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Master Symmetries of the XY Model

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Dedicated to Res Jost and Arthur Wightman

Abstract. Master symmetries, found by Barouch and Fuchssteiner for a finite size XY model with the help of a computer program, are mathematically analyzed for an infinitely extended XY model by a rigorous operator algebraic method with an easy computation. The infinite family of commuting Hamiltonians and the master symmetries generating them form an infinite dimensional Lie group of automorphisms of a C^*-algebra of observables for the model.

1. Introduction

The one-dimensional XY-model in statistical mechanics of the spin 1/2 lattice system is known to be exactly solvable. One possible feature of an exactly solvable quantum model is the existence of a commuting family of explicitly describable operators (constants of motion or symmetry generators) which commute with the Hamiltonian of the model. Barouch and Fuchssteiner [7] found an interesting mechanism of creating such a commuting family, which will be quoted in detail in the next section.

Barouch and Fuchssteiner refers the proof to a computer computation. They give only the first few operators in the commuting family explicitly and no general explicit forms for the operators in an infinite family are given. The complicated expressions for the first few operators do not seem to suggest any general explicit form either. Thus we are not sure about the proof of the claim for the general operators in the infinite family, e.g. the proof of their commutativity.

Also Barouch and Fuchssteiner do not specify the boundary condition for the model. The expression for the master symmetry containing the number j of the lattice site explicitly excludes the possibility of the periodic boundary condition. However the validity of the commutativity can be broken at the boundary for other boundary conditions.

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The aim of this paper is to present a more proper mathematical formulation, to provide a rigorous proof and to give manageable description of a general term (not just the first few operators in an infinite family) for master symmetries and commuting families derived from them.

We show that master symmetries and the commuting families derived from them (after multiplied by i) form an infinite dimensional Lie algebra of derivations defined on a common dense subalgebra $\mathfrak{A}_0$ in the C^*-algebra $\mathfrak{A}$, consisting of all local observables, that their closures are generators of one-parameter groups of automorphisms of $\mathfrak{A}$, forming an infinite-dimensional Lie group and that the subgroup generated by commuting families is abelian. Furthermore, it is found that each member of commuting families derived from all master symmetries under consideration is a finite linear combination of members of the commuting family generated by the basic master symmetry.

It is very likely that the abelian subgroup of automorphisms mentioned above (or its closure) is maximal abelian in the group of all automorphisms of $\mathfrak{A}$.

In Sect. 2, we formulate the main results as Theorems. Theorems 1 and 2 are essentially the claim of Barouch and Fuchssteiner [7], while Theorem 3 is new. While Theorems 1–3 are Lie algebra statements about derivations of the C^*-algebra of observables, Theorem 4 is a result about the exponentiation of the infinite Lie algebra into a group of automorphisms.

In Sect. 3, we review the C^*-algebraic method of solution for the one-dimensional XY-model. In particular, the CAR algebra is introduced via the Jordan Wigner transformation from the spin algebra.

In Sect. 4, master symmetries and commuting families derived from them are explicitly computed as derivations on the CAR algebra.

In Sect. 5, the problem of going back from the CAR algebra to the algebra of observables is solved for derivations and the uniqueness is shown.

In Sect. 6, the exponentiation of derivations into automorphisms is carried out by the technique of Bogoliubov automorphisms of the CAR algebra and then by changing over to automorphisms of the spin algebra.

In Sect. 7, we collect results of the preceding sections into a proof of Theorems in Sect. 2. We also remark in Sect. 6 about the maximal abelian nature of the commuting family of automorphisms on the CAR algebra due to a theorem of Kishimoto [9] and conjecture the same for the spin algebra.

2. Statement of Results

We denote the lattice sites in one dimension by integers $\mathbf{Z}$, the Pauli spin operators at each lattice site $j \in \mathbf{Z}$ by $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, and $\sigma_z^{(j)}$, the C^*-algebra of observables generated by them at all lattice sites $j \in \mathbf{Z}$ by $\mathfrak{A}$, and the subalgebra consisting of all finite polynomials of σ's by $\mathfrak{A}_0$.

The standard expression for the Hamiltonian of the XY-model is given by

$$H = -J \sum_{j \in \mathbf{Z}} H(j), \tag{2.1}$$

$$H(j) = (1+\gamma)\sigma_x^{(j)}\sigma_x^{(j+1)} + (1-\gamma)\sigma_y^{(j)}\sigma_y^{(j+1)} + 2\lambda\sigma_z^{(j)}. \tag{2.2}$$

(In notation of [7], $-J(1+\gamma)=J_x$, $-J(1-\gamma)=J_y$, $-2J\lambda=h$.) We interpret H as the derivation

$$\delta_H A=[H,A]=-J\sum_{j\in\mathbf{Z}}[H(j),A] \tag{2.3}$$

for $A\in\mathfrak{A}_0$. Each term of the sum (2.1) defines an inner derivation, which vanishes for sufficiently large j for any given $A\in\mathfrak{A}_0$ due to the commutativity of σ's at different lattice sites, i.e. $[H(j),A]$ vanishes if j and $j+1$ are outside of supporting sites of A. (The supporting sites of an element A may be defined as the set of all j which appears in an explicit expression for A in terms of σ's.) Therefore (2.3) is actually a finite sum and belongs to $\mathfrak{A}_0$ for any $A\in\mathfrak{A}_0$.

In this note, we explicitly exclude the case $(\lambda,\gamma)=(0,\pm 1)$ (the case of one-dimensional Ising model).

We consider another derivation of a similar nature given by

$$S=\sum_{j\in\mathbf{Z}}S(j), \tag{2.4}$$

$$S(j)=\sigma_x^{(j)}\sigma_y^{(j+1)}-\sigma_y^{(j)}\sigma_x^{(j+1)}. \tag{2.5}$$

We use the following "master symmetry" in the sense of a derivation on $\mathfrak{A}_0$,

$$P=-\sum_{j\in\mathbf{Z}}P(j), \qquad P(j)=(j+(1/2))H(j)-\lambda\sigma_z^{(j)}. \tag{2.6}$$

This is related to the first master symmetry M_0 of Barouch and Fuchssteiner ((3.2) and (3.7) in [7], where (3.7) should read $\lambda=-(1/2)h$) via

$$M_0=JP-(1/2)H, \tag{2.7}$$

which produces the same family of commuting "Hamiltonians" as P except for the constant coefficient J. We note that $P(j)$ of (2.6) can be obtained from $H(j)$ of (2.2) by multiplying the terms $\sigma_x^{(j)}\sigma_x^{(j+1)}$ and $\sigma_y^{(j)}\sigma_y^{(j+1)}$ by $(j+(1/2))$ and the term $\sigma_z^{(j)}$ by j, the purpose of the $\sigma_z^{(j)}$ term being just to cancel (1/2) in the coefficient of $H(j)$ for the $\sigma_z^{(j)}$ term.

The derivation P is called a master symmetry because of the following property.

Theorem 1. *Starting with $H_0=H$ and $S_0=S$, the derivations H_n and S_n on $\mathfrak{A}_0$ are defined recursively by*

$$H_{(n+1)}=i[P,H_n], \tag{2.8}$$

$$S_{(n+1)}=i[P,S_n], \tag{2.9}$$

$n=0,1,2,\ldots$. *Then for any $j,k\in\mathbf{N}\cup\{0\}$,*

$$[H_j,H_k]=[H_j,S_k]=[S_j,S_k]=0. \tag{2.10}$$

More explicit description of H_n and S_n as well as the proof will be given later. We note that the commutator of derivations are defined, for example, by

$$\delta_{H_{n+1}}(A)=i(\delta_P\{\delta_{H_n}(A)\}-\delta_{H_n}\{\delta_P(A)\}). \tag{2.11}$$

By (2.1) and (2.6), we have for $A \in \mathfrak{A}_0$

$$\delta_{H_1}(a) = iJ \sum_{j,k} \{[P(j), [H(k), A]] - [H(k), [P(j), A]]\}$$
$$= iJ \sum_{j,k} [[P(j), H(k)], A], \tag{2.12}$$

where the last equality is by Jacobi identity. This will be denoted by

$$H_1 = iJ \sum_{j,k} [P(j), H(k)]. \tag{2.13}$$

Similar expressions can be written down for other H_n and S_n.

We define $\mathfrak{g}_0$ to be the set of all finite linear combination of H_n, $n=0,1,2,\ldots$ and S_n, $n=0,1,2,\ldots$ with real coefficients. Addition of any element of $\mathfrak{g}_0$ to the master symmetry P does not change H_n and S_n defined above due to the abelian nature of $\mathfrak{g}_0$. In particular M_0 of Barouch and Fuchssteiner and our P produce the same H_n and S_n except for a constant coefficient J^n.

In the same spirit, we define another master symmetry P_1:

$$P_1 = -\sum_{j \in \mathbf{Z}} P_1(j), \tag{2.14a}$$

$$P_1(j) = (j+1)\{(1+\gamma)\sigma_x^{(j)}\sigma_z^{(j+1)}\sigma_x^{(j+2)} + (1-\gamma)\sigma_y^{(j)}\sigma_z^{(j+1)}\sigma_y^{(j+2)}\}$$
$$-2j\sigma_z^{(j)} - (2j+1)\lambda(\sigma_x^{(j)}\sigma_x^{(j+1)} + \sigma_y^{(j)}\sigma_y^{(j+1)}). \tag{2.14b}$$

This differs from the second master symmetry M_1 of Barouch and Fuchssteiner by a constant coefficient and an addition of elements of $\mathfrak{g}_0$:

$$M_1 = 4J^2 P_1 + 2J^2 S_1 - 4J\lambda H. \tag{2.15}$$

We now define an infinite family of master symmetries recursively by

$$P_{j+1} = i[P, P_j], \quad j=1,2,\ldots. \tag{2.16}$$

As before, the P_j $(j=l,2,\ldots)$ are understood as derivations on $\mathfrak{A}_0$.

Each of P_j produces a commuting family of derivations on $\mathfrak{A}_0$, which commute mutually for different j:

$$H_{k+1,j} = i[P_j, H_{k,j}], \quad H_{0,j} = H, \tag{2.17}$$

$$S_{k+1,j} = i[P_j, S_{k,j}], \quad S_{0,j} = S. \tag{2.18}$$

Here $k=0,1,2,\ldots$.

We extend the notation of (2.17) and (2.18) for $j=0$ with $P_0=P$, so that $H_{k,0}=H_k$, $S_{k,0}=S_k$. We have

Theorem 2. *For all* $k,j,l,m=0,1,2,\ldots$,

$$[H_{k,j}, H_{l,m}] = [H_{k,j}, S_{l,m}] = [S_{k,j}, T_{l,m}] = 0. \tag{2.19}$$

We note that M_j in [7] and $4J^{j+1}P_j$, $j=1,2,\ldots$ differ by an element of $\mathfrak{g}_0$ for each j and hence produce the same family of H's and S's as H's and T's of Barouch and Fuchssteiner up to constant coefficients. Thus Theorem 2 is the claim by Barouch and Fuchssteiner, for which we give a straightforward proof. Actually we have the following stronger result, which proves Theorem 2 in view of Theorem 1.

Theorem 3. *For all* $k,j=0,1,2,\ldots,$

$$H_{k,j}\in\mathfrak{g}_0\,,\qquad S_{k,j}\in\mathfrak{g}_0\,. \tag{2.20}$$

We now describe our result about integrating (or exponentiating) these derivations. Let $\mathfrak{g}_1$ be the set of all finite real linear combinations of P_k, $k=0,1,2,\ldots$ and elements of $\mathfrak{g}_0$.

Theorem 4. *For any* $X\in\mathfrak{g}_1$, *the closure of* iX *is the generator of a one-parameter group of automorphisms of* $\mathfrak{A}$, *which will be denoted as* e^{iX}. *For* $X, Y\in\mathfrak{g}_0$, e^{iX} *and* e^{iY} *commute.*

Actually we can describe the group generated by e^{iX}, $X\in\mathfrak{g}_1$ in more detail. This will be a subject of the forthcoming paper.

3. The Method of Solution

1. Introduction of CAR-*Algebra*

We now quote the operator algebraic method of solution for the XY-model [3, 4, 5].

We introduce two involutive automorphisms of $\mathfrak{A}$ defined as follows:

$$\Theta(A)=\lim_{N\to\infty}\left(\prod_{j=-N}^{N}\sigma_z^{(j)}\right)A\left(\prod_{j=-N}^{N}\sigma_z^{(j)}\right), \tag{3.1}$$

$$\Theta_-(A)=\lim_{N\to\infty}\left(\prod_{j=-N}^{0}\sigma_z^{(j)}\right)A\left(\prod_{j=-N}^{0}\sigma_z^{(j)}\right). \tag{3.2}$$

More explicitly, they can be defined by the following action on the generators of $\mathfrak{A}$:

$$\Theta(\sigma_x^{(j)})=-\sigma_x^{(j)}\,,\qquad \Theta(\sigma_y^{(j)})=-\sigma_y^{(j)}\,,\qquad \Theta(\sigma_z^{(j)})=\sigma_z^{(j)}\,. \tag{3.3}$$

$$(j=0,\pm1,\pm2,\ldots).$$

$$\Theta_-(\sigma_\alpha^{(j)})=\begin{cases}\sigma_\alpha^{(j)} & \text{if } j\geqq1\,,\\ \Theta(\sigma_\alpha^{(j)}) & \text{if } j\leqq0\,.\end{cases} \tag{3.4}$$

$$(\alpha=x,y,z)\,.$$

Let $\hat{\mathfrak{A}}$ be the crossed product of $\mathfrak{A}$ by the action Θ_-^n of $n\in\mathbf{Z}_2$. More explicitly, $\hat{\mathfrak{A}}$ is the direct sum

$$\hat{\mathfrak{A}}=\mathfrak{A}+T\mathfrak{A}\,, \tag{3.5}$$

where the element T of $\hat{\mathfrak{A}}$ satisfies

$$T^2=1\,,\qquad T^*=T,\qquad TAT=\Theta_-(A) \tag{3.6}$$

for all $A\in\mathfrak{A}$. The automorphisms Θ and Θ_- of $\mathfrak{A}$ can be extended to automorphisms of $\hat{\mathfrak{A}}$ (again written by the same letters Θ and Θ_-) satisfying $\Theta(T)=\Theta_-(T)=T$.

The algebras $\mathfrak{A}$, $\mathfrak{A}_0$, and $\hat{\mathfrak{A}}$ are decomposed into a sum of Θ-even and Θ-odd parts via the following elementwise decomposition:

$$A=A_++A_-\,,\qquad A_\pm=2^{-1}(A\pm\Theta(A))\,, \tag{3.7}$$

where $\Theta(A_\pm)=\pm A_\pm$ by definition. Accordingly,

$$\mathfrak{A}=\mathfrak{A}_+ + \mathfrak{A}_-, \qquad \mathfrak{A}_0=\mathfrak{A}_{0+}+\mathfrak{A}_{0-}, \qquad \hat{\mathfrak{A}}=\hat{\mathfrak{A}}_+ + \hat{\mathfrak{A}}_-. \tag{3.8}$$

We define the following C^*-subalgebra of $\hat{\mathfrak{A}}$.

$$\mathfrak{A}^{\mathrm{CAR}}=\mathfrak{A}_+ + T\mathfrak{A}_-. \tag{3.9}$$

It is generated by the so-called Jordan-Wigner transform of σ's,

$$c_j = TS_j(\sigma_x^{(j)} - i\sigma_y^{(j)})/2, \tag{3.10a}$$

$$c_j^* = TS_j(\sigma_x^{(j)} + i\sigma_y^{(j)})/2, \tag{3.10b}$$

where

$$S_j=\begin{cases}\sigma_z^{(1)}\dots\sigma_z^{(j-1)} & \text{if } j\geqq 2, \quad (3.11a)\\ 1 & \text{if } j=1, \quad (3.11b)\\ \sigma_z^{(0)}\dots\sigma_z^{(j)} & \text{if } j\leqq 0. \quad (3.11c)\end{cases}$$

These operators satisfy the canonical anticommutation relations (CAR):

$$[c_j, c_k]_+ = [c_j^*, c_k^*]_+ = 0, \qquad [c_j, c_k^*]_+ = \delta_{jk}, \tag{3.12}$$

where $[A,B]_+ = AB+BA$ and δ_{jk} is 0 for $j\neq k$ and 1 for $j=k$.

The automorphism Θ_- of $\hat{\mathfrak{A}}$ leaves $\mathfrak{A}^{\mathrm{CAR}}$ invariant (as a set) and $\hat{\mathfrak{A}}$ can also be viewed as the crossed product of $\mathfrak{A}^{\mathrm{CAR}}$ by the Θ_- action of $\mathbf{Z}_2$. Generators of $\mathfrak{A}$ can be expressed in terms of creation and annihilation operators c_j^* and c_j and T as follows:

$$\sigma_j^{(z)} = 2c_j^*c_j - 1, \qquad \sigma_x^{(j)} = TS_j(c_j + c_j^*), \qquad \sigma_y^{(j)} = TS_j i(c_j - c_j^*). \tag{3.13}$$

Again, $\mathfrak{A}^{\mathrm{CAR}}$ can be split into Θ-even and Θ-odd parts:

$$\mathfrak{A}^{\mathrm{CAR}} = \mathfrak{A}_+^{\mathrm{CAR}} + \mathfrak{A}_-^{\mathrm{CAR}}, \tag{3.14a}$$

$$\mathfrak{A}_+^{\mathrm{CAR}} = \mathfrak{A}_+, \qquad \mathfrak{A}_-^{\mathrm{CAR}} = T\mathfrak{A}_-. \tag{3.14b}$$

2. Selfdual Formulation of CAR *and Bogoliubov Automorphisms* [1, 6]

We introduce the complex Hilbert space

$$\mathscr{H} = l_2(\mathbf{Z}) \oplus l_2(\mathbf{Z}). \tag{3.15}$$

An element h of $\mathscr{H}$ consists of two elements f and g of $l_2(\mathbf{Z})$, denoted by

$$h=\begin{pmatrix} f\\ g\end{pmatrix} \tag{3.16}$$

instead of $h=f\oplus g$. Each $f\in l_2(\mathbf{Z})$ can be described by its components f_j, $j\in\mathbf{Z}$. We denote

$$B(h) = \sum f_j c_j^* + \sum g_j c_j, \tag{3.17}$$

which converges in the C^*-norm of $\mathfrak{A}^{\mathrm{CAR}}$. We introduce the antilinear involution Γ on $\mathscr{H}$ by

$$\Gamma\begin{pmatrix} f\\ g\end{pmatrix} = \begin{pmatrix} \bar{g}\\ \bar{f}\end{pmatrix}, \tag{3.18}$$

where the bar in $\bar{f}$ and $\bar{g}$ denotes the componentwise complex conjugation. Then $\{B(h)\}$ generates $\mathfrak{A}^{\mathrm{CAR}}$ and satisfies the following selfdual form of CAR:

$$[B(h_1)^*, B(h_2)]_+ = (h_1, h_2)\mathbf{1}, \tag{3.19a}$$

$$B(h)^* = B(\Gamma h). \tag{3.19b}$$

Here we are using the notation of physicists for the inner product:

$$(h_1, h_2) = (f_1, f_2) + (g_1, g_2) = \sum_j (\bar{f}_{1j} f_{2j} + \bar{g}_{1j} g_{2j}). \tag{3.20}$$

For any unitary operator U commuting with Γ, there exists the unique automorphism of $\mathfrak{A}^{\mathrm{CAR}}$, denoted by α_U and called a Bogoliubov automorphism, satisfying

$$\alpha_U(B(h)) = B(Uh). \tag{3.21}$$

Examples are $\Theta = \alpha_{-1}$ for $U = -1$ and $\Theta_- = \alpha_{\theta_-}$ where

$$(\theta_- f)_j = \begin{cases} f_j & j \geqq 1, \\ -f_j & j \leqq 0. \end{cases} \tag{3.22}$$

3. XY-*Model Hamiltonian*

The operator $H(j)$ given by (2.2) belongs to $\mathfrak{A}_+ = \mathfrak{A}_+^{\mathrm{CAR}}$ and can be written as a quadratic expression of c_j, c_{j+1}, c_j^* and c_{j+1}^* up to a constant term which does not count as a derivation:

$$H(j) = 2(c_j c_{j+1}^* + c_{j+1} c_j^*) + 2\gamma(c_j c_{j+1} + c_{j+1}^* c_j^*) + 4\lambda c_j^* c_j - 2\lambda. \tag{3.23}$$

The derivation (2.3) can now be explicitly written:

$$\delta_H(B(h)) = B(2J\mathbf{K}h), \tag{3.24a}$$

$$\mathbf{K} = \begin{pmatrix} U + U^* - 2\lambda, & \gamma(U - U^*) \\ -\gamma(U - U^*), & -(U + U^* - 2\lambda) \end{pmatrix}. \tag{3.24b}$$

Here the 2×2 matrix notation for the operator $\mathbf{K}$ on $\mathscr{H}$ corresponds to the 2 components notation (3.6) for elements h of K, U and U^* are shift operators on $l_2(\mathbf{Z})$ to the left and to the right:

$$(Uf)_j = f_{j+1}, \quad (U^* f)_j = f_{j-1}. \tag{3.25}$$

The operator $\mathbf{K}$ is bounded, acting on $\mathscr{H}$ and satisfies

$$\mathbf{K}^* = \mathbf{K}, \quad \Gamma\mathbf{K} = -\mathbf{K}\Gamma. \tag{3.26}$$

The derivation δ_H exponentiate to a one-parameter group of automorphisms $(\alpha_H)_t = e^{i\delta_H}$, $t \in \mathbf{R}$, of $\mathfrak{A}^{\mathrm{CAR}}$, explicitly given as Bogoliubov automorphisms:

$$(\alpha_H)_t = \alpha_{U(t)}, \quad U(t) = \exp i(2J\mathbf{K}t). \tag{3.27}$$

In subsequent computations, it is advantageous to introduce the following Fourier transform. For each $f \in l_2(Z)$ and $h \in \mathscr{H}$, we define functions of $\theta \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{N}$ as follows:

$$f(\theta) = \sum_{n \in \mathbf{Z}} e^{in\theta} f_n, \quad h(\theta) = \sum_{n \in \mathbf{Z}} e^{in\theta} h_n. \tag{3.28}$$

Here $f(\theta)$ is a complex number and $h(\theta)$ is a 2 dimensional vector for each θ. The inverse relations are

$$f_n = (2\pi)^{-1} \int_0^{2\pi} e^{-in\theta} f(\theta) d\theta, \tag{3.29a}$$

$$h_n = (2\pi)^{-1} \int_0^{2\pi} e^{-in\theta} h(\theta) d\theta. \tag{3.29b}$$

The operator $\mathbf{K}$ is diagonalized by this transformation.

$$(\mathbf{K}h)(\theta) = \hat{K}(\theta) h(\theta), \tag{3.30a}$$

$$\hat{K}(\theta) = 2 \begin{pmatrix} \cos\theta - \lambda & -i\gamma \sin\theta \\ i\gamma \sin\theta & \lambda - \cos\theta \end{pmatrix}. \tag{3.30b}$$

We note that

$$\hat{K}(\theta)^2 = k^2(\theta)\mathbf{1}, \quad k(\theta) = 2[(\lambda - \cos\theta)^2 + \gamma^2 \sin^2\theta]^{1/2}. \tag{3.31}$$

We also note that

$$(\Gamma h)(\theta) = \begin{pmatrix} \overline{g(-\theta)} \\ \overline{f(-\theta)} \end{pmatrix} \quad \text{for} \quad h(\theta) = \begin{pmatrix} f(\theta) \\ g(\theta) \end{pmatrix}. \tag{3.32}$$

4. Master Symmetries – Lie Algebra Computation

All the derivations introduced in Sect. 2 are infinite sums of inner derivations by elements of $\mathfrak{A}_+ = \mathfrak{A}_+^{\mathrm{CAR}}$, and hence we compute them first for $\mathfrak{A}^{\mathrm{CAR}}$, using notation of the preceding section.

First, $S(j)$ of (2.5) can be written as

$$S(j) = 2i(c_{j+1}^* c_j - c_j^* c_{j+1}). \tag{4.1}$$

Therefore the derivation $\delta_S(A) = \Sigma[S(j), A]$ for $A = B(h)$ is given by

$$\delta_S(B(h)) = B(\mathbf{S}h), \tag{4.2}$$

$$\mathbf{S} = 2i \begin{pmatrix} U^* - U & 0 \\ 0 & U^* - U \end{pmatrix}, \quad \hat{S}(\theta) = -4 \sin\theta \mathbf{1}. \tag{4.3}$$

Next, $P(j)$ of (2.6) can be written as

$$P(j) = (j + (1/2)) H(j) - \lambda(2c_j^* c_j - 1), \tag{4.4}$$

where $H(j)$ is given by (3.23). Therefore $\delta_P(A) = -\Sigma[P(j), A]$ for $A = B(h)$ is given by

$$\delta_P(B(h)) = B(\mathbf{P}h), \tag{4.5}$$

$$\mathbf{P} = 2 \begin{pmatrix} U_1 + U_1^* - 2\lambda\delta, & \gamma(U_1 - U_1^*) \\ \gamma(U_1^* - U_1), & -(U_1 + U_1^* - 2\lambda\delta) \end{pmatrix}, \tag{4.6}$$

where U_1 and δ are defined by

$$(\delta h)_j = j h_j, \quad (\delta h)(\theta) = (-\mathrm{i}d/d\theta) h(\theta) = -ih'(\theta), \tag{4.7}$$

$$U_1 = (\delta + (1/2)) U, \quad U_1^* = U^*(\delta + (1/2)) = (\delta - (1/2)) U^*. \tag{4.8}$$

Note that $\delta U = U(\delta - 1)$, $(\delta - 1)U^* = U^*\delta$. The last term in (4.4) serves the purpose of canceling out the (1/2) term for -2λ in $H(j)$. The above expression for $\mathbf{P}$ can be written in a compact way

$$\mathbf{P} = \delta\mathbf{K} + \mathbf{K}\delta, \tag{4.9}$$

because $2U_1 = \delta U + U\delta$ and $2U_1^* = \delta U^* + U^*\delta$, where $\mathbf{K}$ is given by (3.24b).

A basic property of $\mathbf{P}$ is the following commutation relation:

$$i[\mathbf{P}, \mathbf{K}] = i[\delta, \mathbf{K}^2] = 2kk'\,\mathbf{1} \tag{4.10}$$

due to (3.31), where k' is the multiplication of the derivative $k'(\theta)$:

$$(k'h)(\theta) = k'(\theta)h(\theta). \tag{4.11}$$

Computing in θ-representation of the test function space $\mathscr{H}$, we obtain

$$i[\mathbf{P}, a\mathbf{1}] = 2a'\mathbf{K}, \tag{4.12}$$

$$\begin{aligned} i[\mathbf{P}, b\mathbf{K}] &= i[P, b]K + bi[P, K] \\ &= (2b'k^2 + 2bkk')\mathbf{1} = 2k(d/d\theta)(kb)\mathbf{1} \end{aligned} \tag{4.13}$$

for any multiplication operators a and b in θ-representations. By repeating these computations, we obtain the following description of the derivations H_j on $\mathfrak{A}^{\mathrm{CAR}}$.

Proposition 4.1.

$$\delta_{H_j}(B(h)) = B(\mathbf{H}_j h), \tag{4.14}$$

where $\mathbf{H}_j$ is the multiplication of the following expressions in θ-representation of $\mathscr{H}$:

$$\hat{H}_j(\theta) = \begin{cases} h_j(\theta)\mathbf{1} & \text{for odd } j, \\ h_j(\theta)K(\theta) & \text{for even } j. \end{cases} \tag{4.15}$$

The functions h_j satisfy

$$h_{2n} = 2Dh_{2n-1}, \qquad h_{2n+1} = 2kDkh_{2n}, \tag{4.16}$$

where $D = (d/d\theta)$. With the initial condition $h_0 = 2J$, they are given by

$$h_{2n} = 2^{2n+1}J(D \cdot k)^{2n}\mathbf{1}, \tag{4.17a}$$

$$h_{2n+1} = 2^{2n+2}Jk \cdot (D \cdot k)^{2n+1}\mathbf{1}, \tag{4.17b}$$

where $n = 0, 1, 2, \ldots$.

Here the product $D \cdot k$ is the product of the differentiation operator $D = (d/d\theta)$ and the multiplication operator k, and it does not mean the differentiation of the function $k(\theta)$ alone. The initial condition is by (3.24a).

Similarly, we obtain the following.

Proposition 4.2.

$$\delta_{S_j}(B(h)) = B(\mathbf{S}_j h), \tag{4.18}$$

where $\mathbf{S}_j$ is the multiplication of the following expression in θ-representation of $\mathscr{H}$:

$$\hat{S}_j(\theta) = \begin{cases} s_j(\theta)\mathbf{1} & \text{for even } j, \\ s_j(\theta)\hat{K}(\theta) & \text{for odd } j. \end{cases} \tag{4.19}$$

The functions s_j *satisfy*

$$s_{2n}=2kDks_{2n-1}\,,\qquad s_{2n+1}=2Ds_{2n}\,. \tag{4.20}$$

With the initial condition $s_0(\theta)=-4\sin\theta$, *they are given by*

$$s_{2n}=-2^{2n+2}(kD)^{2n}\sin\theta\,, \tag{4.21a}$$

$$s_{2n+1}=-2^{2n+3}D(kD)^{2n}\sin\theta\,, \tag{4.21b}$$

where $n=0,1,2,\ldots$.

The initial condition is given by (4.3).

We now compute the second master symmetry P_1. We have

$$\begin{aligned} P_1(j)=&-2(j+1)\{c^*_{j+2}c_j+c^*_jc_{j+2}+\gamma(c^*_jc^*_{j+2}+c_{j+2}c_j)\}\\ &-4jc^*_jc_j+2j+2\lambda(2j+1)(c^*_{j+1}c_j+c^*_jc_{j+1})\,. \end{aligned} \tag{4.22}$$

Therefore

$$\delta_{P_1}(B(h))=B(\mathbf{P}_1h)\,, \tag{4.23}$$

$$\begin{aligned}\mathbf{P}_1&=2\begin{pmatrix} U^{*2}(\delta+1)+U^2(\delta-1)+2\delta-\lambda(U^*(2\delta+1)+U(2\delta-1)), & \gamma(U^2(\delta-1)-U^{*2}(\delta+1)) \\ \gamma(U^{*2}(\delta+1)-U^2(\delta-1)), & -U^{*2}(\delta+1)-U^2(\delta-1)-2\delta+\lambda(U^*(2\delta+1)+U(2\delta-1)) \end{pmatrix}\\ &=(1/2)\{(U+U^*)\mathbf{P}+\mathbf{P}(U+U^*)\}\,. \end{aligned} \tag{4.24}$$

We can now compute all members of the infinite family of master symmetries P_j:

Proposition 4.3.

$$\delta_{P_j}(B(h))=B(\mathbf{P}_jh)\,, \tag{4.25}$$

$$\mathbf{P}_j=\mathbf{L}_j\mathbf{P}+\mathbf{P}\mathbf{L}_j\,, \tag{4.26}$$

where the operators $\mathbf{L}_j$ *satisfy the recursive relation*

$$\mathbf{L}_{j+1}=i[\mathbf{P},\mathbf{L}_j]\,,\qquad j=1,2,\ldots. \tag{4.27}$$

They are the multiplication of the following expressions in θ*-representation:*

$$\hat{L}_j(\theta)=\begin{cases} p_j(\theta)\mathbf{1} & \text{for odd}\, j\,,\\ p_j(\theta)K & \text{for even}\, j\,, \end{cases} \tag{4.28}$$

where the functions p_j *satisfy the recursive relations*

$$p_{2n+1}=2(kDk)p_{2n}\,, \tag{4.29a}$$

$$p_{2n}=2p'_{2n-1}\,. \tag{4.29b}$$

With the initial condition $p_1=\cos\theta$, *they are given by*

$$p_{2n+1}=2^{2n}(kD)^{2n}\cos\theta\,, \tag{4.30a}$$

$$p_{2n}=-2^{2n-1}(Dk)^{2n-2}\sin\theta\,. \tag{4.30b}$$

The initial condition is due to (4.24), in which $(U+U^*)/2$ is the multiplication of $\cos\theta$ in θ-representation.

Proposition 4.4.

$$\delta_{H_{k,j}}(B(h)) = B(\mathbf{H}_{k,j}h), \tag{4.31}$$

where $\mathbf{H}_{k,j}$ *is the multiplication of the following expression in* θ*-representation:*

$$\hat{H}_{k,j}(\theta) = \begin{cases} h_{k,j}(\theta)\mathbf{1} & \text{for odd } j \text{ and odd } k, \\ h_{k,j}(\theta)K & \text{otherwise}, \end{cases} \tag{4.32}$$

$k=1,2,\ldots,j=1,2,\ldots$. *The functions* $h_{k,j}$ *satisfy the following recursion relations:*

$$h_{2n+1,j} = 4(p_jkDk)h_{2n,j} \quad \text{for odd } j, \tag{4.33a}$$

$$h_{2n+2,j} = 4(p_jD)h_{2n+1,j} \quad \text{for odd } j, \tag{4.33b}$$

$$h_{l+1,j} = 4(p_jkDk)h_{l,j} \quad \text{for even } j, \tag{4.34}$$

where $n=0,1,2,\ldots$, $l=0,1,2,\ldots$. *With the initial condition* $h_{0,j}=2J$ *(a constant function), they are given by*

$$h_{2n+1,j} = (4p_jkD)^{2n+1}2Jk \quad \text{for odd } j, \tag{4.35a}$$

$$h_{2n,j} = 4p_jD(4p_jkD)^{2n-1}2Jk \quad \text{for odd } j, \tag{4.35b}$$

$$h_{l,j} = (4p_jkDk)^l 2J \quad \text{for even } j. \tag{4.36}$$

Proposition 4.5.

$$\delta_{S_{k,j}}(B(h)) = B(\mathbf{S}_{k,j}h), \tag{4.37}$$

where $\mathbf{S}_{k,j}$ *is the multiplication of the following expression in* θ*-representation:*

$$\hat{S}_{k,j}(\theta) = \begin{cases} s_{k,j}(\theta)K & \text{for odd } j \text{ and odd } k, \\ s_{k,j}(\theta)\mathbf{1} & \text{otherwise}, \end{cases} \tag{4.38}$$

$k=1,2,\ldots,j=1,2,\ldots$. *The functions* $s_{k,j}$ *satisfy the following recursion relations:*

$$s_{2n,j} = 4p_jkDks_{2n-1,j} \quad \text{for odd } j, \tag{4.39a}$$

$$s_{2n+1,j} = 4p_jDs_{2n,j} \quad \text{for odd } j, \tag{4.39b}$$

$$s_{l+1,j} = 4p_jk^2Ds_{l,j} \quad \text{for even } j. \tag{4.40}$$

With the initial condition $s_{0,j} = -4\sin\theta$, *they are given by*

$$s_{2n,j} = -4^{2n+1}(p_jkD)^{2n}\sin\theta \quad \text{for odd } j, \tag{4.41a}$$

$$s_{2n+1,j} = -4^{2n+2}p_jD(p_jkD)^{2n}\sin\theta \quad \text{for odd } j, \tag{4.41b}$$

$$s_{l,j} = -4^{l+1}(p_jk^2D)^l\sin\theta \quad \text{for even } j. \tag{4.42}$$

5. Extension of Derivations to $\hat{\mathfrak{A}}$

For any finite interval I of integers $\mathbf{Z}$, we define $\hat{\mathfrak{A}}(I)$ to be the subalgebra of $\hat{\mathfrak{A}}$ generated by $\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}, j \in I$, and $c_j, c_j^*, j \in I$, $\mathfrak{A}(I) = \mathfrak{A} \cap \hat{\mathfrak{A}}(I)$ to be the subalgebra of $\mathfrak{A}$ generated by $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, and $\sigma_z^{(j)}$, $j \in I$, and $\mathfrak{A}^{\mathrm{CAR}}(I) = \mathfrak{A}^{\mathrm{CAR}} \cap \hat{\mathfrak{A}}(I)$ to be the subalgebra of $\mathfrak{A}^{\mathrm{CAR}}$ generated by c_j and $c_j^*, j \in I$. We define $\hat{\mathfrak{A}}_0$ and $\mathfrak{A}_0^{\mathrm{CAR}}$ to be the union of $\hat{\mathfrak{A}}(I)$ and $\mathfrak{A}^{\mathrm{CAR}}(I)$ for all finite intervals I of $\mathbf{Z}$, respectively. They are subalgebras of $\hat{\mathfrak{A}}$ and $\mathfrak{A}^{\mathrm{CAR}}$, and

$$\mathfrak{A}_0^{\mathrm{CAR}} = \hat{\mathfrak{A}}_0 \cap \mathfrak{A}^{\mathrm{CAR}}, \tag{5.1}$$

$$\mathfrak{A}_0 = \hat{\mathfrak{A}}_0 \cap \mathfrak{A}, \tag{5.2}$$

where the latter is defined already in Sect. 2.

Let $\mathfrak{A}_+(I)$ denote $\mathfrak{A}_+ \cap \mathfrak{A}(I)$, i.e. the set of Θ-even elements of $\mathfrak{A}(I)$. It coincides with $\mathfrak{A}_+^{\mathrm{CAR}}(I) = \mathfrak{A}_+ \cap \mathfrak{A}^{\mathrm{CAR}}(I)$ for any interval I due to relations such as

$$\sigma_\alpha^{(j)} \sigma_\beta^{(j+n)} = d_\alpha^{(j)} \left(\prod_{k=j}^{j+n-1} (2c_k^* c_k - 1) \right) d_\beta^{(j+n)}, \tag{5.3}$$

$$d_\alpha^{(j)} d_\beta^{(j+n)} = \sigma_\alpha^{(j)} \left(\prod_{k=j}^{j+n-1} \sigma_z^{(k)} \right) \sigma_\beta^{(j+n)}, \tag{5.4}$$

where $(\alpha, \beta) = (x, x)$, (x, y), (y, x), (y, y) and

$$d_x^{(l)} = c_l + c_l^*, \quad d_y^{(l)} = i(c_l - c_l^*). \tag{5.5}$$

Let $\mathscr{L}$ be the set of all derivations δ of $\hat{\mathfrak{A}}_0$ such that there exists an index set $\Xi = \{\alpha\}$, a finite interval I_α in $\mathbf{Z}$ for each α and $X_\alpha = X_\alpha^* \in \mathfrak{A}_+(I_\alpha)$ satisfying the following two conditions:

(A)
$$\delta(A) = i \sum_\alpha [X_\alpha, A], \quad A \in \hat{\mathfrak{A}}_0, \tag{5.6}$$

(B) The set Ξ_j of $\alpha \in \Xi$ such that $j \in I_\alpha$ is finite.

Proposition 5.1. *The sum in* (5.6) *is a finite sum of non-zero elements in* $\hat{\mathfrak{A}}_0$. *The set* $\mathscr{L}$ *is a real Lie algebra of symmetric derivations on* $\hat{\mathfrak{A}}_0$.

Proof. For $A \in \hat{\mathfrak{A}}(I)$, consider I_α such that $I \cap I_\alpha$ is empty. Then $\mathfrak{A}_+(I_\alpha)$ and $\mathfrak{A}(I)$ commute because $I \cap I_\alpha$ is empty and $\mathfrak{A}_+^{\mathrm{CAR}}(I_\alpha)$ and $\mathfrak{A}^{\mathrm{CAR}}(I)$ commute for the same reason. Since $\mathfrak{A}_+(I_\alpha) = \mathfrak{A}_+^{\mathrm{CAR}}(I_\alpha)$ and $\hat{\mathfrak{A}}(I)$ is generated by $\mathfrak{A}(I)$ and $\mathfrak{A}^{\mathrm{CAR}}(I)$, A commutes with X_α. Thus $[X_\alpha, A] = 0$ unless α belongs to the finite set

$$\left(\bigcup_{j \in I} \Xi_j \right), \tag{5.7}$$

and hence the sum in (5.6) is finite and belongs to $\hat{\mathfrak{A}}_0$ for $A \in \hat{\mathfrak{A}}_0$.

Due to $X_\alpha^* = X_\alpha$, δ defined by (5.6) is a symmetric derivation. If δ_1 is defined by (5.6) with Y_β, $\beta \in \Xi'$, instead of X_α, $\alpha \in \Xi$, then we have

$$(\delta + \delta_1)(A) = i \sum_\nu [Z_\nu, A], \tag{5.8}$$

where $\nu \in \Xi \cup \Xi'$ and $Z_\alpha = X_\alpha$ for $\alpha \in \Xi$, $Z_\beta = Y_\beta$ for $\beta \in \Xi'$,

$$(c\delta)(A) = i \sum_\alpha [cX_\alpha, A] \tag{5.9}$$

for any real c. Further,

$$[\delta, \delta_1](A) = i \sum_\mu [W_\mu, A], \tag{5.10}$$

where $\mu=(\alpha,\beta)\in\Xi\times\Xi'\equiv\Xi''$ is restricted to the pair such that $I_\alpha\cap I_\beta$ is non-empty and

$$W_\mu=i[X_\alpha, Y_\beta]\in\mathfrak{A}_+(I_\alpha\cup I_\beta), \tag{5.11}$$

where $I_\alpha\cup I_\beta$ is again a finite interval and hence

$$\begin{aligned}\Xi''_{(\alpha,\beta)}=&\{(\alpha,\beta);\, j\in I_\alpha,\ k\in I_\beta \text{ for some } k\in I_\alpha\}\\ &\cup\{(\alpha,\beta);\, j\in I_\beta,\ k\in I_\alpha \text{ for some } k\in I_\beta\}\end{aligned} \tag{5.12}$$

is finite. Therefore $\mathscr{L}$ is a Lie algebra of symmetric derivations on $\hat{\mathfrak{A}}_0$. Q.E.D.

Proposition 5.2. *A derivation $\delta\in\mathscr{L}$ is determined by its value on $B(h)$ for all $h\in\mathscr{H}$ with a finite number of non-zero components h_n.*

Proof. First the finite linear span of $B(h)$ with such h is the same as the finite linear span of c_j and c_j^+, $j\in\mathbf{Z}$. By the linearity and the derivation property

$$\delta(A_1\dots A_n)=\sum_j A_1\dots A_{j-1}\delta(A_j)A_{j+1}\dots A_n, \tag{5.13}$$

the value of δ on $B(h)$ determines its value on $\mathfrak{A}_0^{\mathrm{CAR}}$.

Next we have

$$\delta(T)=i\sum_\alpha [X_\alpha, T]=bT, \tag{5.14a}$$

$$b=i\sum_\alpha (X_\alpha-\Theta_-(X_\alpha))\in\mathfrak{A}_+(I^0), \tag{5.14b}$$

where the sum can be restricted to $\alpha\in\Xi_0$, for example, and $I^0=\cup\{I_\alpha;\,\alpha\in\Xi_0\}$. Note that, on $\mathfrak{A}(I_\alpha)$, $\Theta_-=\mathrm{id}$ if $I_\alpha>0$ and $\Theta_-=\Theta$ if $I_\alpha<0$, so that $\Theta_-(X_\alpha)=X_\alpha$ in either case because $X_\alpha\in\mathfrak{A}_+(I_\alpha)$. For any $A\in\mathfrak{A}_0^{\mathrm{CAR}}$, the relation $T\Theta_-(A)T=A$ implies

$$\begin{aligned}\delta(A)-\Theta_-\delta(\Theta_-(A))&=\delta(T\Theta_-(A)T)-T\delta(\Theta_-(A))T\\ &=\delta(T)\Theta_-(A)T+T\Theta_-(A)\delta(T)=bT\Theta_-(A)T+T\Theta_-(A)TTbT\\ &=bA+A\Theta_-(b).\end{aligned} \tag{5.15}$$

On the other hand $\delta(1)=\delta(1^2)=1\delta(1)+\delta(1)1=2\delta(1)$ implies $\delta(1)=0$ as is well-known, and hence

$$0=\delta(1)=\delta(T^2)=\delta(T)T+T\delta(T)=bT^2+TbT=b+\Theta_-(b). \tag{5.16}$$

Therefore (5.15) and (5.16) determine the commutator

$$[b,A]=\delta(A)-\Theta_-\delta(\Theta_-(A)) \tag{5.17}$$

for any $A\in\mathfrak{A}_0^{\mathrm{CAR}}$ in terms of the value of δ on $\mathfrak{A}_0^{\mathrm{CAR}}$.

There exists a unique tracial state τ on $\mathfrak{A}^{\mathrm{CAR}}$, induced from the unique tracial state on full matrix algebras. Since

$$\tau(UAU^*)=\tau(U^*UA)=\tau(A) \tag{5.18}$$

for any $A\in\mathfrak{A}^{\mathrm{CAR}}$ and unitary U due to the tracial property of τ, we obtain from (3.2)

$$\tau(\Theta_-(A))=\tau(A) \tag{5.19}$$

for any $A \in \mathfrak{A}^{\mathrm{CAR}}$. Then the formula (5.14b) implies

$$\tau(b) = 0. \tag{5.20}$$

The algebra $\mathfrak{A}^{\mathrm{CAR}}(I^0)$ is isomorphic to the full $2^N \times 2^N$ matrix algebra where N is the cardinal number of the finite set I^0. Let u_{ij}, $i, j = 1, 2, \ldots, 2^N$ be its matrix unit. Any $A \in \mathfrak{A}^{\mathrm{CAR}}(I^0)$ can be written as

$$b = 2^{-N}\left(\sum_{k,j} u_{kj}[u_{jk}, b] + \tau(b)1\right). \tag{5.21}$$

Therefore (5.17) and (5.20) uniquely determine $b \in \mathfrak{A}_+(I^0) \subset \mathfrak{A}_0^{\mathrm{CAR}}$ in terms of the values of δ on $\mathfrak{A}_0^{\mathrm{CAR}}$. Q.E.D.

Remark. In the above proof, we used an explicit form (5.14b) to show that $\tau(b) = 0$. This can be avoided by using the following argument.

We consider the following linear functional on $\hat{\mathfrak{A}}$:

$$\hat{\tau}(A_1 + A_2 T) = \tau(A_1) \qquad (A_1, A_2 \in \mathfrak{A}^{\mathrm{CAR}}). \tag{5.22}$$

Then

$$\hat{\tau}((A_1 + A_2 T)^*(A_1 + A_2 T)) = \tau(A_1^* A_1 + \Theta_-(A_2^*)\Theta_-(A_2)) \geqq 0 \tag{5.23}$$

and hence $\hat{\tau}$ is a state. Furthermore by (5.19) we obtain

$$\begin{aligned} \hat{\tau}((A_1 + A_2 T)(A_1 + A_2 T)^*) &= \tau(A_1 A_1^* + A_2 A_2^*) \\ &= \tau(A_1^* A_1 + A_2^* A_2) = \tau(A_1^* A_1 + \Theta_-(A_2^* A_2)) \\ &= \hat{\tau}((A_1 + A_2 T)^*(A_1 + A_2 T)). \end{aligned} \tag{5.24}$$

This proves that $\hat{\tau}$ is tracial.

From the middle expression in (5.16), we obtain

$$\tau(b) = \hat{\tau}(\delta(T)T) = \hat{\tau}(T\delta(T) + \delta(T)T)/2 = 0.$$

This argument can be used whenever we know that $\delta(T) = bT$ with $b \in \mathfrak{A}^{\mathrm{CAR}}$.

6. Automorphism Group

If all X_α in (5.3) are quadratic polynomials of c_j and c_j^*, $j \in I_\alpha$, then $\delta(B(h)) = B(\mathbf{X}h)$ for a skew symmetric linear operator $\mathbf{X}$ defined on $h \in \mathscr{H}$ with a finite number of nonzero components. Let $\mathscr{L}_B$ be the subset of $\mathscr{L}$ consisting of such derivations satisfying the following two conditions:

(a) $\mathbf{X}$ is bounded.
(b) $[\theta_-, \mathbf{X}]$ is in the trace class.

Here θ_- is the operator defined by (3.22).

Actually (b) is automatic and a stronger statement hold for $\delta \in \mathscr{L}$. Namely, we have

$$\begin{aligned} B((\theta_- \mathbf{X} \theta_- - \mathbf{X})h) &= \Theta_- \delta(\Theta_-(B(h))) - \delta(B(h)) \\ &= i \sum_\alpha (\Theta_-([X_\alpha, \Theta_-(B(h))]) - [X_\alpha, B(h)]) \\ &= i \sum_\alpha [(\Theta_-(X_\alpha) - X_\alpha), B(h)]. \end{aligned} \tag{6.1}$$

If 0 is not in the interval I_α, then $\Theta_-(X_\alpha)=X_\alpha$ for $X_\alpha\in\mathfrak{A}_+(I_\alpha)$ and hence the sum is over α in a finite set Ξ_0. For each $\alpha\in\Xi_0$, $\Theta_-(X_\alpha)-X_\alpha$ is a quadratic expression of a finite number of c_j and c_j^* and hence

$$i[\Theta_-(X_\alpha)-X_\alpha, B(h)]=B(\mathbf{x}_\alpha h), \tag{6.2}$$

where $\mathbf{x}_\alpha$ is of a finite rank. Hence the following stronger property holds:

(b′) $\mathbf{Y}\equiv\theta_-\mathbf{X}\theta_- -\mathbf{X}$ is of a finite rank.

This implies (b) due to $[\theta_-,\mathbf{X}]=\mathbf{Y}\theta_-$.

Proposition 6.1. *The subset $\mathscr{L}_B$ is a Lie subalgebra of $\mathscr{L}$. For any $\delta\in\mathscr{L}_B$, there exists a one-parameter group* $\exp\bar{\delta}t$ *of automorphisms of $\hat{\mathfrak{A}}$ with the closure $\bar{\delta}$ of δ as its generator. It leaves the subalgebras $\mathfrak{A}^{\mathrm{CAR}}$, $\mathfrak{A}$ and $\mathfrak{A}_+$ of $\hat{\mathfrak{A}}$ invariant as sets.*

Proof. If derivations δ_1 and δ_2 in $\mathscr{L}_B$ satisfy $\delta_j(B(h))=B(\mathbf{X}_jh)$, then $\delta=[\delta_1,\delta_2]$ satisfies $\delta(B(h))=B(\mathbf{X}h)$ with $\mathbf{X}=[\mathbf{X}_1,\mathbf{X}_2]$. The linear operator $\mathbf{X}$ satisfies (a) and (b), where we use the Jacobi identity for commutators as well as the ideal property of the trace class operators to obtain (b). Therefore $\mathscr{L}_B$ is a Lie subalgebra of $\mathscr{L}$.

Due to $X_\alpha^*=X_\alpha$, δ is a symmetric derivation and hence

$$\begin{aligned} B(\Gamma\mathbf{X}h)&=B(\mathbf{X}h)^*=\delta(B(h))^*=\delta(B(h)^*)\\ &=\delta(B(\Gamma h))=B(\mathbf{X}\Gamma h) \end{aligned} \tag{6.3}$$

from which we obtain

$$[\mathbf{X},\Gamma]=0. \tag{6.4}$$

Since $\delta(1)=0$, the CAR implies

$$\begin{aligned} 0=\delta([B(h_1)^*,B(h_2)]_+)&=[B(\mathbf{X}h_1)^*,B(h_2)]_+ +[B(h_1)^*,B(\mathbf{X}h_2)]_+\\ &=(\mathbf{X}h_1,h_2)+(h_1,\mathbf{X}h_2) \end{aligned} \tag{6.5}$$

for all h_1,h_2 with a finite number of components. Therefore X is skew symmetric:

$$\mathbf{X}^*=-\mathbf{X}. \tag{6.6}$$

By (6.2) and (6.4),

$$U_t=e^{\mathbf{X}t} \tag{6.7}$$

is a unitary operator commuting with Γ and hence defines a Bogoliubov automorphism α_{U_t} of $\mathfrak{A}^{\mathrm{CAR}}$ satisfying

$$\alpha_{U_t}(B(h))=B(U_th). \tag{6.8}$$

The generator of α_{U_t} then coincides with δ on $B(h)$ and hence on $\mathfrak{A}_0^{\mathrm{CAR}}$.

By the condition (b),

$$\mathbf{Y}=\theta_-\mathbf{X}\theta_- -\mathbf{X}=[\theta_-,\mathbf{X}]\theta_- \tag{6.9}$$

is in the trace class and satisfies

$$\Gamma\mathbf{X}^*\Gamma=-\mathbf{Y} \tag{6.10}$$

due to $[\Gamma, \theta_-]=0$, (6.4) and (6.6). There exists an element $Y \in \mathfrak{A}_+^{\mathrm{CAR}}$ called the bilinear Hamiltonian ($Y=(1/2)(B, \mathbf{Y}B)$ in the notation of [1]) satisfying $Y^* = -Y$ (due to $\mathbf{Y}^* = -\mathbf{Y}$) and

$$[Y, B(h)] = B(\mathbf{Y}h). \tag{6.11}$$

By (7.12) of [1] and $\theta_-\mathbf{Y}\theta_- = -\mathbf{Y}$, it also satisfies

$$\Theta_-(Y) = -Y. \tag{6.12}$$

Using the expansional formalism in [2], we define

$$W_t = 1 + \sum_{n=1}^{\infty} \int_0^t \dots \int_0^{t_{n-1}} Y_{t_n} \dots Y_{t_1} dt_n \dots dt_1 \tag{6.13}$$

where $Y_t \equiv \alpha_{U_t}(Y)$. By using Eqs. (2.17), (2.18), Propositions 3 and 4 of [2] as well as the property $Y^* = -Y$, we see that W_t is unitary. By Theorem 2 of [2], W_t satisfies the cocycle equation:

$$W_s \alpha_{U_s}(W_t) = W_{s+t}. \tag{6.14}$$

Therefore

$$\beta_t(A) = W_t \alpha_{U_t}(A) W_t^*, \qquad t \in \mathfrak{A}^{\mathrm{CAR}} \tag{6.15}$$

defines a continuous one-parameter group of automorphisms of $\mathfrak{A}^{\mathrm{CAR}}$.

Using (4.11) of [2], we obtain for $V_t = e^{\theta_- X \theta_- t} = \theta_- U_t \theta_-$

$$\begin{aligned}(d/dt)\beta_{-t}(B(V_t h)) &= \beta_{-t}(d/ds)\beta_{-s}(B(V_{t+s}h))|_{s=0} \\ &= \beta_{-t}\{-(d/ds)\alpha_{U_s}(B(V_t h))|_{s=0} - [Y, B(V_t h)] + B(dV_t h/dt)\} \\ &= \beta_{-t}\{-B(\mathbf{X}V_t h) - B(\mathbf{Y}V_t h) + B(\theta_- \mathbf{X} \theta_- V_t h)\} = 0.\end{aligned} \tag{6.16}$$

Therefore $\beta_{-t}(B(V_t h))$ is constant in t. Since its value at $t=0$ is $B(h)$, we have

$$\beta_t(B(h)) = B(V_t h) = \alpha_{V_t}(h) = \Theta_- \alpha_{U_t} \Theta_-(B(h)). \tag{6.17}$$

Thus we have obtained the following formula:

$$W_t \alpha_{U_t}(A) W_t^* = \alpha_{V_t}(A) = \Theta_- \alpha_{U_t} \Theta_-(A) \tag{6.18}$$

for $A = B(h)$ and hence for all $A \in \mathfrak{A}^{\mathrm{CAR}}$.

By (6.12) and (6.13), we obtain

$$\Theta_-(W_t) = 1 + \sum_{n=1}^{\infty} \int_0^t \dots \int_0^{t_{n-1}} (-Y'_{t_n}) \dots (-Y'_{t_1}) dt_n \dots dt_1 \tag{6.19}$$

where we have used the relation

$$\Theta_-(Y_t) = \Theta_- \alpha_{U_t} \Theta_-(\Theta_-(Y)) = \alpha_{V_t}(-Y) \tag{6.20}$$

and notation

$$Y'_t = \alpha_{V_t}(Y) = W_t \alpha_{U_t}(Y) W_t^*. \tag{6.21}$$

The quantity $-Y'_t$ corresponds to $(Y*(-Y))(t)$ of (3.7) in [2] with $B=Y$ and $A=-Y$ and hence

$$\Theta_-(W_t) W_t = 1 \tag{6.22}$$

due to $A+B=0$ in (3.10) of [2].

We now define the extension α_t of α_{U_t} from $\mathfrak{A}^{\mathrm{CAR}}$ to $\hat{\mathfrak{A}}$ by

$$\alpha_t(A_1 + A_2 T) = \alpha_{U_t}(A_1) + \alpha_{U_t}(A_2) T W_t. \tag{6.23}$$

To show that α_t is indeed a $*$ automorphism, we have to see that relations $T^2=1$, $T^*=T$ and $TAT=\Theta_-(A)$ for $A\in\mathfrak{A}^{\mathrm{CAR}}$ are preserved, namely it is enough to check the following relations:

$$TW_tTW_t=1\,, \tag{6.24}$$

$$W_t^*T=TW_t\,, \tag{6.25}$$

$$TW_t\alpha_{U_t}(A)TW_t=\alpha_{U_t}(\Theta_-(A))\,. \tag{6.26}$$

Equation (6.24) holds by (6.22) and $\Theta_-(W_t)=TW_tT$. Because W_t is unitary, (6.24) implies $W_t^*=TW_tT$ and this implies (6.25). By (6.25) and (6.18), we obtain

$$\begin{aligned} TW_t\alpha_{U_t}(A)TW_t &= TW_t\alpha_{U_t}(A)W_t^*T \\ &= T\{\Theta_-\alpha_{U_t}\Theta_-(A)\}T=\alpha_{U_t}\Theta_-(A) \end{aligned} \tag{6.27}$$

which shows (6.26).

In the notation of [1], we have

$$\alpha_{U_t}(B,\mathbf{Y}B)=(B,U_t\mathbf{Y}U_{-t}B) \tag{6.28}$$

by (7.12) of [1]. It is an entire analytic function of a complex number t, because $U_z=e^{z\mathbf{X}}$ is an entire analytic function of z (values are bounded operators), $\mathbf{Y}$ is in the trace class and (B, LB) is linear and continuous in L with the bound

$$\|Y_z\|\leqq(1/2)\,\|\mathbf{Y}\|_{\mathrm{tr}}\,e^{2\|\mathbf{X}\|\,|\mathrm{Im}\,z|}\,. \tag{6.29}$$

Therefore W_t of (6.13) has an entire analytic extension:

$$W_z=1+\sum_{n=1}^{\infty}z^n\int_0^1\int_0^{t_1}\dots\int_0^{t_{n-1}}Y_{zt_n}\dots Y_{zt_1}dt_n\dots dt_1\,. \tag{6.30}$$

In particular, T is in the domain of the generator of α_t. Since the generator of α_t is a symmetric derivation coinciding with δ on $\mathfrak{A}^{\mathrm{CAR}}$, it must be an extension of δ by the uniqueness result given in Proposition 5.2 (see Remark after Proof of Proposition 5.2).

Since $\alpha_z(B(h))=B(e^{z\mathbf{X}}h)$ and $\alpha_z(T)=TW_z$ are entire analytic, all elements of $\hat{\mathfrak{A}}_0$ are entire analytic elements of δ. Thus the condition (B2) of Theorem 3.2.50 of [8] is satisfied. Since the generator of α_t satisfies the condition (C1) there, it is satisfied by δ which is a restriction of the generator to $\hat{\mathfrak{A}}_0$. The condition (A1) is also satisfied with $D(\delta)=\hat{\mathfrak{A}}_0$. Therefore the closure of δ is exactly the generator of α_t by Theorem 3.2.50 of [8].

Since α_t maps $B(h)$ to $B(U_th)$ and T to TW_t with $W_t\in\mathfrak{A}_+$, the subalgebras $\mathfrak{A}^{\mathrm{CAR}}$, $\mathfrak{A}$ and $\mathfrak{A}_+$ are invariant under α_t as sets. Q.E.D.

Remark to Theorem 6.1. Except for the first statement (that $\mathscr{L}_B$ is a Lie algebra), Theorem 6.1 holds for δ defined on $\hat{\mathfrak{A}}_0$ such that iX is analytic on the set of all $h\in\mathscr{H}$ with a finite number of non-zero components. The proof is the same because $U_t=e^{\hat{\mathbf{X}}t}$ will have the same property as U_t of (6.7), δ is analytic on $B(h)$ in $\mathfrak{A}_0$ (i.e. when h has a finite number of non-zero components),

$$G(t)\equiv\sup_{-t\leqq s\leqq t}\|e^{i\hat{\mathbf{X}}t}E\|$$

is finite for small $t \geqq 0$ and for the (finite dimensional) projection E on a finite interval of $\mathbf{Z}$ (projection considered in the Hilbert space $\mathscr{H} = l_2(\mathbf{Z}) \oplus l_2(\mathbf{Z})$) supporting the operator $\mathbf{Y}$, and the estimate (6.29) holds with $G(|\mathrm{Im}\, z|)^2$ replacing $e^{2\|\mathbf{X}\| \, |\mathrm{Im}\, z|}$.

Proposition 6.2. *The set of automorphisms*

$$G_0 \equiv \{\exp \bar{\delta};\ \delta \in i\mathfrak{g}_0\}|_{\mathfrak{A}^{\mathrm{CAR}}} \tag{6.31}$$

(restricted to $\mathfrak{A}^{\mathrm{CAR}}$) is maximal abelian in the sense that any automorphism of $\mathfrak{A}^{\mathrm{CAR}}$ commuting with all elements of G_0 is in the closure of G_0 (relative to the pointwise convergence).

Proof. Let E be the spectral projection of $\mathbf{K}$ for its positive spectrum, namely

$$(Eh)(\theta) = E(\theta)h, \ E(\theta) = (2k(\theta))^{-1}(K(\theta) + k(\theta)). \tag{6.32}$$

Due to $\Gamma K \Gamma = -K$, we have

$$\Gamma E \Gamma = 1 - E. \tag{6.33}$$

Thus E is a basis projection in the terminology of [1] and [6].

Let M be the abelian von Neumann algebra of all bounded multiplication operators in θ-representation on the space $E\mathscr{H}$ (rather than $\mathscr{H}$). It is then maximal abelian, as $E(\theta)$ is one-dimensional for each θ.

For the Proof of Proposition 6.2, we need the following Lemma describing $\mathfrak{g}_0$ exactly.

Lemma 6.3. *The derivations δ in $i\mathfrak{g}_0$ are exactly those of the following form:*

$$\delta(B(h)) = B(\hat{\delta}h), (\hat{\delta}h)(\theta) = \hat{\delta}(\theta)h(\theta), \tag{6.34}$$

$$\hat{\delta}(\theta) = i\delta_1(\theta)\mathbf{1} + i\delta_2(\theta)K(\theta), \tag{6.35}$$

where δ_1 and δ_2 are any Laurent polynomial of $e^{i\theta}$ such that δ_1 is a real odd function of θ and δ_2 is a real even function of θ:

$$\overline{\delta_1(\theta)} = \delta_1(\theta), \quad \overline{\delta_2(\theta)} = \delta_2(\theta), \tag{6.36}$$

$$\delta_1(-\theta) = -\delta_1(\theta), \quad \delta_2(-\theta) = \delta_2(\theta). \tag{6.37}$$

Proof of Lemma. By the beginning part of the proof of Proposition 6.1, $\Gamma\hat{\delta}\Gamma = \hat{\delta}$ and $\hat{\delta}^* = -\hat{\delta}$. If $\hat{\delta}$ is of the form given by (6.34) and (6.35), then $\hat{\delta}^* = -\hat{\delta}$ implies (6.36) and (3.32) for Γ implies (6.37).

Since

$$kDk = k^2 D + (1/2)(\partial k^2/\partial\theta), \tag{6.38}$$

the formula for H_j and S_j shows that $\hat{\delta}$'s for H_j and S_j are in fact of the form given by (6.34) and (6.35) with δ_1 and δ_2 Laurent polynomials of $e^{i\theta}$. Since k^2 is at most of second degree and D does not change degrees, we can find the exact degrees as follows:

For $\gamma \neq \pm 1$,

	degree	coefficient
$h_{2n}(\theta)$	$2n$	$2^{2n+1}J(\gamma^2-1)^n(2n)!$
$h_{2n+1}(\theta)$	$2n+2$	$-i2^{2n+2}J(\gamma^2-1)^{n+1}(2n+1)!$
$s_{2n}(\theta)$	$2n+1$	$i2^{2n+1}(\gamma^2-1)^n(2n)!$
$s_{2n+1}(\theta)$	$2n+1$	$-2^{2n+2}(\gamma^2-1)^n(2n+1)!$

For $\gamma^2=1$ and $\lambda \neq 0$,

$h_{2n}(\theta)$	n	$2^{4n+1}J\lambda^n n!^2$
$h_{2n+1}(\theta)$	$n+1$	$-i2^{4n+4}J\lambda^{n+1}n!(n+1)!$
$s_{2n}(\theta)$	$n+1$	$i2^{4n+1}\lambda^n n!(n+1)!$
$s_{2n+1}(\theta)$	$n+1$	$2^{4n+2}\lambda^n(n+1)!^2$

The case $(\lambda, \gamma)=(0, \pm 1)$ has been excluded from the beginning (see Sect. 2).

In addition, $h_{2n}(\theta)$ and $s_{2n+1}(\theta)$ are even functions of θ while $h_{2n+1}(\theta)$ and $s_{2n}(\theta)$ are odd functions of θ. Therefore, by taking the real linear combination, we obtain any real odd polynomial of $e^{i\theta}$ for δ_1 and any real even polynomial of $e^{i\theta}$ for δ_2 (even-odd as a function of θ). Note that a real even function which is of the highest degree k as a Lausent polynomial of $e^{i\theta}$ is unique up to the addition of lower degree polynomials and a constant coefficient (for example $\cos k\theta$) and a real odd function which is of the highest degree k is also unique in the same sense (for example $\sin k\theta$). This proves Lemma 6.3.

We remark that if $\gamma^2=1$, $\lambda \neq 0$, then the S's become redundant.

We now resume the proof of Proposition 6.2. By Stone-Weierstrass theorem, any periodic continuous function of θ can be approximated by Laurent polynomials of $e^{i\theta}$. The closure of G_0 in $\operatorname{Aut}\mathfrak{A}^{\mathrm{CAR}}$ contains all Bogoliubov automorphisms α_U with $U=e^{\mathbf{X}}$ and

$$(\mathbf{X}h)(\theta)=\hat{X}(\theta)h(\theta), \qquad \hat{X}(\theta)=iX_1(\theta)1+iX_2(\theta)K(\theta), \tag{6.39}$$

where X_1 is any odd real periodic continuous function of θ and X_2 is any even real periodic continuous function. Since $\|B(h)\|=\|h\|$, by using approximation in strong operator topology on $\mathscr{H}$, closure of G_0 in $\operatorname{Aut}\mathfrak{A}^{\mathrm{CAR}}$ contains $e^{\mathbf{X}}$ with any even real periodic L^∞-function X_1 and any even real periodic measurable function X_2 such that $X_2(\theta)k(\theta)$ is essentially bounded. Then $(Uk)(\theta)=\hat{U}(\theta)h(\theta)$ with UE exhausting unitary elements of M. Now the maximal abelian property of G_0 follows from the maximal abelian property of M on $E\mathscr{H}$ by Theorem 2 of Kishimoto in [9].

Proposition 6.4. *The subgroup*

$$\hat{G}_0=\{\exp\bar{\delta};\, \delta \in i\mathfrak{g}_0\} \tag{6.40}$$

of $\operatorname{Aut}\hat{\mathfrak{A}}$ *is abelian.*

Proof. By Proposition 6.2, we already know that the action of $\hat{G}_0$ on $\mathfrak{A}_0$ is abelian. Therefore this Proposition follows from the following uniqueness result about extension of automorphisms because $\alpha_1\alpha_2$ and $\alpha_2\alpha_1$ always coincide as automorphisms of $\hat{\mathfrak{A}}^{\mathrm{CAR}}$ for two automorphisms in $\hat{G}_0$ and hence they must coincide also on $\hat{\mathfrak{A}}$ as $\hat{G}_0$ is connected.

Lemma 6.5. *Let $\{\alpha_t\}$, $\{\beta_t\}$, $0\leqq t\leqq 1$, be two continuous families of automorphisms of $\hat{\mathfrak{A}}$ such that they leave $\mathfrak{A}^{\mathrm{CAR}}$ invariant as a set and*

$$\alpha_t(T)T\in\mathfrak{A}^{\mathrm{CAR}}, \qquad \beta_t(T)T\in\mathfrak{A}^{\mathrm{CAR}}. \tag{6.41}$$

If α_t and β_t coincide on $\mathfrak{A}^{\mathrm{CAR}}$ and if $\alpha_0=\beta_0$, then $\alpha_t=\beta_t$ for all t.

Proof. It is enough to show

$$\alpha_t(T)=\beta_t(T). \tag{6.42}$$

For any $A\in\mathfrak{A}^{\mathrm{CAR}}$,

$$\alpha_t(T)A\alpha_t(T)=\alpha_t(T\alpha_{-t}(A)T)=\alpha_t\Theta_-\alpha_{-t}(A), \tag{6.43a}$$

$$\beta_t(T)A\beta_t(T)=\beta_t(T\beta_{-t}(A)T)=\beta_t\Theta_-\beta_{-t}(A). \tag{6.43b}$$

Since $\alpha_t=\beta_t$ on $\mathfrak{A}^{\mathrm{CAR}}$, we obtain

$$\beta_t(T)\alpha_t(T)A\alpha_t(T)\beta_t(T)=\beta_t\Theta_t\beta_{-t}\alpha_t\Theta_-\alpha_{-t}(A)=A. \tag{6.44}$$

Since

$$\alpha_t(T)^2=\alpha_t(T^2)=\alpha_t(1)=1, \tag{6.45a}$$

$$\beta_t(T)^2=\beta_t(T^2)=\beta_t(1)=1, \tag{6.45b}$$

(6.44) implies

$$[\beta_t(T)\alpha_t(T), A]=0. \tag{6.46}$$

By (6.41)

$$\beta_t(T)\alpha_t(T)=(\beta_t(T)T)(T\alpha_t(T))=(\beta_t(T)T)(\alpha_t(T)T)^*\in\mathfrak{A}^{\mathrm{CAR}}. \tag{6.47}$$

Therefore $\beta_t(T)\alpha_t(T)$ is a scalar operator $\lambda\mathbf{1}$. By multiplying $\alpha_t(T)$ and using (6.45a), we have

$$\beta_t(T)=\lambda\alpha_t(T) \tag{6.48}$$

for some complex number λ. By (6.45a) and (6.45b), $\lambda^2=1$ and hence

$$\beta_t(T)=\pm\alpha_t(T). \tag{6.49}$$

By continuity, the sign is common for all t and, since it is $+1$ for $t=0$ by assumption, we obtain (6.42) for all t. Q.E.D.

7. Conclusion

Proofs of Theorems in Sect. 2 are essentially given in the preceding sections. We summarize it below.

The commutativity of all $H_{k,j}$ and $S_{k,j}$ as derivations on $\mathfrak{A}^{\mathrm{CAR}}$ are immediate from concrete expressions given to them in Propositions 4.1, 4.2, 4.4, and 4.5. Since these derivations are of the class $\mathscr{L}$ treated in Proposition 5.2, the uniqueness of extension from $\mathfrak{A}^{\mathrm{CAR}}$ to $\hat{\mathfrak{A}}$ shown in Proposition 5.2 proves the commutativity of these derivations on $\hat{\mathfrak{A}}$ and hence on $\mathfrak{A}$. This proves Theorems 1 and 2.

Since these derivations are of the form given in Lemma 6.3 (as follows from explicit forms given by Propositions 4.4 and 4.5), $H_{k,j}$ and $S_{k,j}$ are contained in $\mathfrak{g}_0$ by Lemma 6.3. This proves Theorem 3.

The second half of Theorem 4 is Proposition 6.4. The first half of Theorem 4 will follow from Remark to Proposition 6.1 if we prove that all $h \in \mathscr{H}$ with a finite number of non-zero components are analytic vectors of any $X \in \mathfrak{g}_1$.

For such an h,

$$h(\theta) = \sum_{|j| \leqq m} h_j e^{ij\theta}, \tag{7.1}$$

where each h_j is a constant of θ. Let $\alpha = \max_j \|h_j\|_{\mathscr{H}}$. Any $X \in \mathfrak{g}_1$ is of the form

$$X = l_1(\theta)D + l_2(\theta), \qquad l_p(\theta) = \sum_{|j| \leqq k} l_{pj} e^{ij\theta}, \tag{7.2}$$

where $p = 1, 2$, each l_{pj} is a 2×2 matrix, constant of θ, and $D = d/d\theta$. Let $\beta = \max_{p,j} \|l_{pj}\|$. Then, $X^n h$ is a sum of $(4k+2)^n(2m+1)$ terms ($(4h+2)$ is the number of terms in X and $(2m+1)$ is the number of terms in h) of the form

$$\{e^{ij_n\theta}(D) \dots e^{ij_2\theta}(D) e^{ij_1\theta}(D) e^{ij\theta}\} L_n \dots L_1 h_j, \tag{7.3}$$

where $|j| \leqq m$, $|j_l| \leqq k$ for all l, (D) is either D or 1, and each L_l is l_{1j_l} or l_{2j_l}. The expression in the parenthesis of (7.3) has its absolute value bounded by

$$|j|\,|j+j_1|\,|j+j_1+j_2| \dots |j+j_1+\dots+j_{n-1}| \leqq \prod_{l=0}^{n-1} (m+lk). \tag{7.4}$$

We also have

$$\|L_n \dots L_1 h_j\| \leqq \beta^n \alpha. \tag{7.5}$$

These estimates imply

$$\|X^n h\| \leqq (4k+2)^n \beta^n (2m+1) \alpha \prod_{l=0}^{n-1} (m+lk).$$

Hence

$$\sum z^n \|X^n h\| / n!$$

is convergent if

$$|z|(4k+2)\beta k < 1.$$

Therefore, any h is an analytic vector of X.

Proposition 6.2 leads to an obvious conjecture that G_0 might be maximal abelian on $\mathfrak{A}$.

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