldots,t_N]$. Then $(K_{\alpha\beta,k+1})$ is a cocycle modulo $J^{(k+1)} + \bar{I}^{(k+2)}$ and κ_{k+1} is a lift of the cohomology class

$$\begin{aligned}&\bigl[(K_{\alpha\beta,k+1} \mod \bigl(J^{k+1} + \bar{I}^{(k+2)}\bigr)\bigr]\\ &\quad \in H^1\bigl(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)\bigr) \otimes k[[t_1,\ldots,t_N]]/\bigl(J^{k+1} + \bar{I}^{(k+1)}\bigr)\end{aligned}$$

in $H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)) \otimes k[t_1,\ldots,t_N]$.

(vi) For any $k \geq 0$, set $A^{(k)}_\alpha = \sum_{i\leq k} A_{\alpha,i}$. Then

$$K_{\alpha\beta,k+1} \equiv \mathrm{d}G^{(k+1)}_{\alpha\beta} - G^{(k+1)}_{\alpha\beta} A^{(k)}_\beta + A^{(k)}_\alpha G^{(k+1)}_{\alpha\beta} \mod \bigl(J^{k+1} + \bar{I}^{(k+2)}\bigr). \tag{42}$$

In these properties, $G^{(k)}_{\alpha\beta}$ is considered as an endomorphism of $\mathcal{E}_k$ over $U_{\alpha\beta} \times V_k$ given by its matrix with respect to two bases: e_α for the source, e_β for the target, where $\mathcal{E}_k$ is the vector bundle over $X \times V_k$ defined by the 1-cocycle $(G^{(k)}_{\alpha\beta})$. Similarly $(A^{(k)}_\alpha)$ is understood as a 1-cochain with values in $\mathcal{E}\mathrm{nd}(\mathcal{E}_k) \otimes \Omega^1(D)$, and in formula (42), $A^{(k)}_\alpha$ (resp. $A^{(k)}_\beta$) is represented by its matrix in the basis e_α (resp. e_β). The base changes $G_{\alpha\beta,k+1}$ acting on both sides of (42), reduce to $G_{\alpha\beta,0}$, since the only nonzero terms in (42) are of degree $k+1$, and everything is reduced modulo $(t_1,\ldots,t_N)^{k+2}$. Thus (42) defines $(K_{\alpha\beta,k+1})$ as a 1-cochain with values in $\mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)$. Going over to the proof, we first remark that $G_{\alpha\beta,0}, A_{\alpha,0}$ are already known, and we have to indicate the choice of $G_{\alpha\beta,k}, A_{\alpha,k}$ inductively on $k \geq 0$, the other data $F_{\alpha\beta\gamma,k}, \bar{f}_k, K_{\alpha\beta,k}, \kappa_k$ being recovered via formulas (41), (42). To initialize the induction, first look at (41) with $k = 0$. Then $F_{\alpha\beta\gamma,1} = 0$ by (ii), which implies

$$G_{\alpha\beta,1}G_{\beta\gamma,0}G_{\gamma\alpha,0} + G_{\alpha\beta,0}G_{\beta\gamma,1}G_{\gamma\alpha,0} + G_{\alpha\beta,0}G_{\beta\gamma,0}G_{\gamma\alpha,1} = 0. \tag{43}$$

The latter equation expresses the fact that $(G_{\alpha\beta,1})$ is a 1-cocycle with values in $\mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes (W'')^*$. As in Section 2, we can write $G_{\alpha\beta,1} = \sum a^{(i)}_{\alpha\beta} g_{\alpha\beta} s_i$, where $[(a^{(i)}_{\alpha\beta})]$ for $i = 1,\ldots,N''$ form the basis of W'' dual to $s_1,\ldots,s_{N''}$. Here and further on, we adopt the following convention: all the $G_{\alpha\beta,k}$ (resp. $G^{(k)}_{\alpha\beta}$) are regarded as 1-cochains with values in $\mathcal{E}\mathrm{nd}(\mathcal{E})$ (resp. $\mathcal{E}\mathrm{nd}(\mathcal{E}_k)$) given by matrices with respect to two bases: e_α for the source, e_β for the target. We denote by $\mathcal{E}_k$ the vector bundle over $X \times V_k$ defined by the cocycle $G^{(k)}_{\alpha\beta}$.

Hence, looking at the first term $G_{\alpha\beta,1}G_{\beta\gamma,0}G_{\gamma\alpha,0}$ of the sum in (43), we see that it represents the matrix of $G_{\alpha\beta,1}$ with respect to one and the same basis e_α for the source and the target. The same applies to the other two summands in (43), thus (43) is the cocycle condition

$$a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = 0$$

put down via matrices of the three summands in the basis e_α.

We will adopt the same convention for cochains with values in $\mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)$ or in $\mathcal{E}\mathrm{nd}(\mathcal{E}_k) \otimes \Omega^1(D)$. The $A_{\alpha,k}$ (resp $A_\alpha^{(k)}$) will be considered as matrices representing cochains in $\mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)$ (resp. $\mathcal{E}\mathrm{nd}(\mathcal{E}_k) \otimes \Omega^1(D)$) in the basis e_α over U_α. Now write (42) for $k = 0$:

$$K_{\alpha\beta,1} = \mathrm{d}G_{\alpha\beta,1} - G_{\alpha\beta,1}A_{\beta,0} + A_{\alpha,0}G_{\alpha\beta,1}; \tag{44}$$

we take into account that $I^{(1)} = J^{(1)} = 0$ and that $\mathrm{d}G_{\alpha\beta,0} - G_{\alpha\beta,0}A_{\beta,0} + A_{\alpha,0}G_{\alpha\beta,0} = 0$, the latter equation being a form of (12) in which $G_{\alpha\beta,0}$ are considered as matrices of endomorphisms of $\mathcal{E}$ written with respect to two bases: e_α for the source, e_β for the target, and $(\mathrm{d}G_{\alpha\beta,0})$ is a cocycle representing $\mathrm{At}^D(\mathcal{E})$.

The r.h.s. of (44), with the same convention that $G_{\alpha\beta,1}$ are matrices of endormorphisms of $\mathcal{E}$ with respect to the two bases, is just the cochain $(\mathrm{d}a_{\alpha\beta} + [A_\alpha, a_{\alpha\beta}]) \in \check{C}^1(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))$. As in (22), we can rewrite it as $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(a)$, where $a = (G_{\alpha\beta,1})$, and this representation makes obvious that $(K_{\alpha\beta,1})$ is a 1-cocycle. The differential d_1 of the spectral sequence being induced by $\nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}$, we see that the cocycle $(K_{\alpha\beta,1})$ is a coboundary if and only if

$$\begin{aligned}[a] = [G_{\alpha\beta,1}] \in \ker\bigl(&H^1\bigl(X, \mathcal{E}\mathrm{nd}(\mathcal{E})\bigr) \otimes \bigl(W''\bigr)^* \\ &\to H^1\bigl(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D)\bigr) \otimes \bigl(W''\bigr)^*\bigr).\end{aligned}$$

Assuming that $(K_{\alpha\beta,1})$ is a coboundary, we choose $(A_{\alpha,1})$ as a solution to

$$\tilde{K}_{\alpha\beta,1} = G_{\alpha\beta,0}A_{\beta,1} - A_{\alpha,1}G_{\alpha\beta,0}. \tag{45}$$

Such a solution can be chosen as a linear form in $s_1, \ldots, s_{N''}$. Single out one such solution and denote it $(A''_{\alpha,1}) = (A''_{\alpha,1}(s_1, \ldots, s_{N''}))$. Let $(A'^{(i)}_{\alpha,1}), i = 1, \ldots, N'$ be a basis of $H^0(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1(D))$ dual to the coordinates $t_1, \ldots, t_{N'}$ on W'. Then set

$$A_{\alpha,1} = A''_{\alpha,1}(s_1, \ldots, s_{N''}) + \sum_{i=1}^{N'} A'^{(i)}_{\alpha,1} t_i.$$

Now assume that the forms (40) have been constructed up to degree $k \geq 0$ and define them for degree $k+1$. Start by $F_{\alpha\beta\gamma,k+1}$, which we define, as in the proof of Theorem 1.7, to be a lift to $\check{Z}^2(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes k[s_1, \ldots, s_{N''}]$, of the homogeneous component of degree $k+1$ in $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha}$, which is a cocycle modulo $\bar{I}^{(k+1)} + (s_1, \ldots, s_{N''})^{k+1}$ by the proof of Lemma 1.9.

Then we set $\bar{f}_{k+1}$ equal to any lift of the cohomology class $(F_{\alpha\beta\gamma,k+1}) \in H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes k[[s_1, \ldots, s_{N''}]]/\bar{I}^{(k+1)}$ to $H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes k[s_1, \ldots, s_{N''}]$. By construction, $(F_{\alpha\beta\gamma,k+1})$ is a coboundary modulo $\bar{I}^{(k+2)} + (s_1, \ldots, s_{N''})^{k+2}$, so there exists a cochain in

$$\check{C}^1\bigl(\mathfrak{U}, \mathcal{E}\mathrm{nd}(\mathcal{E})\bigr) \otimes k[s_1, \ldots, s_{N''}]/\bigl(\bar{I}^{(k+2)} + (s_1, \ldots, s_{N''})^{k+2}\bigr)$$

whose coboundary is $(F_{\alpha\beta\gamma,k+1}) \mod (\bar{I}^{(k+2)} + (s_1,\dots,s_{N''})^{k+2})$, and $(G_{\alpha\beta,k+1})$ is defined as any lift of this cochain to $\check{C}^1(\mathfrak{U},\mathcal{E}\mathrm{nd}(\mathcal{E}))\otimes k[s_1,\dots,s_{N''}]$ which is homogeneous of degree $k+1$ in $s_1,\dots,s_{N''}$. Consider now the expression

$$\begin{aligned}\tilde{K}_{\alpha\beta,k+1} &= \mathrm{d}G^{(k+1)}_{\alpha\beta} - G^{(k+1)}_{\alpha\beta}A^{(k)}_{\beta} + A^{(k)}_{\alpha}G^{(k+1)}_{\alpha\beta}\\ &= \mathrm{d}G^{(k)}_{\alpha\beta} - G^{(k)}_{\alpha\beta}A^{(k-1)}_{\beta} + A^{(k-1)}_{\alpha}G^{(k)}_{\alpha\beta} + \mathrm{d}G_{\alpha\beta,k+1}\\ &\quad - G_{\alpha\beta,k+1}A^{(k-1)}_{\beta} + A^{(k-1)}_{\alpha}G_{\alpha\beta,k+1} - G^{(k+1)}_{\alpha\beta}A_{\beta,k} + A_{\alpha,k}G^{(k+1)}_{\alpha\beta}.\end{aligned}$$

By the induction hypothesis, $\tilde{K}_{\alpha\beta,k} = \mathrm{d}G^{(k)}_{\alpha\beta} - G^{(k)}_{\alpha\beta}A^{(k-1)}_{\beta} + A^{(k-1)}_{\alpha}G^{(k+1)}_{\alpha\beta}$ is a cocycle modulo $J^k + \bar{I}^{(k+1)}$ and is a coboundary modulo $J^{k+1} + \bar{I}^{(k+1)} + (t_1,\dots,t_N)^{k+1}$. From (42), in order that $\tilde{K}_{\alpha\beta,k+1}$ has no homogeneous components of order $< k+1$ modulo $J^{k+1} + \bar{I}^{(k+1)} + (t_1,\dots,t_N)^{k+1}$, we have to set $(A_{\alpha,k})$ to be a solution of

$$\begin{aligned}&G^{(k+1)}_{\alpha\beta}A_{\beta,k} - A_{\alpha,k}G^{(k+1)}_{\alpha\beta}\\ &\quad\equiv \tilde{K}_{\alpha\beta,k} \mod \big(J^{k+1} + \bar{I}^{k+1} + (t_1,\dots,t_N)^{k+1}\big),\end{aligned} \tag{46}$$

where $G^{(k+1)}_{\alpha\beta}$ can be replaced by $G_{\alpha\beta,0}$, so that (46) is an equation for the cochain $(G_{\alpha\beta,0}A_{\beta,k})$ with values in $\mathcal{E}\mathrm{nd}(\mathcal{E})\otimes\Omega^1(D)$. Thus, we come to the following inductive procedure: define $K_{\alpha\beta,k+1}$ as the homogeneous form of degree $k+1$ in $\tilde{K}_{\alpha\beta,k+1}$. Assuming it is a cocycle modulo $(J^{k+1} + \bar{I}^{(k+2)})$, we define κ_{k+1} as a lift to $H^1(X,\mathcal{E}\mathrm{nd}(\mathcal{E})\otimes\Omega^1(D))\otimes k[t_1,\dots,t_N]$ of the cohomology class $[(K_{\alpha\beta,k+1}) \mod J^{k+1} + \bar{I}^{(k+2)}]$. Then $J^{(k+2)}$ is well-defined and $(K_{\alpha\beta,k+1})$ becomes a coboundary modulo $J^{(k+2)} + \bar{I}^{(k+2)} + (t_1,\dots,t_N)^{k+2}$. Hence, we can construct $(A_{\alpha,k+1})$ as a lift to $\check{C}^0(\mathfrak{U},\mathcal{E}\mathrm{nd}(\mathcal{E})\otimes\Omega^1(D))\otimes k[t_1,\dots,t_N]$ of a solution $(A_{\alpha,k+1})$ of the equation

$$\begin{aligned}&G_{\alpha\beta,0}A_{\beta,k+1} - A_{\alpha,k+1}G_{\alpha\beta,0}\\ &\quad\equiv \tilde{K}_{\alpha\beta,k+1} \mod \big(J^{k+2} + \bar{I}^{(k+2)} + (t_1,\dots,t_N)^{k+2}\big).\end{aligned}$$

Thus, we have to verify that $(K_{\alpha\beta,k+1})$ is a cocycle. □

LEMMA 3.7. $(K_{\alpha\beta,k+1})$ *defined as the homogeneous component of degree* $k+1$ *of* $\tilde{K}_{\alpha\beta,k+1}$, *is a* 1*-cocycle modulo* $J^{k+1} + \bar{I}^{(k+2)}$.

Proof. By the induction hypothesis, we have

$$\begin{aligned}\mathrm{d}G^{(k)}_{\alpha\beta} &\equiv G^{(k)}_{\alpha\beta}A^{(k-1)}_{\beta} - A^{(k-1)}_{\alpha}G^{(k)}_{\alpha\beta} \mod \big(J^k + \bar{I}^{(k+1)}\big),\\ G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha} &\equiv 1 + F_{\alpha\beta\gamma,k+1} \mod \bar{I}^{(k+1)},\end{aligned}$$

and by construction,

$$\begin{aligned}&G_{\alpha\beta,k+1}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha} + G^{(k)}_{\alpha\beta}G_{\beta\gamma,k+1}G^{(k)}_{\gamma\alpha} + G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G_{\gamma\alpha,k+1}\\ &\quad\equiv -F_{\alpha\beta\gamma,k+1} \mod \big(\bar{I}^{(k+2)} + (s_1,\dots,s_{N''})^{k+2}\big),\end{aligned}$$

$$K_{\alpha\beta,k+1} \equiv \mathrm{d}G^{(k+1)}_{\alpha\beta} - G^{(k+1)}_{\alpha\beta}A^{(k)}_{\beta} + A^{(k)}_{\alpha}G^{(k+1)}_{\alpha\beta} \mod \big(J^{k+1}+\bar{I}^{(k+1)}\big).$$

Denote $G^{(k+1)}_{\alpha\beta}, G^{(k)}_{\alpha\beta}, G_{\alpha\beta,k+1}, A^{(k)}_{\alpha}, K_{\alpha\beta,k+1}$ by $G_{\alpha\beta}, G'_{\alpha\beta}, G''_{\alpha\beta}, A_\alpha, K_{\alpha\beta}$, respectively.

We have

$$\begin{aligned}
(47)\ & K_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} + G_{\alpha\beta}K_{\beta\gamma}G_{\gamma\alpha} + G_{\alpha\beta}G_{\beta\gamma}K_{\gamma\alpha}\\
&\equiv \mathrm{d}G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} + G_{\alpha\beta}\mathrm{d}G_{\beta\gamma}G_{\gamma\alpha} + G_{\alpha\beta}G_{\beta\gamma}\mathrm{d}G_{\gamma\alpha} - G_{\alpha\beta}A_\beta G_{\beta\gamma}G_{\gamma\alpha}\\
&\quad + A_\alpha G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} - G_{\alpha\beta}G_{\beta\gamma}A_\gamma G_{\gamma\alpha} + G_{\alpha\beta}A_\beta G_{\beta\gamma}G_{\gamma\alpha}\\
&\quad - G_{\alpha\beta}\mathrm{d}G_{\beta\gamma}G_{\gamma\alpha}A_\alpha + G_{\alpha\beta}\mathrm{d}G_{\beta\gamma}A_\gamma G_{\gamma\alpha}\\
&\equiv \mathrm{d}G'_{\alpha\beta}G'_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}\mathrm{d}G'_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}G'_{\beta\gamma}\mathrm{d}G'_{\gamma\alpha} + \mathrm{d}G''_{\alpha\beta}G'_{\beta\gamma}G'_{\gamma\alpha}\\
&\quad + G'_{\alpha\beta}\mathrm{d}G''_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}G'_{\beta\gamma}\mathrm{d}G''_{\gamma\alpha} + \mathrm{d}G'_{\alpha\beta}G''_{\beta\gamma}G'_{\gamma\alpha} + \mathrm{d}G'_{\alpha\beta}G'_{\beta\gamma}G''_{\gamma\alpha}\\
&\quad + G''_{\alpha\beta}\mathrm{d}G'_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}\mathrm{d}G'_{\beta\gamma}G''_{\gamma\alpha} + G''_{\alpha\beta}G'_{\beta\gamma}\mathrm{d}G'_{\gamma\alpha}\\
&\equiv \mathrm{d}\big(G'_{\alpha\beta}G'_{\beta\gamma}G'_{\gamma\alpha}\big) - G'_{\alpha\beta}G''_{\beta\gamma}\mathrm{d}G'_{\gamma\alpha} + \mathrm{d}\big(G''_{\alpha\beta}G'_{\beta\gamma}G'_{\gamma\alpha}\\
&\quad + G'_{\alpha\beta}G''_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}G'_{\beta\gamma}G''_{\gamma\alpha}\big)\\
&\equiv \mathrm{d}(F_{\alpha\beta\gamma,k+1}) - \mathrm{d}(F_{\alpha\beta\gamma,k+1})\\
&\equiv 0 \mod \big(J^{k+1}+\bar{I}^{(k+2)}\big).
\end{aligned}$$

This ends the proof. □

Coming back to the proof of the theorem, we define f_k as any lift to $\mathbb{H}^2(\mathcal{C}^\bullet)\otimes k[t_1,\dots,t_N]$, homogeneous of degree k in $t_1,\dots,t_N$, of the cohomology class of the cochain

$$(48)\qquad \big((K_{\alpha\beta,k}),(F_{\alpha\beta\gamma,k})\big) \mod \big(J^k+\bar{I}^{k+1}\big) \\ \in \check{C}^2\big(\mathfrak{U},\mathcal{C}^\bullet\big)\otimes k[[t_1,\dots,t_N]]/\big(J^k+\bar{I}^{(k+1)}\big),$$

which we are assuming to be a cocycle. Then quotienting by I makes (48) a coboundary of $((A_{\alpha,k}),(G_{\alpha\beta,k}))$, and the pair $(G^{(k)}_{\alpha\beta},(A^{(k)}_{\alpha}))$ defines $(\mathcal{E}_k,\nabla_k)$ over $X\times V_k$. It remains to prove that (48) is a cocycle with values in $\mathcal{C}^\bullet\otimes k[t_1,\dots,t_N]/(J^k+\bar{I}^{k+1})$. One part of this, namely, the equation

$$\check{d}(K_{\alpha\beta,k}) = \nabla_{\mathcal{E}\mathrm{nd}(\mathcal{E})}(F_{\alpha\beta\gamma,k})$$

is verified by the computation (47). The second part $\check{d}(F_{\alpha\beta\gamma,k})=0$ is guaranteed by Lemma 1.9.

4. Integrable connections

4.1. Higher order deformations of integrable connections. From now on, we take into account the fact that $(\mathcal{E},\nabla)$ is an integrable connection with fixed divisor of poles D and consider deformations of $(\mathcal{E},\nabla)$ preserving the integrability and the divisor of poles. In Theorem 2.8, we characterized the first order deformations of $(\mathcal{E},\nabla)$ in terms of the hypercohomology group

$\mathbb{H}^1(X,\mathcal{F}^\bullet) = \mathbb{H}^1(X,\mathcal{R}^\bullet)$. Now we will consider the second order deformation and respectively, the first obstruction. So, we search for the extension

$$\begin{aligned}\tilde{g}_{\alpha\beta} &= \big(1+\varepsilon a_{\alpha\beta,1}+\varepsilon^2 a_{\alpha\beta,2}\big)g_{\alpha\beta},\\ \tilde{A}_\alpha &= A_\alpha+\varepsilon A_{\alpha,1}+\varepsilon^2 A_{\alpha,2}\end{aligned}\tag{49}$$

of $(g_{\alpha\beta},A_\alpha)$ to $V=\operatorname{Spec} k[\varepsilon]/(\varepsilon^3)$. To order 1, we have the conditions (24):

$$\check{d}(a_{\alpha\beta,1})=0,\qquad \check{d}(A_{\alpha,1})=\nabla(a_{\alpha\beta,1}),\qquad \nabla(A_{\alpha,1})=0.\tag{50}$$

Expanding (13) to order 2, we obtain in addition to (6) and (23), the equation

$$\nabla A_{\alpha,2} = -A_{\alpha,1}\wedge A_{\alpha,1}.\tag{51}$$

Note that $\nabla(A_{\alpha,1})=0$ implies that $\nabla(A_{\alpha,1}\wedge A_{\alpha,1})=0$. One easily verifies the following relations

$$\begin{aligned}\nabla(A_{\alpha,1}\wedge A_{\alpha,1}) &= 0,\\ \check{d}(A_{\alpha,1}\wedge A_{\alpha,1}) &= -\nabla(A_{\alpha,1}a_{\alpha\beta,1}-a_{\alpha\beta,1}A_{\beta,1}),\\ \check{d}(A_{\alpha,1}a_{\alpha\beta,1}-a_{\alpha\beta,1}A_{\beta,1}) &= \nabla(a_{\alpha\beta,1}a_{\beta\gamma,1}\circlearrowleft),\end{aligned}$$

where $\circlearrowleft$ denotes the skew-symmetrization on the subscripts α,β,γ. These three equations express the fact that the triple

$$\big((a_{\alpha\beta,1}a_{\beta\gamma,1}\circlearrowleft),(A_{\alpha,1}a_{\alpha\beta,1}-a_{\alpha\beta,1}A_{\beta,1}),(A_{\alpha,1}\wedge A_{\alpha,1})\big)\in\check{C}^2\big(\mathfrak{U},\mathcal{R}^\bullet\big)$$

is a cocycle with respect to the differential $D=\nabla\pm\check{d}$. Then the conditions saying that (49) is an integrable connection with fixed divisor of poles D modulo ε^3, that is, formulas (29), (31) and (51), mean that the cocycle defined above is the coboundary of the cochain $((a_{\alpha\beta,2}),(\mathcal{A}_{\alpha,2}))$:

$$D(a_2,\mathcal{A}_2)=\big((a_{\alpha\beta,1}a_{\beta\gamma,1}\circlearrowleft),(A_{\alpha,1}a_{\alpha\beta,1}-a_{\alpha\beta,1}A_{\beta,1}),(A_{\alpha,1}\wedge A_{\alpha,1})\big).$$

As the cocycle (52) represents the Yoneda square of $[a_1,\mathcal{A}_1]$, we deduce the following proposition.

Proposition 4.1. *The first order deformation $(\mathcal{E}_1,\nabla_1)$ of $(\mathcal{E},\nabla)$ defined by the cocycle $((a_{\alpha\beta,1}),(\mathcal{A}_{\alpha,1}))$ extend to an integrable connection $(\mathcal{E}_2,\nabla_2)$ over $X\times V_2$ with fixed divisor of poles D if and only if the Yoneda square $[a_1,\mathcal{A}_1]\circ[a_1,\mathcal{A}_1]$ is zero in $\mathbb{H}^2(\mathcal{R}^\bullet)$.*

Thus, the integrable case looks similar to the non-integrable one (compare to Proposition 1.6), provided we replace the 2-term complex $\mathcal{C}^\bullet$ by $\mathcal{R}^\bullet$. As far as only the hypercohomology $\mathbb{H}^1$ and $\mathbb{H}^2$ are concerned, we can also truncate $\mathcal{R}^\bullet$ at the level 2: $\mathbb{H}^i(\mathcal{R}^\bullet)=\mathbb{H}^i(\tilde{\mathcal{R}}^\bullet)$, for $i=0,1,2$, where $\tilde{\mathcal{R}}^\bullet=[\mathcal{R}^0\to\mathcal{R}^1\to\ker(\mathcal{R}^2\to\mathcal{R}^3)]$.

4.2. Kuranishi space of integrable connections. Now, we turn to the construction of the Kuranishi space of integrable connections with fixed divisor of poles D. Its construction is completely similar to the one in the non-integrable case, so instead of giving a proof of the next theorem, we will only supply some remarks indicating modifications that should be brought to the proof of Theorem 3.6 in order to get the proof in the integrable case.

The spectral sequence $E_1^{p,q} = H^q(X, \mathcal{R}^p)$ converging to $\mathbb{H}^\bullet(\mathcal{R}^\bullet)$ is not concentrated on two vertical strings, so here $\mathbb{H}^2(\mathcal{R}^\bullet)$ has a filtration consisting of three nonzero summands which are subquotients of $H^0(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^2(*D)), H^1(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^1(D), H^2(X, \mathcal{E}\mathrm{nd}(\mathcal{E}))$. Hence, we have to add to the forms (40) two more homogeneous forms of degree k, say

$$\begin{aligned} L_{\alpha,k}(t_1, \dots, t_N) &\in \Gamma\big(U_\alpha, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^2(*D)\big) \otimes k[t_1, \dots, t_N], \\ l_k(t_1, \dots, t_N) &\in H^0\big(X, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^2(*D)\big) \otimes k[t_1, \dots, t_N], \end{aligned} \tag{52}$$

and modify according the conditions (i),…,(vi) to which the forms (40), (52) should satisfy. Remark also that the long exact cohomology sequence for $\mathcal{C}^\bullet$ introduced in the proof of Theorem 3.6 remains exact only in its 4 terms when $\mathcal{C}^\bullet$ is replaced by $\mathcal{R}^\bullet$.

THEOREM 4.2. *Let X be a complete scheme of finite type over k or a complex space* (*in which case $k = \mathbb{C}$*), *∇ an integrable connection on $\mathcal{E}$ with fixed divisor of poles D, $\mathcal{R}^\bullet$ the complex of sheaves on X defined above, $W = \mathbb{H}^1(X, \mathcal{R}^\bullet)$, $(\delta_1, \dots, \delta_N)$ a basis of W and $(t_1, \dots, t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of* 0 *in W, and $(\mathcal{E}_1, \nabla_1)$ the universal first deformation of $(\mathcal{E}, \nabla)$ over $X \times W_1$ in the class of integrable connections with fixed divisor of poles D. Then there exists a formal power series*

$$f(t_1, \dots, t_N) = \sum_{k=2}^{\infty} f_k(t_1, \dots, t_N) \in \mathbb{H}^2\big(X, \mathcal{R}^\bullet\big)[[t_1, \dots, t_N]],$$

where f_k is homogeneous of degree k $(k \geq 2)$, with the following property. Let I be the ideal of $k[[t_1, \dots, t_N]]$ generated by the image of the map $f^* : \mathbb{H}^2(X, \mathcal{R}^\bullet)^* \to k[[t_1, \dots, t_N]]$, adjoint to f. Then for any $k \geq 2$, the pair $(\mathcal{E}_1, \nabla_1)$ extends to an integrable connection $(\mathcal{E}_k, \nabla_k)$ on $X \times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I \otimes k[[t_1, \dots, t_N]]/(t_1, \dots, t_N)^{k+1}$.

REMARK 4.3. The complex $\mathcal{R}^\bullet$ may be replaced by its subcomplex $0 \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^1(D) \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^2(2D) \to \dots$. Theorem 3.6 will remain valid if we replace $\mathcal{R}^\bullet$ in its statement by this smaller complex.

In the case where ∇ is an integrable logarithmic connection, we can reduce $\mathcal{R}^\bullet$ further to $\mathcal{L}^\bullet = [0 \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^1(\log D) \to \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_X^2(\log D) \to \dots]$. We now go over to integrable logarithmic connections.

4.3. Integrable logarithmic connections.

DEFINITION 4.4. Let X be a nonsingular complex projective variety, S a normal crossing divisor with smooth components. An integrable logarithmic connection E on X is a pair $(\mathcal{E},\nabla)$ where $\mathcal{E}$ is a torsion free coherent sheaf of $\mathcal{O}_X$-modules on X and $\nabla:\mathcal{E}\to\mathcal{E}\otimes\Omega^1_X(\log S)$ is $\mathbb{C}$-linear and satisfies the Leibniz rule and the integrability condition $\nabla^2=0$ (see in the beginning of Section 2).

Let $\mathcal{D}_X$ be the sheaf of algebraic differential operators on X and let $\mathcal{D}_X[\log S]$ be the $\mathcal{O}_X$-subalgebra generated by the germs of tangent vector fields which preserve the ideal sheaf of the reduced scheme S. According to [Ni], a logarithmic connection on X with singularities over S can be interpreted as a $\mathcal{D}_X[\log S]$-module which is coherent and torsion free as an $\mathcal{O}_X$-module.

REMARK 4.5. A nonsingular integrable connection on X is simply a $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module.

DEFINITION 4.6. An infinitesimal deformation of an integrable logarithmic connection $\mathcal{E}$ is a pair $(\mathcal{E}_V,\alpha)$, where $\mathcal{E}_V$ is a family of logarithmic connections parameterized by $V=\mathrm{Spec}(\mathbb{C}[\varepsilon])/\varepsilon^2$, with an isomorphism $\alpha:\mathcal{E}_V/\varepsilon\mathcal{E}_V\to\mathcal{E}$.

We define $T_{\mathcal{E}}$ as the set of all equivalence classes of infinitesimal deformations of $\mathcal{E}$. Let the sheaf $\mathcal{K}_{\mathcal{E}}$ be the kernel of $\nabla:\mathcal{E}\mathrm{nd}(\mathcal{E})\otimes\Omega^1(\log S)\to\mathcal{E}\mathrm{nd}(\mathcal{E})\otimes\Omega^2(\log S)$. As the curvature of ∇ is 0, the image of $\nabla:\mathcal{E}\to\mathcal{E}\otimes\Omega^1(\log S)$, is contained in $\mathcal{K}_{\mathcal{E}}$. If $A\in H^0(X,\mathcal{K}_{\mathcal{E}})$, then $\nabla+\varepsilon A$ is a family of logarithmic connections on the underlying sheaf $\mathcal{E}$ parameterized by V. This gives a linear map $p:H^0(X,\mathcal{K}_{\mathcal{E}})\to T_{\mathcal{E}}$.

THEOREM 4.7. *If an integrable logarithmic connection $\mathcal{E}$ is locally free, the vector space $T_{\mathcal{E}}$ of infinitesimal deformations of $\mathcal{E}$ (which equals the tangent space at $[\mathcal{E}]$ to the moduli scheme $\mathcal{M}$ of stable integrable logarithmic connections when $\mathcal{E}$ is stable) is canonically isomorphic to the first hypercohomology $\mathbb{H}^1(\mathcal{C}_{\mathcal{E}})$ of the complex $\mathcal{C}_{\mathcal{E}}=(\nabla:\ \mathcal{E}\mathrm{nd}(\mathcal{E})\to\mathcal{K}_{\mathcal{E}})$, which is in turn equal to the first hypercohomology of the logarithmic de Rham complex $\mathcal{L}^\bullet=(\mathcal{E}\mathrm{nd}(\mathcal{E})\otimes\Omega^\bullet_X(\log S),\nabla)$ associated to $\mathcal{E}\mathrm{nd}(\mathcal{E})$.*

Proof. See [Ni]. □

We deduce the construction of the Kuranishi space of integrable logarithmic connections over X.

4.4. Kuranishi space of integrable logarithmic connections.

THEOREM 4.8. *Let X be a smooth projective variety over an algebraically closed field k (or on $\mathbb{C}$), $\mathcal{E}$ a vector bundle on X, ∇ an integrable logarithmic connection on $\mathcal{E}$, $\mathcal{L}^\bullet$ the complex of sheaves on X defined in Theorem 4.7,*

$W=\mathbb{H}^1(X,\mathcal{L}^\bullet)$, $(\delta_1,\dots,\delta_N)$ a basis of W and $(t_1,\dots,t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of 0 in W, and $(\mathcal{E}_1,\nabla_1)$ the universal first order deformation of $(\mathcal{E},\nabla)$ over $X\times W_1$ in the class of integrable logarithmic connections with fixed divisor of poles D. Then there exists a formal power series

$$f(t_1,\dots,t_N)=\sum_{k=2}^{\infty} f_k(t_1,\dots,t_N)\in\mathbb{H}^2\big(X,\mathcal{L}^\bullet\big)[[t_1,\dots,t_N]],$$

where f_k is homogeneous of degree k $(k\geq 2)$, with the following property. Let I be the ideal of $k[[t_1,\dots,t_N]]$ generated by the image of the map $f^*:\mathbb{H}^2(X,\mathcal{L}^\bullet)^*\to k[[t_1,\dots,t_N]]$, adjoint to f. Then for any $k\geq 2$, the pair $(\mathcal{E}_1,\nabla_1)$ extends to an integrable logarithmic connection $(\mathcal{E}_k,\nabla_k)$ on $X\times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I\otimes k[[t_1,\dots,t_N]]/(t_1,\dots,t_N)^{k+1}$.

5. Parabolic connections

Let X be a smooth projective curve of genus g. We set

$$T_n:=\big\{(t_1,\dots,t_n)\in\overbrace{X\times\cdots\times X}^{n}\;\big|\;t_i\neq t_j \text{ for } i\neq j\big\}$$

for a positive integer n. For integers d,r with $r>0$, we set

$$\Lambda_r^{(n)}(d):=\left\{\big(\lambda_j^{(i)}\big)_{0\leq j\leq r-1}^{1\leq i\leq n}\in\mathbb{C}^{nr}\;\middle|\;d+\sum_{i,j}\lambda_j^{(i)}=0\right\}.$$

Take an element $t=(t_1,\dots,t_n)\in T_n$ and $\lambda=(\lambda_j^{(i)})_{1\leq i\leq n,0\leq j\leq r-1}\in\Lambda_r^{(n)}(d)$.

Definition 5.1. $(E,\nabla,\{l_*^{(i)}\}_{1\leq i\leq n})$ is said to be a (t,λ)-parabolic connection of rank r if

(1) E is a rank r algebraic vector bundle on X, and
(2) $\nabla:E\to E\otimes\Omega_C^1(\log(t_1+\cdots+t_n)$ is a connection, and
(3) for each t_i, $l_*^{(i)}$ is a filtration of $E|_{t_i}=l_0^{(i)}\supset l_1^{(i)}\supset\cdots\supset l_{r-1}^{(i)}\supset l_r^{(i)}=0$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)})=1$ and $(\mathrm{Res}_{t_i}(\nabla)-\lambda_j^{(i)}\mathrm{id}_{E|_{t_i}})(l_j^{(i)})\subset l_{j+1}^{(i)}$ for $j=0,\dots,r-1$.

Remark 5.2. By condition (3) above and [EV-1], we have

$$\deg E=\deg\big(\det(E)\big)=-\sum_{i=1}^{n}\mathrm{Tr}\,\mathrm{Res}_{t_i}(\nabla)=-\sum_{i=1}^{n}\sum_{j=0}^{r-1}\lambda_j^{(i)}=d.$$

Let T be a smooth algebraic scheme which is a covering of the moduli stack of n-pointed smooth projective curves of genus g over $\mathbb{C}$ and take a universal family $(\mathcal{C},\tilde{t}_1,\dots,\tilde{t}_n)$ over T.

DEFINITION 5.3. We denote the pull-back of $\mathcal{C}$ and $\tilde{t}$ with respect to the morphism $T \times \Lambda_r^{(n)}(d) \to T$ by the same characters $\mathcal{C}$ and $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_n)$. Then $D(\tilde{t}) := \tilde{t}_1 + \cdots + \tilde{t}_n$ becomes a family of Cartier divisors on $\mathcal{C}$ flat over $T \times \Lambda_r^{(n)}(d)$. We also denote by $\tilde{\lambda}$ the pull-back of the universal family on $\Lambda_r^{(n)}(d)$ by the morphism $T \times \Lambda_r^{(n)}(d) \to \Lambda_r^{(n)}(d)$. We define a functor $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t}, r, d)$ from the category of locally noetherian schemes over $T \times \Lambda_r^{(n)}(d)$ to the category of sets by

$$\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t}, r, d)(S) := \left\{ \left(E, \nabla, \{l_j^{(i)}\}\right) \right\} / \sim,$$

where

(1) E is a vector bundle on $\mathcal{C}_S = \mathcal{C} \times_{T \times \Lambda_r^{(n)}(d)} S$ of rank r,

(2) $\nabla : E \to E \otimes \Omega^1_{\mathcal{C}_S/S}(D(\tilde{t})_S)$ is a relative connection,

(3) $E|_{(\tilde{t}_i)_S} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ is a filtration by subbundles such that $(\mathrm{Res}_{(\tilde{t}_i)_S}(\nabla) - (\tilde{\lambda}_j^{(i)})_S)(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $0 \le j \le r-1$, $i = 1, \dots, n$,

(4) for any geometric point $s \in S$, $\dim(l_j^{(i)}/l_{j+1}^{(i)}) \otimes k(s) = 1$ for any i, j and $(E, \nabla, \{l_j^{(i)}\}) \otimes k(s)$ is α-stable.

Here $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{l_j'^{(i)}\})$ if there exist a line bundle $\mathcal{L}$ on S and an isomorphism $\sigma : E \xrightarrow{\sim} E' \otimes \mathcal{L}$ such that $\sigma|_{t_i}(l_j^{(i)}) = l_j'^{(i)}$ for any i, j and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega^1_{\mathcal{C}/T}\big(D(\tilde{t})\big) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma \otimes \mathrm{id}} \\ E' \otimes \mathcal{L} & \xrightarrow{\nabla'} & E' \otimes \Omega^1_{\mathcal{C}/T}\big(D(\tilde{t})\big) \otimes \mathcal{L} \end{array}$$

commutes.

We now can construct the moduli space of this functor.

THEOREM 5.4. *There exists a relative fine moduli scheme*

$$M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t}, r, d) \to T \times \Lambda_r^{(n)}(d)$$

of $\boldsymbol{\alpha}$-stable parabolic connections of rank r and degree d, which is smooth, irreducible and quasi-projective and has an algebraic symplectic structure. The fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{t}_x, \lambda)$ over $(x, \lambda) \in T \times \Lambda_r^{(n)}(d)$ is the irreducible moduli space of $\boldsymbol{\alpha}$-stable $(\tilde{t}_x, \lambda)$ parabolic connections whose dimension is $2r^2(g-1) + nr(r-1) + 2$ if it is nonempty.

Proof. See [I]. □

Let $(\tilde{E},\tilde{\nabla},\{\tilde{l}_j^{(i)}\})$ be a universal family on $\mathcal{C}\times_T M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t},r,d)$. We define a complex $\mathcal{G}^\bullet$ by

$$\mathcal{G}^0 := \big\{s\in\mathcal{E}\mathrm{nd}(\tilde{E})| s|_{\tilde{t}_i\times M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t},r,d)}\big(\tilde{l}_j^{(i)}\big)\subset\tilde{l}_j^{(i)} \text{ for any } i,j\big\},$$
$$\mathcal{G}^1 := \big\{s\in\mathcal{E}\mathrm{nd}(\tilde{E})\otimes\Omega^1_{\mathcal{C}/T}\big(D(\tilde{t})\big)\ \big|$$
$$\mathrm{Res}_{\tilde{t}_i\times M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t},r,d)}(s)\big(\tilde{l}_j^{(i)}\big)\subset\tilde{l}_{j+1}^{(i)} \text{ for any } i,j\big\},$$
$$\nabla_{\mathcal{G}^\bullet}:\mathcal{G}^0\longrightarrow\mathcal{G}^1;\qquad \nabla_{\mathcal{G}^\bullet}(s)=\tilde{\nabla}\circ s-s\circ\tilde{\nabla}.$$

As in the previous section, we can construct the Kuranishi space of (t,λ)-parabolic connections on a smooth projective curve in using the hypercohomology of $\mathcal{G}^\bullet$.

Theorem 5.5. *Let X be a smooth projective curve over k, $(\mathcal{E},\nabla,\{l_*^{(i)}\})$ a (t,λ)-parabolic connection on X, $\mathcal{G}^\bullet$ the complex of sheaves on X defined above, $W=\mathbb{H}^1(X,\mathcal{G}^\bullet)$, $(\delta_1,\dots,\delta_N)$ a basis of W and $(t_1,\dots,t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of 0 in W, and $(\mathcal{E}_1,\nabla_1,\{l_*^{(i)}\}_1)$ the universal first order deformation of $(\mathcal{E},\nabla,\{l_*^{(i)}\})$ over $X\times W_1$ in the class of (t,λ)-parabolic connections. Then there exists a formal power series*

$$f(t_1,\dots,t_N)=\sum_{k=2}^{\infty}f_k(t_1,\dots,t_N)\in\mathbb{H}^2\big(X,\mathcal{G}^\bullet\big)[[t_1,\dots,t_N]],$$

where f_k is homogeneous of degree k $(k\geq 2)$, with the following property. Let I be the ideal of $k[[t_1,\dots,t_N]]$ generated by the image of the map $f^*:\mathbb{H}^2(X,\mathcal{G}^\bullet)\to k[[t_1,\dots,t_N]]$, adjoint to f. Then for any $k\geq 2$, the triple $(\mathcal{E}_1,\nabla_1,\{l_*^{(i)}\}_1)$ extends to a (t,λ)-parabolic connection $(\mathcal{E}_k,\nabla_k,\{l_*^{(i)}\}_k)$ on $X\times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I\otimes k[[t_1,\dots,t_N]]/(t_1,\dots,t_N)^{k+1}$.

We now want to construct the Kuranishi space of T-parabolic bundles. Let T be a finite set of smooth points $\{P_1,\dots,P_n\}$ of X and W a vector bundle on X.

Definition 5.6. By a quasi-parabolic structure on a vector bundle W at a smooth point P of X, we mean a choice of a flag

$$W_P=F_1(W)_P\supset F_2(W)_P\supset\cdots\supset F_l(W)_P=0,$$

in the fibre W_P of W at P. A parabolic structure at P is a pair consisting of a flag as above and a sequence $0\leq\alpha_1<\alpha_2<\cdots<\alpha_l<1$ of weights of W at P.

The integers $k_1=\dim F_1(W)_P-\dim F_2(W)_P,\dots,k_l=\dim(F_l(W)_P)$ are called the multiplicities of $\alpha_1,\dots,\alpha_l$. A T-parabolic structure on W is the triple consisting of a flag at P, some weights α_i, and their multiplicities k_i.

A vector bundle W endowed with a T-parabolic structure is called a T-parabolic bundle.

DEFINITION 5.7. A T-parabolic bundle W_1 on X is a T-parabolic subbundle of a T-parabolic bundle W_2 on X, if W_1 is a subbundle of W_2 and at each smooth point P of T, the weights of W_1 are a subset of those of W_2. Further, if we take the weight α_{j_0} such that $1 \leq j_0 \leq m$, and the weight β_{k_0} for the greatest integer k_0 such that $F_{j_0}(W_1)_P \subset F_{k_0}(W_2)_P$, then $\alpha_{j_0} = \beta_{k_0}$.

DEFINITION 5.8. The parabolic degree of a T-parabolic vector bundle W on X is

$$\operatorname{par}\deg(W) := \deg(W) + \sum_{P \in I} \sum_{i=1}^{r} k_i(P)\alpha_i(P).$$

DEFINITION 5.9. A T-parabolic bundle W is stable (resp. semistable) if for any proper nonzero T-parabolic subbundle $W' \subset W$ the inequality

$$\operatorname{par}\deg W' < (\text{resp. } \leq) \frac{\operatorname{par}\deg W \operatorname{rk}(W')}{\operatorname{rk} W}$$

holds.

We have a forgetful map g from (t, λ) parabolic connections to T-parabolic bundles. We thus can construct the Kuranishi space of T-parabolic bundles by following an analogous argument to the one given above. We first introduce the Higgs field $\Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X(D)$ defined as follows:

$$\forall p \in X, \forall f \in \mathcal{O}_{X,p}, \forall s \in \mathcal{E}_P, \quad \Phi(fs) = f\Phi(s).$$

We afterwards consider a parabolic bundle $\mathcal{E}$ with fixed weights and parabolic points $P_1, \ldots, P_N$. We set $L = K \otimes \mathcal{O}(P_1, \ldots, P_N)$, the line bundle associated to the canonical divisor together with the divisor of poles $D = P_1 + \cdots + P_N$. The sheaf of rational 1-forms on X is identified with the sheaf of rational sections of the canonical bundle having single poles at points $P_1, \ldots, P_N$. We replace t_i by P_i, for $i = 1, \ldots, N$ and $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t}, r, d)$ by M^s_T. We define a complex $\mathcal{B}^\bullet$ by

$$\begin{aligned}
\mathcal{B}^0 &:= \big\{ s \in \mathcal{E}\mathrm{nd}(\tilde{E}) | s|_{\tilde{P}_i \times M^s_{Z,\mathcal{C}/T}(\tilde{P},r,d)} \big(\tilde{l}^{(i)}_j\big) \subset \tilde{l}^{(i)}_j \text{ for any } i, j \big\}, \\
\mathcal{B}^1 &:= \big\{ s \in \mathcal{E}\mathrm{nd}(\tilde{E}) \otimes \Omega^1_{\mathcal{C}/T}\big(D(\tilde{P}i)\big) \ | \\
&\qquad \operatorname{Res}_{\tilde{P}_i \times M^s_{Z,\mathcal{C}/T}(\tilde{P},r,d)}(s)\big(\tilde{l}^{(i)}_j\big) \subset \tilde{l}^{(i)}_{j+1} \text{ for any } i, j \big\},
\end{aligned}$$

$$\operatorname{ad}\Phi_{\mathcal{B}^\bullet} : \mathcal{B}^0 \longrightarrow \mathcal{B}^1; \qquad \operatorname{ad}\Phi_{\mathcal{B}^\bullet}(s) = \tilde{\Phi} \circ s - s \circ \tilde{\Phi}.$$

From this, we deduce the construction of the Kuranishi space of T-parabolic bundles on a smooth projective curve.

THEOREM 5.10. *Let X be a smooth projective curve over k or a complex space* (*in which case $k=\mathbb{C}$*), *$\mathcal{E}$ a T-parabolic bundle on X, $\mathcal{B}^\bullet$ the complex of sheaves on X defined as above, $W=\mathbb{H}^1(X,\mathcal{B}^\bullet)$, $(\delta_1,\dots,\delta_N)$ a basis of W and $(t_1,\dots,t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of 0 in W, and $\mathcal{E}_1$ the universal first order deformation of $\mathcal{E}$ over $X\times W_1$. Then there exists a formal power series*

$$f(t_1,\dots,t_N)=\sum_{k=2}^{\infty} f_k(t_1,\dots,t_N)\in\mathbb{H}^2\big(X,\mathcal{B}^\bullet\big)[[t_1,\dots,t_N]],$$

where f_k is homogeneous of degree k $(k\geq 2)$, with the following property. Let I be the ideal of $k[[t_1,\dots,t_N]]$ generated by the image of the map $f^*:\mathbb{H}^2(X,\mathcal{B}^\bullet)^*\to k[[t_1,\dots,t_N]]$, adjoint to f. Then for any $k\geq 2$, $\mathcal{E}_1$ extends to a T-parabolic bundle $\mathcal{E}_k$ on $X\times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I\otimes k[[t_1,\dots,t_N]]/(t_1,\dots,t_N)^{k+1}$.

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