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L^1-CONVERGENCE OF GREEDY ALGORITHM BY GENERALIZED WALSH SYSTEM

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ABSTRACT. In this paper we consider the generalized Walsh system and a problem L^1- convergence of greedy algorithm of functions after changing the values on small set.

1. INTRODUCTION AND PRELIMINARIES

Let a denote a fixed integer, $a \geq 2$ and put $\omega_a = e^{\frac{2\pi i}{a}}$. Now we will give the definitions of generalized Rademacher and Walsh systems [2].

Definition 1.1. The Rademacher system of order a is defined by

$$\varphi_0(x) = \omega_a^k \quad if \quad x \in \left[\frac{k}{a}, \frac{k+1}{a}\right), \quad k = 0, 1, \cdots, a-1, \quad x \in [0,1)$$

and for $n \geq 0$

$$\varphi_n(x+1) = \varphi_n(x) = \varphi_0(a^n x).$$

Definition 1.2. The generalized Walsh system of order a is defined by

$$\psi_0(x) = 1,$$

and if $n = \alpha_1 a^{n_1} + \cdots + \alpha_s a^{n_s}$ where $n_1 > \cdots > n_s$, then

$$\psi_n(x) = \varphi_{n_1}^{\alpha_1}(x) \cdot \cdots \cdot \varphi_{n_s}^{\alpha_s}(x).$$

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Let's denote the generalized Walsh system of order a by Ψ_a.

Note that Ψ_2 is the classical Walsh system.

The basic properties of the generalized Walsh system of order a are obtained by Chrestenson, Pely, Fine, Young, Vatari, Vilenkin and others (see [2, 14, 15, 17]).

In this paper we consider L^1- convergence of greedy algorithm with respect to Ψ_a system. Now we present the definition of greedy algorithm.

Let X be a Banach space with a norm $||\cdot|| = ||\cdot||_X$ and a basis $\Phi = \{\phi_k\}_{k=1}^{\infty}$, $||\phi_k||_X = 1$, $k = 1, 2, \ldots$.

For a function $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} a_k(f)\phi_k \quad .$$

Definition 1.3. Let an element $f \in X$ be given. Then the m-th greedy approximant of the function f with regard to the basis Φ is given by

$$G_m(f, \phi) = \sum_{k \in \Lambda} a_k(f)\phi_k,$$

where $\Lambda \subset \{1, 2, \cdots\}$ is a set of cardinality m such that

$$|a_n(f)| \geq |a_k(f)|, \quad n \in \Lambda, \quad k \notin \Lambda.$$

In particular we'll say that the greedy approximant of $f \in L^p[0,1]$, $p \geq 0$ converges with regard to the Ψ_a , if the sequence $G_m(x, f)$ converges to $f(t)$ in L^p norm. This new and very important direction invaded many mathematician's attention (see [3]-[6], [8, 9, 16]).

Körner [9] constructed an L^2 function (then a continuous function) whose greedy algorithm with respect to trigonometric systems diverges almost everywhere.

Temlyakov in [16] constructed a function f that belongs to all L^p, $1 \leq p < 2$ (respectively $p > 2$), whose greedy algorithm concerning trigonometric systems divergence in measure (respectively in L^p, $p > 2$), e.i. the trigonometric system are not a quasi-greedy basis for L^p if $1 < p < \infty$.

In [6] Gribonval and Nielsen proved that for any $1 < p < \infty$ there exits a function $f(x) \in L^p[0,1)$ whose greedy algorithm with respect to Ψ_2- classical Walsh system diverges in $L^p[0,1]$. Moreover, similar result for Ψ_a system follows from Corollary 2.3. (see [6]). Note also that in [4] and [5] this result was proved for $L^1[0,1]$.

The following question arises naturally: is it possible to change the values of any function f of class L^1 on small set, so that a greedy algorithm of new modified function concerning Ψ_a system converges in the L^1 norm?

The classical **C**-property of Luzin is well-known, according to which every measurable function can be converted into a continuous one be changing it on a set of arbitrarily small measure. This famous result of Luzin [10] dates back to 1912.

Note that Luzin's idea of modification of a function improving its properties was substantially developed later on.

In 1939, Men'shov [11] proved the following fundamental theorem.

Theorem (Men'shov's C-strong property). *Let $f(x)$ be an a.e. finite measurable function on $[0,2\pi]$. Then for each $\varepsilon > 0$ one can define a continuous function $g(x)$ coinciding with $f(x)$ on a subset E of measure $|E| > 2\pi - \varepsilon$ such that its Fourier series with respect to the trigonometric system converges uniformly on $[0,2\pi]$.*

Further interesting results in this direction were obtained by many famous mathematicians (see for example [1, 12, 13]).

Particulary in 1991 Grigorian obtain the following result [7]:

Theorem (L^1-strong property). *For each $\varepsilon > 0$ there exits a measurable set $E \subset [0,2\pi]$ of measure $|E| > 2\pi - \varepsilon$ such that for any function $f(x) \in L^1[0,2\pi]$ one can find a function $g(x) \in L^1[0,2\pi]$ coinciding with $f(x)$ on E so that its Fourier series with respect to the trigonometric system converges to $g(x)$ in the metric of $L^1[0,2\pi]$.*

In this paper we prove the following:

Theorem 1.4. *For any $\varepsilon \in (0,1)$ and for any function $f \in L^1[0,1)$ there is a function $g \in L^1[0,1)$, with $mes\{x \in [0,1) \ ; \ g \neq f\} < \varepsilon$, such that the nonzero fourier coefficients by absolute values monotonically decreasing.*

Theorem 1.5. *For any $0 < \varepsilon < 1$ and each function $f \in L^1[0,1)$ one can find a function $g \in L^1[0,1)$, $mes\{x \in [0,1) \ ; \ g \neq f\} < \varepsilon$, such that its fourier series by Ψ_a system L^1 convergence to $g(x)$ and the nonzero fourier coefficients by absolute values monotonically decreasing, i.e. the greedy algorithm by Ψ_a system L^1-convergence.*

The Theorems 1.1 and 1.2 follows from next more general Theorem 1.3, which in itself is interesting:

Theorem 1.6. *For any $0 < \varepsilon < 1$ there exists a measurable set $E \subset [0,1)$ with $|E| > 1 - \varepsilon$ and a series by Ψ_a system of the form*

$$\sum_{i=1}^{\infty} c_i \psi_i(x), \quad |c_i| \downarrow 0$$

such that for any function $f \in L^1[0,1)$ one can find a function $g \in L^1[0,1)$,

$$g(x) = f(x); \quad \textit{if} \ \ x \in E$$

and the series of the form

$$\sum_{n=1}^{\infty} \delta_n c_n \psi_n(x), \quad \textit{where} \ \ \delta_n = 0 \ \ \textit{or} \ \ 1,$$

which convergence to $g(x)$ in $L^1[0,1)$ metric and

$$\left|\left|\sum_{n=1}^{m} \delta_n c_n \psi_n(x\right|\right|_1 \leq 12 \cdot ||f||_1, \quad \forall m \geq 1.$$

Remark 1.7. Theorems 1.6 for classical Walsh system Ψ_2 was proved by Grigorian [8].

Remark 1.8. From Theorem 1.5 follows that generalized Walsh system Ψ_a has L^1-strong property.

2. Basic lemmas

First we present some properties of Ψ_a system (see Definition 1.2).

Property 1. Each nth Rademacher function has period $\frac{1}{a^n}$ and

$$\varphi_n(x) = const \in \Omega_a = \{1, \omega_a, \omega_a^2, \cdots ., \omega_a^{a-1}\}, \tag{2.1}$$

if $x \in \Delta_{n+1}^{(k)} = \left[\frac{k}{a^{n+1}}, \frac{k+1}{a^{n+1}}\right)$, $k = 0, \cdots, a^{n+1}-1$, $n = 1, 2, \cdots ..$

It is also easily verified, that

$$(\varphi_n(x))^k = (\varphi_n(x))^m, \quad \forall n, k \in \mathcal{N}, where \;\; m = k \; (mod \; a)$$

Property 2. It is clear, that for any integer n the Walsh function $\psi_n(x)$ consists of a finite product of Rademacher functions and accepts values from Ω_a.

Property 3. Let $\omega_a = e^{\frac{2\pi i}{a}}$. Then for any natural number m we have

$$\sum_{k=0}^{a-1} \omega_a^{k \cdot m} = \begin{cases} a \;, \;\; if \;\; m \equiv 0 (\text{mod } a), \\ \\ 0, \quad if \;\; m \neq 0 (\text{mod } \;\; a) \;. \end{cases} \tag{2.2}$$

Property 4. The generalized Walsh system Ψ_a, $a \geq 2$ is a complete orthonormal system in $L^2[0,1)$ and basis in $L^p[0,1)$, $p > 1$ [14]).

Property 5. From definition 2 we have

$$\psi_i(x) \cdot \psi_j(a^s x) = \psi_{j \cdot a^s + i}(x) \;, \text{ where } 0 \leq i \;, \; j < a^s,$$

and particulary

$$\psi_{a^k + j}(x) = \varphi_k(x) \cdot \psi_j(x), \quad if \quad 0 \leq j \leq a^k - 1. \tag{2.3}$$

Now for any $m = 1, 2, \cdots$ and $1 \leq k \leq a^m$ we put $\Delta_m^{(k)} = \left[\frac{k-1}{a^m}, \frac{k}{a^m}\right)$ and consider the following function

$$I_m^{(k)}(x) = \begin{cases} 1 \;, \text{ if } x \in [0,1) \setminus \Delta_m^{(k)} \;, \\ 1 - a^m \;, \text{ if } x \in \Delta_m^{(k)}, \end{cases}$$

and periodically extend these functions on R^1 with period 1.

By $\chi_E(x)$ we denote the characteristic function of the set E, i.e.

$$\chi_E(x) = \begin{cases} 1 \;, \text{ if } x \in E \;, \\ 0 \;, \text{ if } x \notin E \;. \end{cases} \tag{2.4}$$

Then, clearly

$$I_m^{(k)}(x) = \psi_0(x) - a^m \cdot \chi_{\Delta_m^{(k)}}(x) \;, \tag{2.5}$$

and for the natural numbers $m \geq 1$ and $1 \leq i \leq a^m$

$$a_i(\chi_{\Delta_m^{(k)}}) = \int_0^1 \chi_{\Delta_m^{(k)}}(x) \cdot \overline{\psi_i}(x)dx = \mathcal{A} \cdot \frac{1}{a^m}, \quad 0 \leq i < a^m. \tag{2.6}$$

$$b_i(I_m^{(k)}) = \int_0^1 I_m^{(k)}(x)\overline{\psi_i}(x)dx = \begin{cases} 0 \ , \ \text{if } i = 0 \text{ and } i \geq a^k \ , \\ -\mathcal{A} \ , \ \text{if } 1 \leq i < a^k \end{cases}$$

where $\mathcal{A} = const \in \Omega_a$ and $|\mathcal{A}| = 1$.

Hence

$$\chi_{\Delta_m^{(k)}}(x) = \sum_{i=0}^{a^k-1} b_i(\chi_{\Delta_m^{(k)}})\psi_i(x) \ ,$$

$$I_m^{(k)}(x) = \sum_{i=1}^{a^k-1} a_i(I_m^{(k)})\psi_i(x) \ . \tag{2.7}$$

Lemma 2.1. *For any numbers $\gamma \neq 0$, $N_0 > 1$, $\varepsilon \in (0,1)$ and interval by order a $\Delta = \Delta_m^{(k)} = [\frac{k-1}{a^m}, \frac{k}{a^m})$, $\quad i = 1, \cdots, a^m$ there exists a measurable set $E \subset \Delta$ and a polynomial $P(x)$ by Ψ_a system of the form*

$$P(x) = \sum_{k=N_0}^{N} c_k\psi_k(x)$$

which satisfy the conditions:

1) *coefficients* $\{c_k\}_{k=N_0}^N$ *equal* 0 *or* $-\mathcal{K} \cdot \gamma \cdot |\Delta|$,

where $\mathcal{K} = const \in \Omega_a$, $|\mathcal{K}| = 1$,

2) $$|E| > (1-\varepsilon) \cdot |\Delta|,$$

3) $$P(x) = \begin{cases} \gamma, & \text{if} \ \ x \in E; \\ 0, & \text{if} \ \ x \notin \Delta. \end{cases}$$

4) $$\frac{1}{2} \cdot |\gamma| \cdot |\Delta| < \int_0^1 |P(x)|dx < 2 \cdot |\gamma| \cdot |\Delta|.$$

5) $$\max_{N_0 \leq m \leq N} \int_0^1 \Big| \sum_{k=N_0}^{m} c_k\psi_k(x)\Big| < a \cdot |\gamma| \cdot \sqrt{\frac{|\Delta|}{\varepsilon}}.$$

Proof. We take a natural numbers $\nu_0 \ \ s$ so that

$$\nu_0 = \left[\log_a \frac{1}{\varepsilon}\right] + 1; \quad s = [\log_a N_0] + m. \tag{2.8}$$

Define the coefficients c_n, a_i, b_j and the function $P(x)$ in the following way:

$$P(x) = \gamma \cdot \chi_{\Delta_m^{(k)}}(x) \cdot I_{\nu_0}^{(1)}(a^s x), \quad x \in [0,1] \ , \tag{2.9}$$

$$c_n = c_n(P) = \int_0^1 P(x)\overline{\psi_n}(x)dx \ , \ \forall n \geq 0,$$

$$a_i = a_i(\chi_{\Delta_m^{(k)}}) \ , \ 0 \le i < a^m \ , \quad b_j = b_j(I_{\nu_0}^{(1)}) \ , \ 1 \le j < a^{\nu_0} \ .$$

Taking into account (2.1)-(2.2), (2.3)-(2.4), (2.6)-(2.7) for $P(x)$ we obtain

$$P(x) = \gamma \cdot \sum_{i=0}^{a^m-1} a_i \psi_i(x) \cdot \sum_{j=1}^{a^{\nu_0}-1} b_j \psi_j(a^s x) =$$

$$= \gamma \cdot \sum_{j=1}^{a^{\nu_0}-1} b_j \cdot \sum_{i=0}^{a^m-1} a_i \psi_{j\cdot a^s+i}(x) = \sum_{k=N_0}^{N} c_k \psi_k(x) \ ,$$

where

$$c_k = c_k(P) = \begin{cases} -\mathcal{K} \cdot \frac{\gamma}{a^m} \text{ or } 0 \ , & \text{if } k \in [N_0, N] \\ 0 \ , & \text{if } k \notin [N_0, N], \end{cases} \tag{2.10}$$

$$\mathcal{K} \in \Omega_a, \quad |\mathcal{K}| = 1, \quad N = a^{s+\nu_0} + a^m - a^s - 1. \tag{2.11}$$

Set

$$E = \{x \in \Delta : P(x) = \gamma\} \ .$$

By (2.4), (2.5) and (2.9) we have

$$|E| = a^{-m}(1 - a^{-\nu_0}) > (1 - \epsilon)|\Delta|,$$

$$P(x) = \begin{cases} \gamma \ , \ \text{if } x \in E \ , \\ \gamma(1 - a^{\nu_0}) \ , \ \text{if } x \in \Delta \setminus E \ , \\ 0 \ , \ \text{if } x \notin \Delta \ . \end{cases}$$

Hence and from (2.8) we get

$$\int_0^1 |P(x)| dx = 2 \cdot |\gamma||\Delta| \cdot (1 - a^{-\nu_0}),$$

and taking into account that $a \ge 2$ we have

$$\frac{1}{2} \cdot |\gamma| \cdot |\Delta| < \int_0^1 |P(x)| dx < 2 \cdot |\gamma| \cdot |\Delta|.$$

From relations (2.8), (2.10) and (2.11) we obtain

$$\begin{aligned} & \max_{N_0 \le m \le N} \int_0^1 \left| \sum_{k=N_0}^{m} c_k \psi_k(x) \right| dx \\ < \ & \left[\int_0^1 |P(x)|^2 dx \right]^{\frac{1}{2}} \\ \le \ & \left[\sum_{k=N_0}^{N} c_k^2 \right]^{\frac{1}{2}} = |\gamma| \cdot |\Delta| \cdot \sqrt{a^{\nu_0+s} + a^m} = |\gamma| \cdot \sqrt{|\Delta|} \cdot \sqrt{a^{\nu_0} + 1} \\ < \ & |\gamma| \cdot \sqrt{|\Delta|} \cdot \sqrt{\frac{a}{\varepsilon}} \\ < \ & a \cdot |\gamma| \cdot \sqrt{\frac{|\Delta|}{\varepsilon}}. \end{aligned}$$

$\square$

Lemma 2.2. *For any given numbers* $N_0 > 1$, $(N_0 \in \mathcal{N})$, $\varepsilon \in (0,1)$ *and each function* $f(x) \in L^1[0,1)$, $||f||_1 > 0$ *there exists a measurable set* $E \subset [0,1)$, *function* $g(x) \in L^1[0,1)$ *and a polynomial by* Ψ_a *system of the form*

$$P(x) = \sum_{k=N_0}^{N} c_k \psi_{n_k}(x), \quad n_k \uparrow$$

satisfying the following conditions:

1) $$|E| > 1 - \varepsilon,$$

2) $$f(x) = g(x), \qquad x \in E,$$

3) $$\frac{1}{2}\int_0^1 |f(x)|dx < \int_0^1 |g(x)|dx < 3\int_0^1 |f(x)|dx.$$

4) $$\int_0^1 |P(x) - g(x)|dx < \varepsilon.$$

5) $$\varepsilon > |c_k| \geq |c_{k+1}| > 0.$$

6) $$\max_{N_0 \leq m \leq N} \int_0^1 \left| \sum_{k=N_0}^{m} c_k \psi_{n_k}(x) \right| dx < 3\int_0^1 |f(x)|dx.$$

Proof. Consider the step function

$$\varphi(x) = \sum_{\nu=1}^{\nu_0} \gamma_\nu \cdot \chi_{\Delta_\nu}(x), \tag{2.12}$$

where Δ_ν are a-dyadic, not crosse intervals of the form $\Delta_m^{(k)} = [\frac{k-1}{a^m}, \frac{k}{a^m})$, $k = 1, 2, \cdots, a^m$ so that

$$0 < |\gamma_\nu|^2 |\Delta_\nu| < \frac{\varepsilon^3}{16a^2} \cdot \left(\int_0^1 |f(x)|dx\right)^2. \tag{2.13}$$

$$0 < |\gamma_1||\Delta_1| < \cdots < |\gamma_\nu||\Delta_\nu| < \cdots < |\gamma_{\nu_0}||\Delta_{\nu_0}| < \frac{\varepsilon}{2}.$$

$$\int_0^1 |f(x) - \varphi(x)|dx < \min\{\frac{\varepsilon}{4}; \frac{\varepsilon}{4}\int_0^1 |f(x)|dx\}. \tag{2.14}$$

Applying Lemma 2.1 successively, we can find the sets $E_\nu \subset [0,1)$ and a polynomial

$$P_\nu(x) = \sum_{k=N_{\nu-1}}^{N_\nu - 1} c_k \psi_{n_k}(x), \quad 1 \leq \nu \leq \nu_0,$$

which, for all $1 \leq \nu \leq \nu_0$, satisfy the following conditions:

$$|c_k| = |\gamma_\nu| \cdot |\Delta_\nu|, \qquad k \in [N_{\nu-1}, N_\nu) \tag{2.15}$$

$$|E_\nu| > (1-\varepsilon)\cdot|\Delta_\nu|, \tag{2.16}$$

$$P_\nu(x) = \begin{cases} \gamma_\nu & : \quad x \in \ E_\nu \\ 0 & : \quad x \notin \ \Delta_\nu, \end{cases}$$

$$\frac{1}{2}|\gamma_\nu|\cdot|\Delta_\nu| < \int_0^1 |P_\nu(x)|dx < 2|\gamma_\nu|\cdot|\Delta_\nu|. \tag{2.17}$$

$$\max_{N_{\nu-1}\le m\le N_\nu} \int_0^1 \Big|\sum_{k=N_0}^{m} c_k\psi_{n_k}(x)\Big| < a\cdot|\gamma_\nu|\cdot\sqrt{\frac{|\Delta_\nu|}{\varepsilon}}. \tag{2.18}$$

Define a set E, a function $g(x)$ and a polynomial $P(x)$ in the following away:

$$P(x) = \sum_{\nu=1}^{\nu_0} P_\nu(x) = \sum_{k=N_0}^{N} c_k\psi_{n_k}(x), \quad N = N_{\nu_0}-1. \tag{2.19}$$

$$g(x) = P(x) + f(x) - \varphi(x). \tag{2.20}$$

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu. \tag{2.21}$$

From (2.12),(2.14), (2.16)-(2.17), (2.19)-(2.21) we have

$$|E| > 1-\varepsilon\ ,$$

$$f(x) = g(x)\ , \qquad \text{for } x\in E,$$

$$\frac{1}{2}\int_0^1 |f(x)|dx < \int_0^1 |g(x)|dx < 3\int_0^1 |f(x)|dx.$$

By (2), (2.14), (2.15) and (2.20) we get

$$\int_0^1 |P(x)-g(x)|dx = \int_0^1 |f(x)-\varphi(x)|dx < \varepsilon.$$

$$\varepsilon > |c_k| \ge |c_{k+1}| > 0, \quad \text{for} \quad k = N_0, N_0+1, \cdots, N-1.$$

That is, assertions 1)-5) of Lemma 2.2 actually hold. We now verify assertion 6). For any number m, $N_0 \le m \le N$ we can find j, $1\le j\le \nu_0$ such that $N_{j-1} < m \le N_j$. then by (2.24) and (2.30) we have

$$\sum_{k=N_0}^{m} c_k\psi_{n_k}(x) = \sum_{n=1}^{j-1} P_n(x) + \sum_{k=N_{j-1}}^{m} c_k\psi_{n_k}(x).$$

hence and from relations (2.13), (2.14), (2.17), (2.18) we obtain

$$\begin{aligned}
&\int_0^1 \left| \sum_{k=N_0}^{m} c_k \psi_{n_k}(x) \right| dx \\
\leq\ & \sum_{\nu=1}^{\nu_0} \int_0^1 |P_\nu(x)| dx + \int_0^1 \left| \sum_{k=N_{j-1}}^{m} c_k \psi_{n_k}(x) \right| dx \\
<\ & 2 \int_0^1 |\varphi(x)| dx + a \cdot |\gamma_j| \cdot \sqrt{\frac{|\Delta_j|}{\varepsilon}} \\
<\ & 3 \int_0^1 |f(x)| dx.
\end{aligned}$$

□

3. Main results

Proof. Let

$$\{f_n(x)\}_{n=1}^{\infty} \tag{3.1}$$

be a sequence of all step functions, values and constancy interval endpoints of which are rational numbers. Applying Lemma 2.2 consecutively, we can find a sequences of functions $\{\overline{g}_n(x)\}$ of sets $\{E_n\}$ and a sequence of polynomials

$$\overline{P}_n(x) = \sum_{k=N_{n-1}}^{N_n - 1} c_{m_k} \psi_{m_k}(x), \quad N_0 = 1, \quad |c_{m_k}| > 0$$

which satisfy the conditions:

$$|E_n| > 1 - \varepsilon \cdot 4^{-8(n+2)} \tag{3.2}$$

$$f_n(x) = \overline{g}_n(x), \text{ for } \quad \text{all} \ \ x \in E_n, \tag{3.3}$$

$$\frac{1}{2} \int_0^1 |f_n(x)| dx < \int_0^1 |\overline{g}_n(x)| dx < 3 \int_0^1 |f_n(x)| dx. \tag{3.4}$$

$$\int_0^1 |\overline{P}_n(x) - \overline{g}_n(x)| dx < 4^{-8(n+2)}.$$

$$\max_{N_{n-1} \leq M \leq N_n} \int_0^1 \left| \sum_{k=N_{n-1}}^{M} c_{m_k} \psi_{m_k}(x) \right| dx < 3 \int_0^1 |f_n(x)| dx. \tag{3.5}$$

$$\frac{1}{n} > |c_{m_k}| > |c_{m_{k+1}}| > |c_{m_{N_n}}| > 0. \tag{3.6}$$

Set

$$\sum_{k=1}^{\infty} c_{m_k} \psi_{m_k}(x) = \sum_{n=1}^{\infty} \overline{P}_n(x) = \sum_{n=1}^{\infty} \sum_{k=N_{n-1}}^{N_n - 1} c_{m_k} \psi_{m_k}(x),$$

and

$$E=\bigcap_{n=1}^{\infty} E_n. \tag{3.7}$$

It is easy to see that (see (3.2)), $|E|>1-\varepsilon$.

Now we consider a series

$$\sum_{i=1}^{\infty} c_i\psi_i(x)$$

where $c_i=c_{m_k}\quad i\in[m_k,m_{k+1})$. From (3.6) it follows that $|c_i|\downarrow 0$.

Let given any function $f(x)\in L^1[0,1)$ then we can choose a subsequence $\{f_{s_n}(x)\}_{n=1}^{\infty}$ from (3.1) such that

$$\lim_{N\to\infty}\int_0^1\left|\sum_{n=1}^{N} f_{s_n}(x)-f(x)\right|dx=0, \tag{3.8}$$

$$\int_0^1|f_{s_n}(x)|dx\le\epsilon\cdot 4^{-8(n+2)}, n\ge 2,$$

where

$$\epsilon=\min\{\frac{\varepsilon}{2},\int_E|f(x)|dx\}. \tag{3.9}$$

We set

$$g_1(x)=\overline{g}_{s_1}(x),\quad P_1(x)=\overline{P}_{s_1}(x)=\sum_{k=N_{s_1-1}}^{N_{s_1}-1} c_{m_k}\psi_{m_k}(x) \tag{3.10}$$

It is easy to see that

$$\int_0^1|f(x)-f_{k_1}(x)|<\frac{\epsilon}{2}$$

Taking into account (3.4), (3.5) and (3.10) we have

$$\max_{N_{s_1-1}\le M\le N_{s_1}}\int_0^1\left|\sum_{k=N_{s_1-1}}^{M} c_{m_k}\psi_{m_k}(x)\right|dx<3\int_0^1|f_{s_1}(x)|dx<6\int_0^1|g_1(x)|dx.$$

Then assume that numbers $\nu_1,\nu_2,\cdots,\nu_{q-1}$ ($\nu_1=s_1$), functions $g_n(x)$, $f_{\nu_n}(x)$, $n=1,2,\cdots,q-1$ and polynomials

$$P_n(x)=\sum_{k=M_n}^{\overline{M}_n} c_{m_k}\psi_{m_k}(x),\quad M_n=N_{\nu_n-1},\quad \overline{M}_n=N_{\nu_n}-1,$$

are chosen in such a way that the following condition is satisfied:

$$g_n(x)=f_{s_n}(x),\quad x\in E_{\nu_n},\quad 1\le n\le q-1, \tag{3.11}$$

$$\int_0^1|g_n(x)|dx<4^{-3n}\epsilon,\quad 1\le n\le q-1,$$

$$\int_0^1\left|\sum_{k=2}^{n}(P_k(x)-g_k(x))\right|dx<4^{-8(n+1)}\epsilon,\quad 1\le n\le q-1, \tag{3.12}$$

$$\max_{M_n \le M \le \overline{M}_n} \int_0^1 \left| \sum_{k=M_n}^{M} c_{m_k} \psi_{m_k}(x) \right| dx < 4^{-3n}\epsilon, \quad 1 \le n \le q-1. \tag{3.13}$$

We choose a function $f_{\nu_q}(x)$ from the sequence (3.1) such that

$$\int_0^1 \left| f_{\nu_q}(x) - \left[f_{s_q}(x) - \sum_{k=2}^{n} (P_k(x) - g_k(x)) \right] \right| dx < 4^{-8(q+2)}\epsilon. \tag{3.14}$$

This with (3.8) imply

$$\int_0^1 \left| f_{\nu_q}(x) - \sum_{k=2}^{n} (P_k(x) - g_k(x)) \right| dx < 4^{-8q-1}\epsilon,$$

and taking into account relation (3.14) we get

$$\int_0^1 |f_{\nu_q}(x)| dx < 4^{-8q}\epsilon.$$

We set

$$P_q(x) = \overline{P}_{\nu_q}(x) = \sum_{k=M_q}^{\overline{M}_q} c_{m_k} \psi_{m_k}(x), \tag{3.15}$$

where

$$M_q = N_{\nu_q - 1}, \quad \overline{M}_q = N_{\nu_q} - 1,$$

$$g_q(x) = f_{s_q}(x) + [\overline{g}_{\nu_q}(x) - f_{\nu_q}(x)] \tag{3.16}$$

By (3.3)-(3.5), (3.12)-(3.16) we have

$$g_q(x) = f_{s_q}(x), \quad x \in E_{\nu_q}, \tag{3.17}$$

$$\begin{aligned} &\int_0^1 |g_q(x)| dx \\ \le\ &\int_0^1 \left| f_{\nu_q}(x) - \left[f_{s_q}(x) - \sum_{k=2}^{n} (P_k(x) - g_k(x)) \right] \right| dx \\ &+ \int_0^1 |\overline{g}_{\nu_q}(x)| dx + \int_0^1 \left| \sum_{k=2}^{n} (P_k(x) - g_k(x)) \right| dx \\ <\ &4^{-3n}\epsilon, \end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&\int_0^1 \left| \sum_{k=2}^{q} (P_k(x) - g_k(x)) \right| dx \\
\leq\ & \int_0^1 \left| f_{\nu_q}(x) - \left[f_{s_q}(x) - \sum_{k=2}^{n} (P_k(x) - g_k(x)) \right] \right| dx \\
& + \int_0^1 |\overline{P}_{\nu_q}(x) - \overline{g}_{\nu_q}(x))| dx \\
<\ & 4^{-8(n+1)} \epsilon,
\end{aligned}
$$

$$
\max_{M_q \leq M \leq \overline{M}_q} \int_0^1 \left| \sum_{k=M_q}^{M} c_{m_k} \psi_{m_k}(x) \right| dx \leq 3 \int_0^1 |f_{\nu_q}(x)| dx < 4^{-3n} \epsilon. \tag{3.19}
$$

Thus, by induction we can choose the sequences of sets $\{E_q\}$, functions $\{g_q(x)\}$ and polynomials $\{P_q(x)\}$ such that conditions (3.17) - (3.19) are satisfied for all $q \geq 1$. Define a function $g(x)$ and a series in the following away:

$$
g(x) = \sum_{n=1}^{\infty} g_n(x), \tag{3.20}
$$

$$
\sum_{n=1}^{\infty} \delta_n c_n \psi_n(x) = \sum_{n=1}^{\infty} \left[\sum_{k=M_n}^{\overline{M}_n} c_{m_k} \psi_{m_k}(x) \right], \tag{3.21}
$$

where

$$
\delta_n = \begin{cases} 1\ , \ \text{if } i = m_k, \quad \text{where} \quad k \in \bigcup_{q=1}^{\infty} [M_q, \overline{M}_q] \\ \\ 0, \quad \text{in the other case} \ . \end{cases}
$$

Hence and from relations (3.4), (3.7), (3.11), (3.20) ,

$$
g(x) = f(x), \qquad x \in E, \quad g(x) \in L^1[0,1),
$$

$$
\frac{1}{2} \int_0^1 |f(x)| dx < \int_0^1 |g(x)| dx < 4 \int_0^1 |f(x)| dx. \tag{3.22}
$$

Taking into account (3.15), (3.18)-(3.21) we obtain that the series (3.21) convergence to $g(x)$ in $L^1[0,1)$ metric and consequently is its Fourier series by Ψ_a system, $a \geq 2$.

From Definition 1.3, and from relations (3.9), (3.13), (3.22) for any natural number m there is N_m so that

$$
\begin{aligned}
||G_m(g)||_1 = ||S_m(g)||_1 &= \int_0^1 \left| \sum_{n=1}^{\infty} \delta_n c_n \psi_n(x) \right| dx \\
&\leq 4 \int_0^1 |f(x)| dx \\
&\leq \sum_{n=1}^{\infty} \left(\max_{M_n \leq M \leq \overline{M}_n} \int_0^1 \left| \sum_{k=M_n}^{M} c_{m_k} \psi_{m_k}(x) \right| dx \right) \\
&\leq 2 \int_0^1 |g_1(x)| dx + \epsilon \cdot \sum_{n=2}^{\infty} 4^{-n} \\
&\leq 3 \int_0^1 |g(x)| dx \leq 12 \int_0^1 |f(x)| dx = 12||f||_1.
\end{aligned}
$$

□

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