Degeneration of Fermat hypersurfaces in positive characteristic

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ABSTRACT. We work over an algebraically closed field k of positive characteristic p. Let q be a power of p. Let A be an $(n + 1) \times (n + 1)$ matrix with coefficients a_{ij} in k, and let X_A be a hypersurface of degree q + 1 in the projective space \mathbf{P}^n defined by $\sum a_{ij}x_ix_j^q = 0$. It is well-known that if the rank of A is n + 1, the hypersurface X_A is projectively isomorphic to the Fermat hypersurface of degree q + 1. We investigate the hypersurfaces X_A when the rank of A is n, and determine their projective isomorphism classes.

1. Introduction

We work over an algebraically closed field k of positive characteristic p. Let q be a power of p. Let n be a positive integer. We denote by $M_{n+1}(k)$ the set of square matrices of size n+1 with coefficients in k. For a nonzero matrix $A = (a_{ij})_{0 \le i,j \le n} \in M_{n+1}(k)$, we denote by X_A the hypersurface of degree q+1 defined by the equation

$$\sum a_{ij} x_i x_j^q = 0$$

in the projective space \mathbf{P}^n with homogeneous coordinates (x_0, x_1, \ldots, x_n) . The following is well-known ([2], [10], [14], see also §4 of this paper).

PROPOSITION 1. Let $A = (a_{ij})_{0 \le i,j \le n} \in M_{n+1}(k)$ and $X_A \subset \mathbf{P}^n$ be as above. Then the following conditions are equivalent:

- (i) rank(A) = n + 1,
- (ii) X_A is smooth,
- (iii) X_A is isomorphic to the Fermat hypersurface of degree q + 1, and
- (iv) there exists a linear transformation of coordinates $T \in GL_{n+1}(k)$ such that ${}^{t}TAT^{(q)} = I_{n+1}$, where ${}^{t}T$ is the transpose of T, $T^{(q)}$ is the matrix obtained from T by raising each coefficient to its q-th power, and I_{n+1} is the identity matrix.

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Thanh HOAI HOANG

The Fermat hypersurface of degree q + 1 defined over an algebraically closed field of positive characteristic p has been a subject of numerous papers. It has many interesting properties, such as supersingularity ([15], [16], [17]), or unirationality ([13], [15], [16]). Moreover, the hypersurface X_A associated with the matrix A with coefficients a_{ij} in the finite field \mathbf{F}_{q^2} , which is called a Hermitian variety, has also been studied for many applications, such as coding theory ([8]). (The general results on Hermitian varieties are due to Segre [11]; see also [6]). Therefore it is important to extend these studies to degenerate cases.

In the case where characteristic $p \neq 2$, the following is well-known and can be found in any standard textbook on quadratic forms: the hypersurface defined by the quadratic form $\sum a_{ij}x_ix_j = 0$ is projectively isomorphic to the hypersurface defined by

$$x_0^2 + \dots + x_{r-1}^2 = 0,$$

where *r* is the rank of $A = (a_{ij})$. This result has been extended the case of characteristic 2 (see [3]). Therefore we have a question what is the normal form of the hypersurfaces defined by a form $\sum a_{ij}x_ix_j^q = 0$. When *A* satisfies ${}^{t}A = A^{(q)}$ and hence this form is the Hermitian form over \mathbf{F}_q , the hypersurface X_A is projectively isomorphic over \mathbf{F}_{q^2} to

$$x_0^{q+1} + \dots + x_{r-1}^{q+1} = 0,$$

where r is the rank of A ([5]).

In this paper, we classify the hypersurfaces X_A associated with the matrices A of rank *n* over an algebraically closed field. Note that two hypersurfaces X_A , $X_{A'}$ associated with the matrices A, A' are projectively isomorphic if and only if there exists a linear transformation $T \in GL_{n+1}(k)$ such that $A' = {}^{t}TAT^{(q)}$. In this case, we write $A \sim A'$.

We define I_s to be the $s \times s$ identity matrix, and E_r to be the $r \times r$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In particular, $E_1 = (0)$ and E_0 is the 0×0 matrix. Throughout this paper, a blank in a block decomposition of a matrix means that all the components of the block are 0. Our main result is as follow.

THEOREM 1. Let $A = (a_{ij})_{0 \le i,j \le n}$ be a nonzero matrix in $M_{n+1}(k)$, and let X_A be the hypersurface of degree q + 1 defined by $\sum a_{ij}x_ix_j^q = 0$ in the projective space \mathbf{P}^n with homogeneous coordinates (x_0, x_1, \ldots, x_n) . Suppose that the rank of A is n. Then the hypersurface X_A is projectively isomorphic to one of the hypersurfaces X_s associated with the matrices

$$W_s = \left(\begin{array}{c|c} I_s \\ \hline \\ E_{n-s+1} \end{array}\right),$$

where $0 \le s \le n$. Moreover, if $s \ne s'$, then X_s and $X_{s'}$ are not projectively isomorphic.

COROLLARY 1. If A is a general point of $\{A \in M_{n+1}(k) | rank(A) = n\}$, then $A \sim W_{n-1}$.

COROLLARY 2. Suppose that $n \ge 2$, s < n and $(n, s) \ne (2, 0)$. Then X_s is rational.

We also determine the automorphism group

$$\operatorname{Aut}(X_s) = \{g \in PGL_{n+1}(k) \mid g(X_s) = X_s\},\$$

of the hypersurface X_s for each s. For $M \in GL_{n+1}(k)$, we denote by $[M] \in PGL_{n+1}(k)$ the image of M by the natural projection.

THEOREM 2. Let X_s be the hypersurface associated with the matrix W_s in the projective space \mathbf{P}^n . The projective automorphism group $\operatorname{Aut}(X_s)$ with $s \le n-2$ is the group consisting of [M], with

$$M = \left(\begin{array}{c|c} T & {}^t \mathbf{a} & 0 \\ \hline 0 & d & 0 \\ \hline \mathbf{c} & e & 1 \end{array} \right),$$

where $T \in GL_{n-1}(k)$, **a**, **c** are row vectors of dimension n-1, $d, e \in k$, and they satisfy the following conditions:

(i) $[T] \in \operatorname{Aut}(X_s^{n-2}), \ {}^{t}TW_s'T^{(q)} = \delta W_s', \ \delta = \delta^q \neq 0, \ where \ X_s^{n-2} \ is the hypersurface defined in \mathbf{P}^{n-2} \ by the matrix$

$$W'_{s} = \left(\begin{array}{c|c} I_{s} \\ \hline & E_{n-s-1} \end{array}\right)$$

Thanh HOAI HOANG

(ii) $d = \delta$, (iii) $[\mathbf{a}W'_s + d(0, ..., 0, 1)] \cdot T^{(q)} = \delta(0, ..., 0, 1)$, (iv) ${}^{t}TW'_s \cdot {}^{t}\mathbf{a}^{(q)} + {}^{t}\mathbf{c}d^{q} = 0$, and (v) $[\mathbf{a}W'_s + d(0, ..., 0, 1)] \cdot {}^{t}\mathbf{a}^{(q)} + ed^{q} = 0$.

Moreover, we have

$$\operatorname{Aut}(X_n) = \left\{ \begin{bmatrix} T_n \\ \hline \mathbf{u} & 1 \end{bmatrix} \middle| \begin{array}{c} {}^tT_n T_n^{(q)} = \lambda I_n, \ T_n \in GL_n(k), \ \lambda \neq 0, \\ \mathbf{u} \ is \ a \ row \ vector \ of \ dimension \ n \end{array} \right\},$$

and

We give a brief outline of our paper. In §2, we prove Theorem 1 and its corollaries. In §3, we prove Theorem 2. In §4, we recall the proof of Proposition 1 because this proposition plays an important role in the proof of Theorem 1. In §5, we investigate the plane curve X_A associated with the matrix A of rank ≤ 2 in the projective plane \mathbf{P}^2 , and recovers Homma's unpublished work [9] (see Remark 5).

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2. Proofs of Theorem 1 and its corollaries

We present several preliminary lemmas. The following remark may be helpful in reading the proof of lemmas.

REMARK 1. Let

$$T = \begin{pmatrix} t_{00} & \cdots & t_{0n} \\ \vdots & \ddots & \vdots \\ t_{n0} & \cdots & t_{nn} \end{pmatrix}$$

be an invertible matrix. Suppose that $\sum a_{ij}x_ix_j^q = 0$ is the equation associated to a matrix $A = (a_{ij})_{0 \le i,j \le n}$. Then the operation

$$A \mapsto {}^{t}TAT^{(q)}$$

on the matrix is equivalent to the transformation of the coordinates

$$x_i \mapsto \sum_{j=0}^n t_{ij} x_j,$$

where $0 \le i \le n$.

LEMMA 1. Put

$$G_{s,r} = \begin{pmatrix} I_s & & & \\ \hline & E_r & & \\ \hline \mathbf{a} & 0 \cdots 0 & 1 & \\ 0 & 0 & & \\ \vdots & \vdots & E_{n-s-r+1} \\ 0 & 0 & & \\ \end{pmatrix},$$

and

$$G_{s,r+2} = \begin{pmatrix} I_s & & \\ \hline & E_{r+2} & \\ \hline \mathbf{a}^{(q^2)} & 0 \cdots 0 & 1 \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \\ \hline 0 & 0 & \\ \end{bmatrix},$$

where $s \ge 1$, $r \ge 0$, $n - s - r - 1 \ge 0$, and **a** is a nonzero row vector of dimension *s*. Then

$$G_{s,r} \sim G_{s,r+2}$$
.

PROOF. By the transformation

$$T_{G} = \begin{pmatrix} I_{s} & -{}^{t}\mathbf{a} & \\ \hline & I_{r} & & \\ \hline & & 1 & \\ \hline & & 1 & \\ \hline & & & 1 & \\ \hline & & & & I_{n-s-r-1} \end{pmatrix},$$

we have

$$^{t}T_{G}G_{s,r}T_{G}^{(q)}=G_{s,r+2}.$$

REMARK 2. Lemma 1 holds when r = 0 or n - s - r - 1 = 0. In particular, when n - s - r - 1 = 0, we have $G_{s,r+2} = W_s$.

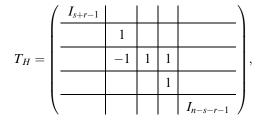
LEMMA 2. Put

	(D_{s-1}	$-{}^{t}\mathbf{a}'' \ 0 \cdots 0$)	
$H_{s,r} =$		$-\mathbf{a}'$				
		0				
		÷	E_r			
		0				
			001	1		,
				1		
				0		
				:	$E_{n-s-r+1}$	
	(0)	

where $s \ge 1$, $r \ge 2$, $n - s - r - 1 \ge 1$, $D_{s-1} \in M_{s-1}(k)$, \mathbf{a}' and \mathbf{a}'' are row vectors of dimension s - 1. Then

$$H_{s,r} \sim H_{s,r+2}.$$

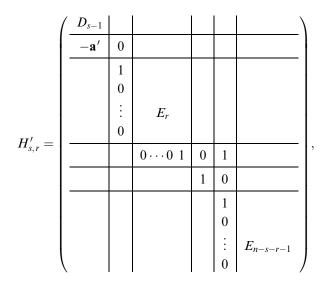
PROOF. By the transformation



we have

$$^{t}T_{H}H_{s,r}T_{H}^{(q)} = H_{s,r+2}.$$

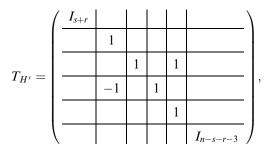
LEMMA 3. Put



where $s \ge 1$, $r \ge 2$, $n - s - r - 3 \ge 1$, $D_{s-1} \in M_{s-1}(k)$, and \mathbf{a}' is a row vector of dimension s - 1. Then

$$H_{s,r}' \sim H_{s,r+2}'$$

PROOF. By the transformation



we have

$${}^{t}T_{H'}H_{s,r}'T_{H'}^{(q)} = H_{s,r+2}'.$$

REMARK 3. Lemma 2 and 3 will be used only in the case where n - s + 1 is odd. Hence, we do not need to prove the case n - s - 1 = 0 in Lemma 2 nor the case n - s - 3 = 0 in Lemma 3.

LEMMA 4. Put

$$P_s = \begin{pmatrix} I_s & & \\ \hline \mathbf{a} & & \\ 0 & & \\ \vdots & E_{n-s+1} \\ 0 & & \end{pmatrix},$$

where $s \ge 1$, $n - s + 1 \ge 1$, and **a** is a nonzero row vector of dimension s. Then

- (1) If n-s+1 is even, then $P_s \sim W_s$.
- (2) If n s + 1 is odd, then

$$P_{s} \sim B_{s-1} = \begin{pmatrix} D_{s-1} \\ b_{s-1} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \end{pmatrix},$$

where $D_{s-1} \in M_{s-1}(k)$, \mathbf{b}_{s-1} is the row vector of dimension s-1. In particular, if s = 1 and n is odd, then $P_1 \sim W_0$.

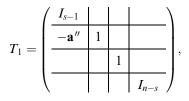
PROOF. (1) Suppose that n - s + 1 is even. Using Lemma 1 and Remark 2, we have

$$P_s = G_{s,0} \sim G_{s,n-s+1} = W_s.$$

(2) Next, suppose that n-s+1 is odd. By interchanging the coordinates x_0, \ldots, x_{s-1} , and scalar multiplication of the coordinates x_s, \ldots, x_n if nessesary, we can show that

$$P_{s} \sim P_{s}' = \begin{pmatrix} I_{s-1} & & & \\ \hline 1 & & \\ \hline a' & 1 & 0 & \\ \hline & & 1 & \\ & & 0 & \\ & & \vdots & E_{n-s} \\ & & 0 & \\ \end{pmatrix},$$

with \mathbf{a}' being a row vector of dimension s-1. By the transformation



with $\mathbf{a}''^{(q)} = \mathbf{a}'$, we have

$$Q_{s} = {}^{t}T_{1}P_{s}'T_{1}^{(q)} = \begin{pmatrix} \begin{array}{c|c} D_{s-1} & -{}^{t}\mathbf{a}'' & & \\ \hline -\mathbf{a}' & 1 & & \\ \hline 1 & 0 & & \\ \hline & & 1 & & \\ & & 0 & \\ & & \vdots & E_{n-s} \\ & & 0 & \\ \end{array} \right)$$

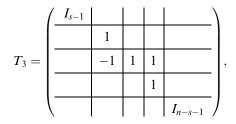
where $D_{s-1} = I_{s-1} + {}^{t}\mathbf{a}'' \cdot \mathbf{a}'$. If n-s+1=1, by the transformation

$$T_2 = \left(\begin{array}{c|c} I_{n-1} & | \\ \hline 1 & \\ \hline a'' & -1 & 1 \end{array}\right),$$

we have

$${}^{t}T_{2}Q_{n}T_{2}^{(q)}=B_{n-1}.$$

Suppose that n - s + 1 > 1. Note that, since we are in the case where n - s + 1 is odd, we have $n - s + 1 \ge 3$. By the transformation



we have

$$Q_{s}' = {}^{t}T_{3}Q_{s}T_{3}^{(q)} = \begin{pmatrix} \begin{array}{c|c} D_{s-1} & -{}^{t}\mathbf{a}'' & | & | \\ \hline -\mathbf{a}' & 0 & | \\ \hline 1 & 0 & | \\ \hline 1 & 0 & | \\ \hline & 1 & 1 & | \\ \hline & & 1 & 1 \\ \hline & & 0 & | \\ \hline & & 0 & | \\ \hline & & & 0 & | \\ \hline \end{array} \right) = H_{s,2}.$$

Using Lemma 2, we have

$$Q'_{s} = H_{s,2} \sim H_{s,n-s} = Q''_{s} = \begin{pmatrix} D_{s-1} & -{}^{t}\mathbf{a}'' & 0 \cdots & 0 & \\ \hline -\mathbf{a}' & & & & \\ 0 & & & & \\ \vdots & E_{n-s} & & \\ 0 & & & \\ \hline & 0 & 0 & 0 & 1 & 1 \\ \hline & & & 1 & 0 \end{pmatrix}.$$

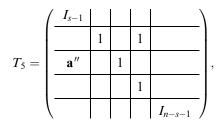
Then by the transformation

$$T_4 = \left(\begin{array}{c|c} I_{n-1} & & \\ \hline & 1 & \\ \hline & -1 & 1 \end{array} \right),$$

we have

$$R_{s} = {}^{t}T_{4}Q_{s}''T_{4}^{(q)} = \begin{pmatrix} \begin{array}{c|c} D_{s-1} & -{}^{t}\mathbf{a}'' & 0 \cdots & 0 \\ \hline -\mathbf{a}' & & & \\ 0 & & & \\ \vdots & & E_{n-s+2} \\ 0 & & & \end{pmatrix}.$$

If s = 1, $R_1 \sim W_0$. Suppose that s > 1. By the transformation



we obtain

$$R'_{s} = {}^{t}T_{5}R_{s}T_{5}^{(q)} = \begin{pmatrix} \begin{matrix} D_{s-1} & & & \\ \hline -\mathbf{a}' & 0 & & & \\ \hline & 1 & 0 & 1 & & \\ \hline & 1 & 0 & & \\ \hline & & 1 & 0 & & \\ \hline & & & 1 & & \\ \hline & & & 0 & & \\ \hline & & & \vdots & E_{n-s-1} \end{pmatrix}.$$

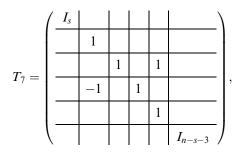
If n - s - 1 = 1, by the transformation

$$T_6 = \begin{pmatrix} I_{n-2} & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & -1 & 1 \end{pmatrix},$$

we have

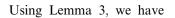
$${}^{t}T_{6}R_{n-2}'T_{6}^{(q)}=B_{n-3}$$

Suppose that n - s - 1 > 1. Then by the transformation



we have

$$R_{s}'' = {}^{t}T_{7}R_{s}'T_{7}^{(q)} = \begin{pmatrix} \begin{matrix} D_{s-1} & & & & & \\ & -\mathbf{a}' & 0 & & & & \\ \hline & -\mathbf{a}' & 0 & & & \\ \hline & 1 & & & & \\ 0 & E_{2} & & & \\ \hline & 0 & 1 & 0 & 1 & \\ \hline & 0 & 1 & 0 & 1 & \\ \hline & & 1 & 0 & & \\ \hline & & 1 & 0 & & \\ \hline & & & 1 & 0 & \\ \hline & & & 1 & 0 & \\ \hline & & & & 1 & \\ \hline & & & & 0 & \\ \hline & & & & 0 & \\ \hline & & & & 0 & \\ \hline \end{array} \right) = H_{s,2}'$$



$$R_{s}'' = H_{s,2}' \sim H_{s,n-s-2}' = R_{s}''' = \begin{pmatrix} D_{s-1} & & & & \\ -\mathbf{a}' & 0 & & & \\ & 1 & & & \\ 0 & & & \\ \vdots & E_{n-s-2} & & \\ 0 & & & \\ \hline & 0 & 0 & 0 & 1 & \\ \hline & 0 & 0 & 0 & 1 & \\ \hline & 1 & 0 & \\ \hline & & 1 & 0 & \\ \hline \end{pmatrix}.$$

It is easy to see that

$$^{t}T_{6}R_{s}^{\prime\prime\prime}T_{6}^{(q)}=B_{s-1}.$$

LEMMA 5. Put

$$B_s = \begin{pmatrix} D_s & & \\ \hline \mathbf{b}_s & & \\ 0 & & \\ \vdots & E_{n-s+1} \\ 0 & & \\ \end{pmatrix},$$

where $s \ge 1$, $n - s + 1 \ge 1$, $D_s \in M_s(k)$, and \mathbf{b}_s is a row vector of dimension s. Suppose that the rank of B_s is n. Then

$$B_s \sim W_s = \left(\begin{array}{c|c} I_s \\ \hline & E_{n-s+1} \end{array} \right),$$

or

$$B_{s} \sim B_{s-1} = \begin{pmatrix} D_{s-1} \\ b_{s-1} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \end{pmatrix},$$

where $D_{s-1} \in M_{s-1}(k)$, and \mathbf{b}_{s-1} is a row vector of dimension s-1.

PROOF. Suppose that det $D_s \neq 0$. By Proposition 1, there exists a linear transformation of coordinates $T_D \in GL_s(k)$ such that ${}^{t}T_D D_s T_D^{(q)} = I_s$. By the transformation

$$T = \left(\begin{array}{c|c} T_D \\ \hline \\ I_{n-s+1} \end{array}\right),$$

we have

$${}^{t}TB_{s}T^{(q)} = \begin{pmatrix} I_{s} \\ b_{s}' \\ 0 \\ \vdots \\ 0 \\ 0 \\ \end{bmatrix},$$

where $\mathbf{b}'_s = \mathbf{b}_s T_D^{(q)}$. If $\mathbf{b}'_s = 0$, then $B_s \sim W_s$. Suppose that $\mathbf{b}'_s \neq 0$. By Lemma 4, we have $B_s \sim W_s$, or $B_s \sim B_{s-1}$.

Suppose that det $D_s = 0$. Then one row of the matrix D_s is a linear combination of the other rows. By interchanging coordinates x_0, \ldots, x_{s-1} if nessesary, we can assume that the *s*-th row is a linear combination of the other rows. We write the matrix D_s as

$$D_s = \left(\begin{array}{c|c} P & {}^t \mathbf{g} \\ \hline \mathbf{h} & d \end{array} \right),$$

where $P \in M_{s-1}(k)$, **g**, **h** are row vectors of dimension s-1, $d \in k$, and that satisfy $\mathbf{h} = \mathbf{w}P$, $d = \mathbf{w}^t \mathbf{g}$ with **w** being a row vector of dimension s-1. Then

$$B_{s} \sim B'_{s} = \begin{pmatrix} P & {}^{t}\mathbf{g} \\ \hline \mathbf{h} & d \\ \hline \mathbf{f} & e \\ 0 & 0 \\ \vdots & \vdots & E_{n-s+1} \\ 0 & 0 & \\ \end{pmatrix},$$

where **f** is a row vector of dimension s - 1, and $e \in k$. By the transformation

we obtain

$$B_{s}'' = {}^{t}T'B_{s}'T'^{(q)} = \begin{pmatrix} P & -P \cdot {}^{t}\mathbf{w}^{(q)} + {}^{t}\mathbf{g} \\ \hline \\ \hline \\ \mathbf{f} & -\mathbf{f} \cdot {}^{t}\mathbf{w}^{(q)} + e \\ 0 & 0 \\ \vdots & \vdots & E_{n-s+1} \\ 0 & 0 & \\ \end{pmatrix}.$$

Put

$$Q = \left(\begin{array}{c|c} P & -P \cdot {}^{t} \mathbf{w}^{(q)} + {}^{t} \mathbf{g} \\ \hline \mathbf{f} & -\mathbf{f} \cdot {}^{t} \mathbf{w}^{(q)} + e \end{array}\right).$$

Because the rank of B'_s is *n*, we have det $Q \neq 0$. Let $Q' \in GL_s(k)$ such that $QQ'^{(q)} = I_s$,

$$Q' = \left(\begin{array}{c|c} P' & {}^t \mathbf{g}' \\ \hline \mathbf{f}' & e' \end{array}\right),$$

where $P' \in M_{s-1}(k)$, \mathbf{g}' , \mathbf{f}' are row vectors of dimension s-1, $e' \in k$. By the transformation

$$T'' = \left(\begin{array}{c|c} P' & {}^t \mathbf{g}' \\ \hline \mathbf{f}' & e' \\ \hline & & I_{n-s+1} \end{array} \right),$$

we obtain

$${}^{t}T''B_{s}''T''^{(q)} = \begin{pmatrix} \begin{array}{c|c} {}^{t}P' & & \\ \hline g' & 0 & \\ \hline & 1 & \\ & 0 & \\ & \vdots & E_{n-s+1} \\ & 0 & \\ \end{array} \end{pmatrix}.$$

Putting $D_{s-1} = {}^{t}P'$ and $\mathbf{b}_{s-1} = \mathbf{g}'$, we have $B_{s}'' \sim B_{s-1}$.

Remark 4. When s = 1, we have

$$B_{s-1} = B_0 = E_{n+1} = W_0$$

Now we prove Theorem 1 and Corollary 1.

PROOF. Because the rank of the matrix A is n, Proposition 1 implies that the hypersurface X_A is singular. By using a linear transformation of coordinates if nessesary, we can assume that X_A has a singular point $(0, \ldots, 0, 1)$. Then we have $a_{in} = 0$ for any $0 \le i \le n$. The matrix A is now of the form

$$A = \left(\begin{array}{c|c} D_n \\ \hline \\ \mathbf{b}_n \end{array}\right) = B_n$$

where $D_n \in M_n(k)$, and \mathbf{b}_n is a row vector of dimension *n*. Using Lemma 5 repeatedly and Remark 4, we have that the hypersurface X_A is isomorphic to one of the hypersurfaces defined by the matrixes W_s with $0 \le s \le n$.

If A is general, then $det(D_n) \neq 0$, and hence by the first paragraph of the proof of Lemma 5 and Lemma 4, we have $A \sim W_{n-1}$.

Next we prove that $s \neq s'$ implies $W_s \neq W_{s'}$. For this, we introduce some notions. Let X_s^n be the hypersurface defined by the matrix W_s in the projective space \mathbf{P}^n . The defining equation of X_s^n can be written as

$$F_q x_n + F_{q+1} = 0$$

where

$$F_q = \begin{cases} 0 & \text{if } s = n \\ x_{n-1}^q & \text{if } s < n, \end{cases}$$

and

$$F_{q+1} = \begin{cases} x_0^{q+1} + \dots + x_{n-1}^{q+1} & \text{if } s = n \\ x_0^{q+1} + \dots + x_{s-1}^{q+1} + x_s^q x_{s+1} + \dots + x_{n-2}^q x_{n-1} & \text{if } s < n \end{cases}$$

It is easy to see that X_s^n has only one singular point $P_0 = (0, ..., 0, 1)$. The variety of lines in \mathbf{P}^n passing through P_0 can be naturally identified with the hypersurface $\mathscr{H}_{\infty} = \{x_n = 0\}$ in \mathbf{P}^n by the correspondence $Q \in \mathscr{H}_{\infty}$ to the line $\overline{QP_0}$. Let φ be the map defined by

$$\varphi: \mathbf{P}^n \setminus \{P_0\} \to \mathbf{P}^{n-1}$$

$$P \mapsto \overline{PP_0}.$$

Let $\overline{X_s^n} = \varphi(X_s^n \setminus \{P_0\})$. For $Q = (y_0, \dots, y_{n-1}, 0) \in \mathscr{H}_{\infty}$, we consider the line

$$l = \overline{QP_0} = \{ (\lambda y_0, \dots, \lambda y_{n-1}, \mu) \, | \, (\lambda, \mu) \in \mathbf{P}^1 \}$$

We have $l \in \overline{X_s^n}$ if and only if there exists $P = (p_0, \dots, p_{n-1}, p_n) \in X_s^n \setminus \{P_0\}$ satisfying $P \in I$, i.e. there exists an element $\mu \in k$ such that

$$(p_0,\ldots,p_{n-1},p_n)=(y_0,\ldots,y_{n-1},\mu),$$

for some $P \in X_s^n \setminus \{P_0\}$, or equivalently there exists an element $\mu \in k$ such that

$$F_q(y_0,\ldots,y_{n-1})\mu + F_{q+1}(y_0,\ldots,y_{n-1}) = 0.$$

Then

$$\varphi^{-1}(l) \cap (X_s^n \setminus \{P_0\}) = \begin{cases} \emptyset & \text{if } F_q(y_0, \dots, y_{n-1}) = 0 \text{ and} \\ F_{q+1}(y_0, \dots, y_{n-1}) \neq 0, \\ \{a \text{ single point}\} & \text{if } F_q(y_0, \dots, y_{n-1}) \neq 0, \\ l \setminus \{P_0\} & \text{if } F_q(y_0, \dots, y_{n-1}) = 0 \text{ and} \\ F_{q+1}(y_0, \dots, y_{n-1}) = 0. \end{cases}$$

Putting $V_s = \{F_q = 0, F_{q+1} = 0\} \subset \mathbf{P}^{n-1}$, and $H_s = \{F_q = 0\} \subset \mathbf{P}^{n-1}$, we have

$$V_s = \begin{cases} X_s^{n-2} & \text{if } s \le n-2, \\ \text{nonsingular Fermat hypersurface in } \mathbf{P}^{n-1} & \text{if } s = n, \\ \text{nonsingular Fermat hypersurface in } \mathbf{P}^{n-2} & \text{if } s = n-1, \end{cases}$$

where X_s^{n-2} is the hypersurface in \mathbf{P}^{n-2} associated with the matrix

$$\left(\begin{array}{c|c} I_s \\ \hline \\ \hline \\ E_{n-s-1} \end{array}\right).$$

For any $s \neq s'$, suppose that X_{s}^{n} and $X_{s'}^{n}$ are isomorphic and let $\psi : X_{s}^{n} \to X_{s'}^{n}$ be an isomorphism. Because each of X_{s}^{n} and $X_{s'}^{n}$ has only one singular point

 P_0 , we have $\psi(P_0) = P_0$, and hence ψ induces an isomorphism $\overline{\psi}$ from $\overline{X_s^n}$ to $\overline{X_{s'}^n}$. For any line $l \in \overline{X_s^n}$ and $l' \in \overline{X_{s'}^n}$ such that $\overline{\psi}(l) = l'$, we have

$$\#(\varphi^{-1}(l) \cap (X_s^n \setminus \{P_0\})) = \#(\varphi^{-1}(l') \cap (X_{s'}^n \setminus \{P_0\})).$$

Thus $V_s \cong V_{s'}$ and $H_s \cong H_{s'}$. Hence, for any $s \neq s'$, if $V_s \ncong V_{s'}$ or $H_s \ncong H_{s'}$ then $X_s^n \ncong X_{s'}^n$.

In the case n = 1, we have that X_0^1 consists of two points, and X_1^1 consists of a single point. In the case n = 2, we have that X_0^2 consists of two irreducible components, X_1^2 is irreducible, and X_2^2 consists of (q + 1) lines. Hence, in the case n = 1 and n = 2, we see that $s \neq s'$ implies $W_s \neq W_{s'}$. By induction on n, we have the proof.

Next, we prove Corollary 2.

PROOF. Under the condition $n \ge 2$, s < n and $(n, s) \ne (2, 0)$, we have x_{n-1} does not divide F_{q+1} , and hence V_s is of codimension 2 in \mathbf{P}^{n-1} . By induction on n, X_s^n is irreducible. The morphism

$$arphi|_{X^n_s \setminus \{P_0\}} : X^n_s \setminus \{P_0\} o \mathscr{H}_\infty \cong \mathbf{P}^{n-1}$$

is birational with the inverse rational map

$$Q = (y_0, \dots, y_{n-1}, 0) \mapsto \left(y_0, \dots, y_{n-1}, -\frac{F_{q+1}(y_0, \dots, y_{n-1})}{y_{n-1}^q}\right).$$

3. Proof of Theorem 2

For any $s \le n-2$, the matrix W_s can be written

$$W_s = \left(\begin{array}{c|c} W'_s & \\ \hline 0 \cdots 0 & 1 & 0 \\ \hline & 1 & 0 \end{array} \right).$$

For any $g \in Aut(X_s)$, we have $g(P_0) = P_0$ because X_s has only one singular point $P_0 = (0, ..., 0, 1)$. The automorphism g is defined by a matrix of the form

$$M = \begin{pmatrix} T & {}^{t}\mathbf{a} & 0\\ \hline \mathbf{b} & d & 0\\ \hline \mathbf{c} & e & 1 \end{pmatrix},$$

where $T \in M_{n-1}(k)$, **a**, **b**, **c** are row vectors of dimension n-1, and $d, e \in k$. We have ${}^{t}MW_{s}M^{(q)} = \delta W_{s}$ for some $0 \neq \delta \in k$, which implies Thanh HOAI HOANG

(1)

$${}^{t}TW'_{s} \cdot {}^{t}\mathbf{a}^{(q)} + {}^{t}\mathbf{c}d^{q} = 0 \tag{3}$$

$$[\mathbf{a}W'_{s} + d(0, \dots, 0, 1)] \cdot {}^{t}\mathbf{a}^{(a)} + ed^{q} = 0$$
⁽⁴⁾

$$\mathbf{b} = 0 \tag{5}$$

$$d^{q} = \delta \tag{6}$$

By (1), we see that T is a matrix defining an automorphism of X_s^{n-2} in \mathbf{P}^{n-2} . Because $s \le n-2$, by (2) we have $d = \delta$. Hence, we can calculate T by induction on *n*. The vectors \mathbf{a} , \mathbf{c} and d, e can be find by using the equations (2)-(6). Conversely, it is easy to show that if the matrix M satisfies the conditions (i)–(v) then it defines a projective automorphism of X_s . The projective automorphism groups of X_n and X_{n-1} are easy to calculate.

4. **Proof of Proposition 1**

For reader's convenience, we give a proof of Proposition 1, which is based on arguments of [12], chapter VI. The implications (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i) are clear. We prove (i) \Rightarrow (iv). For $B \in GL_{n+1}(k)$, consider the map f_B defined by

$$f_B: GL_{n+1}(k) \to GL_{n+1}(k)$$

 $T \mapsto {}^t TBT^{(q)}.$

Because the differential of the Frobenius map $F: T \mapsto T^{(q)}$ is identically zero, we can deduce that

$$d(f_B) = d({}^tT)BT^{(q)}.$$

Therefore, the tangent map of f_B is surjective for any $B \in GL_{n+1}(k)$. Hence, f_B is generically surjective, and the image of f_B contains a non-empty open subset U_B . Let A be any matrix of $M_{n+1}(k)$ such that the hypersurface X_A is nonsingular, i.e. $A \in GL_{n+1}(k)$. Because $GL_{n+1}(k)$ is irreducible, we have $U_A \cap U_I \neq \emptyset$, where I is the identity matrix of size n+1. There exist $T_1, T_2 \in GL_m(k)$ such that $f_A(T_1) = f_I(T_2)$. Putting $T = T_1T_2^{-1}$, we have ${}^{t}TAT^{(q)} = I.$ \square

5. The case of plane curves

Next we will study the plane curves X_A associated with matrices A of rank ≤ 2 in the projective plane \mathbf{P}^2 .

THEOREM 3. Let $A = (a_{ij})_{0 \le i,j \le 2} \in M_3(k)$ be a nonzero matrix and let X_A be the curve defined by $\sum a_{ij}x_ix_j^q = 0$ in \mathbf{P}^2 . Suppose that the rank of A is smaller than 3.

(i) When the rank of A is 1, the curve X_A is projectively isomorphic to one of the following curves

$$Z_0: x_0^{q+1} = 0,$$
 or $Z_1: x_0^q x_1 = 0.$

(ii) When the rank of A is 2, the curve X_A is projectively isomorphic to one of the following curves

$$X_0: x_0^q x_1 + x_1^q x_2 = 0, \quad X_1: x_0^{q+1} + x_1^q x_2 = 0, \quad or \quad X_2: x_0^{q+1} + x_1^{q+1} = 0.$$

PROOF. In the case the rank of A is 2. By Theorem 1, the plane curve X_A is projectively isomorphic to one of the plane curves X_0 , or X_1 , or X_2 .

In the case rank of A is 1. With the same argument of the proof of Theorem 1, we can assume that the matrix A is as following form

$$A = \begin{pmatrix} a_{00} & a_{01} & 0\\ a_{10} & a_{11} & 0\\ a_{20} & a_{21} & 0 \end{pmatrix}.$$

By interchanging with x_0 and x_1 if nessesary, we can assume that $(a_{01}, a_{11}, a_{21}) \neq (0, 0, 0)$. Because rank of A is 1, there exists $\lambda \in k$ such that $(a_{00}, a_{10}, a_{20}) = \lambda(a_{01}, a_{11}, a_{21})$. The curve X_A is defined by the equation

$$(a_{00}x_0 + a_{10}x_1 + a_{20}x_2)(x_0^q + \lambda x_1^q) = 0.$$

It is easy to show that X_A is projectively isomorphic to the curve Z_0 or Z_1 .

REMARK 5. In fact, the case when the plane curve X_A of degree p + 1 has been proved by Homma in [9].

Note that the plane curve X_1 has a special property such that the tangent line of X_1 at every smooth point passes through the point (0, 1, 0). Therefore, the plane curve X_1 is strange. Moreover, this curve is irreducible and nonreflexive. In [1], Ballico and Hefez proved that a reduced irreducible nonreflexive plane curve of degree q + 1 is isomorphic to one of the following curves:

- (1) $X_I: x_0^{q+1} + x_1^{q+1} + x_2^{q+1} = 0,$
- (2) a nodal curve whose defining equation is given in [4] and [7], or
- (3) strange curves.

Let \mathscr{L} be the space of all reduced irreducible projective plane curves of degree q+1, which is open in the space $\mathscr{P} \cong \mathbf{P}^{\binom{q+3}{2}}$ of all projective plane curves of degree q+1. Let \mathscr{L}_* be the locus of \mathscr{P} consisting of curves isomorphic to X_I , and let \mathscr{L}_1 be the locus of \mathscr{P} consisting of strange curves. Let (ξ_J) be the homogeneous coordinates of \mathscr{P} where $J = (j_0, j_1, j_2)$ ranges over the set of all ordered triples on non-negative integer such that $j_0 + j_1 + j_2 = q + 1$. The point (ξ_J) corresponds to the curve $\sum \xi_J x^J = 0$ where $x^J = x_0^{j_0} x_1^{j_1} x_2^{j_2}$. Then the locus of all curves defined by the equation of the form $\sum a_{ij} x_i x_j^q = 0$ is the linear subspace of \mathscr{P} defined by $\xi_J = 0$, unless $J \in \{(q+1,0,0), (0,q+1,0), (0,0,q+1,0), (0,0,q+1,0), (0,0,q,1), (0,1,q)\}$. By Theorem 3, we have that because Z_0 , Z_1 , X_0 , X_2 are reducible, the closure $\overline{\mathscr{L}_*}$ of \mathscr{L}_* in \mathscr{L} consists of curves isomorphic to X_I or to X_1 , and the intersection of $\overline{\mathscr{L}_*}$ and \mathscr{L}_1 consist of curves isomorphic to X_1 .

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