ON A CLASS OF ADMISSIBLE PARTITIONS[1]

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0. Introduction and summary. In many multi-decision problems the existence of admissible decision functions (for definitions we refer to [5]) depends upon the existence of corresponding partitions of the sample space into regions of specified shape. Usually the requirements for such statistical partitions differ from those relating to less restricted partitions of a set S according to a vector of finite measures on the measurable subsets of S, usually referred to under the general title of the "ham sandwich problem" (see, e.g., [3], where further references may be found). Nevertheless, as indicated here, the solutions to a wide class of division problems rest very heavily on the fundamental result of Lyapunov and generalizations of this (see, e.g., [4]).

Several multi-decision problems (Section 4 below) relating to the mean μ of a k-variate normal distribution $N(\mu, \Sigma)$ reduce to the problem of locating (hence called "topothetical," cf. [2]) the parameter point μ into one of $k + 1$ convex k-dimensional polyhedral cones $\omega_1, \omega_2, \cdots, \omega_{k+1}$ (hereafter referred to as "cones") with common vertex μ_0 which form a partition of the parameter space E_k of μ (see (1) below). Let us identify the sample space E_k of an observation X from $N(\mu, \Sigma)$ with the parameter space E_k. It was shown in [2] that the family R_ω of all translations $R(\tau) = (R_1(\tau), \cdots, R_{k+1}(\tau))$ of the system $\omega = (\omega_1, \cdots, \omega_{k+1})$ (see Definition 2) defines a class of admissible procedures, henceforth referred to as partitions; the decision d_i that $\mu \varepsilon \omega_i$ is taken when the actual observation $x \varepsilon R_i$. Furthermore, there exists a unique partition $R(\tau_0) \varepsilon R_\omega$ which is minimax. The minimax character of $R(\tau_0)$ amounts to the following proposition: There exists a unique, in R_ω, partition of E_k into $k + 1$ cones with the same probability content under the normal k-variate distribution (Corollary 5.1 of [2]). The distribution may be assumed spherical normal (unit variance in any direction) without any loss of generality, since a nonsingular linear transformation T such that $T\Sigma T' = I$ preserves the shape of the partition ω.

The purpose of this note is to extend the above result to the case of arbitrary probability contents for such conical regions (Theorem 1), and at the same time show how the corresponding partitions are related to classes of admissible partitions for a family of classification and topothetical problems relating to the normal mean μ. Several problems which have been extensively studied in the statistical literature emerge as special cases of our general topothetical problem (Section 4).

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1. Preliminary results. For the proof of the main result (Theorem 1), we require certain preliminary results, which are of some interest in themselves. First some notation and definitions.

Let σ denote an arbitrary but fixed k-simplex with vertices $\mu_1, \cdots, \mu_{k+1}$ and center (i.e., the center of the hypersphere passing through the points $\mu_1, \cdots, \mu_{k+1}$) the common vertex μ_0 of the cones $\omega_1, \cdots, \omega_{k+1}$, such that for each $i = 1, \cdots, k+1$, if d denotes the usual distance function in E_k,

$$\omega_i = \{\mu \in E_k : \delta_{ij}(\mu) \leqq 0, j \neq i, j = 1, \cdots, k+1\} \tag{1}$$

where for $i \neq j$,

$$\delta_{ij}(\mu) = d^2(\mu, \mu_i) - d^2(\mu, \mu_j) = (2\mu - \mu_i - \mu_j)'(\mu_j - \mu_i). \tag{2}$$

Remark. The simplex σ is characterized by the property that the k bounding faces of ω_i are the perpendicular bisectors of the edges of σ emanating from the vertex μ_i; the hyperplane $\delta_{ij}(\mu) = 0$ is perpendicular to the edge (μ_i, μ_j). However, the exposition below shows that any k-simplex whose edges through vertex μ_i are perpendicular to the bounding hyperplanes of ω_i would suffice for our purposes. Note also that convexity of ω_i is essential for the existence of σ above.

Definition 1. For any k-simplex σ^* the corresponding classification problem of choosing one of its $k+1$ vertices as the true mean μ on the basis of an observation X from $N(\mu, I)$ will be called the σ^*-classification problem.

Assumption. Throughout this paper we assume a simple loss function, i.e., 0 or 1 according to whether a correct or incorrect decision is taken. Therefore the risk function becomes the probability of error.

Definition 2. Any partition R in R_ω, defined as a translation of the system $\omega = (\omega_1, \cdots, \omega_{k+1})$, will be called a similar partition to ω. The class R_ω coincides with the totality of partitions $R = (R_1, \cdots, R_{k+1})$ where

$$R_i = \{x \in E_k : \delta_{ij}(x) \leqq c_i - c_j, j \neq i\}, \qquad i = 1, \cdots, k+1,$$

and $c = (c_1, \cdots, c_{k+1})$ is a constant vector of non-negative components (cf. [1]).

The following two lemmas summarize relevant results obtained in [2]:

Lemma 1. *The class of conical partitions R_ω is*

(a) *the minimal complete class of partitions (procedures) for the σ-classification problem* (cf. *Theorem* 6.7.1 *of* [1]),

(b) *an admissible class of partitions for the topothetical problem of locating μ into one of the cones $\omega_1, \cdots, \omega_{k+1}$* (*Theorem* 6.4 *of* [2]).

Lemma 2. *The (unique) admissible minimax partition for the σ-classification problem is such*

(a) *for the topothetical problem of locating μ into one of the conical regions $\omega_i(\sigma) = \{\mu \in E_k : \delta_{ij}(\mu) \leqq -d^2(\mu_i, \mu_j), j \neq i\}$, $i = 1, \cdots, k+1$, with vertices μ_i, respectively, the complement of $\omega_1(\sigma) + \omega_2(\sigma) + \cdots + \omega_{k+1}(\sigma)$ constituting an indifference region* (*Theorem* 5.1 *of* [2]),

(b) *for the topothetical problem of locating μ into one of the subsets $\omega_1^*, \cdots, \omega_{k+1}^*$*

of the $\omega_1(\sigma), \cdots, \omega_{k+1}(\sigma)$, *respectively, provided* ω_i^* *contains the point* μ_i *and the complement of* $\omega_1^* + \omega_2^* + \cdots + \omega_{k+1}^*$ *constitutes an indifference region.*

(Follows immediately from (a) and the following Lemma.)

LEMMA 3. *For each* $r > 0$, *denote by* σ_r *the homothetic k-simplex of* σ *with center of similitude the point* μ_0, *vertices* $\mu_1(r), \cdots, \mu_{k+1}(r)$ *and homothetic ratio* r (i.e., $|\mu_0 - \mu_i(r)| = r\,|\mu_0 - \mu_i|$). *Define* $\omega_i(\sigma_r)$ *analogously to* $\omega_i(\sigma)$ *of Lemma* 2. *Then for each* $i = 1, \cdots, k+1$ *and every partition* $R = (R_1, \cdots, R_{k+1}) \,\varepsilon\, R_\omega$ *we have*

$$\min_{\mu \varepsilon \omega_i(\sigma_r)} P[X \,\varepsilon\, R_i \mid \mu] = P[X \,\varepsilon\, R_i \mid \mu = \mu_i(r)].$$

Lemma 3 is equivalent to Lemma 5.3 of [2].

LEMMA 4. *For each* $r > 0$, *define* σ_r *as in Lemma* 3. *Let* $p_i(\delta)$ *denote the probability of correctly taking decision* D_i *that* $\mu = \mu_i(r)$ *in the* σ_r*-classification problem when using procedure* δ. *Then*

(i) *for any vector* $\alpha = (\alpha_1, \cdots, \alpha_{k+1})$ *of positive components with* $\alpha_1 + \cdots + \alpha_{k+1} = 1$, *there exists a unique partiton* R^r *similar to* ω (i.e., $R^r \,\varepsilon\, R_\omega$) *with*

$$p_i(r) \equiv p_i(R^r) = \alpha_i, \qquad i = 1, \cdots, k; \tag{3}$$

(ii) $p_{k+1}(r)$ *is a (strictly) increasing and continuous function of* r *in* $(0, \infty)$ *and*

$$\lim_{r \to 0} p_{k+1}(r) = \inf_{r>0} p_{k+1}(r) = \alpha_{k+1}. \tag{4}$$

PROOF. Let $p(\delta) = (p_1(\delta), \cdots, p_{k+1}(\delta))$. It may be shown [4] that the set of points $p(\delta)$ for all decision functions δ constitutes a convex and compact subset M of E_{k+1} contained in the unit hypercube K and containing all the corners of K with coordinates adding to 1. The "upper" surface U of M corresponds to the set of admissible procedures, which by Lemma 1 (a) are the similar partitions R_ω to ω. The line parallel to the $k+1$-coordinate axis through the point $(\alpha_1, \cdots, \alpha_k, 0)$ intersects U in a single point, namely, the $p(r)$ corresponding to the admissible partition R^r which satisfies (3).

For (ii) note first that $p_{k+1}(r) > \alpha_{k+1}$ for $r > 0$, since otherwise the completely randomized ("guess") procedure δ_0 with $p_i(\delta) = \alpha_i$, $i = 1, \cdots, k+1$, would also be admissible.

Now observe that the (unique) admissible partitions $R^r = (R_1^r, \cdots, R_{k+1}^r)$ and $R^{r'} = (R_1^{r'}, \cdots, R_{k+1}^{r'})$ for the σ_r- and $\sigma_{r'}$-classification problems, respectively, which satisfy

$$p_i(r) = p_i(r') = \alpha_i, \qquad i = 1, \cdots, k, \tag{5}$$

have the following relation: the partitioning point $\tau_{r'}$, which defines $R^{r'}$ as the translation of $\tau_{r'}$ of ω, lies in R_{k+1}^r whenever $r' < r$. To see this note that if $\tau_{r'}$ were in the complement $\bar{R}_{k+1}^r$ of R_{k+1}^r, then at least one of the $R_i^{r'}$, say $R_{i*}^{r'}$ would be a proper subset of R_{i*}^r. But then, by Lemma 3, we would have

$$P[X \,\varepsilon\, R_{i*}^r \mid \mu = \mu_{i*}(r')] < P[X \,\varepsilon\, R_{i*}^r \mid \mu = \mu_{i*}(r)] = \alpha_{i*},$$

and hence also

$$P[X \varepsilon R_{i^*}^{r'} \mid \mu = \mu_{i^*}(r')] < \alpha_{i^*}, \qquad \text{for some} \quad i^* = 1, \cdots, k,$$

which contradicts (5). Therefore $\tau_{r'}$ lies in the interior of R_{k+1}^r, and, by the same argument of Lemma 3,

$$p_{k+1}(r') = P[X \varepsilon R_{k+1}^{r'} \mid \mu = \mu_{k+1}(r')] < P[X \varepsilon R_{k+1}^{r'} \mid \mu = \mu_{k+1}(r)]$$
$$< P[X \varepsilon R_{k+1}^{r} \mid \mu = \mu_{k+1}(r)] = p_{k+1}(r).$$

Since the continuity of $p_{k+1}(r)$ is an immediate consequence of the continuity of the normal distribution, the monotonicity and continuity of $p_{k+1}(r)$ have been established.

Finally (4) follows from the fact that $p_{k+1}(r)$ is bounded below by α_{k+1}; for when $r = 0$, the vertices $\mu_1(0), \cdots, \mu_{k+1}(0)$ coincide with the point μ_0 and clearly the σ_0-classification problem degenerates. However, for the topothetical problem of locating μ into one of the cones $\omega_1, \cdots, \omega_{k+1}$ intersecting at μ_0, if $\alpha_i(\delta)$ denotes the minimum probability of correctly taking decision d_i that $\mu \varepsilon \omega_i$ when the decision rule δ is used, then no δ with $\alpha_i(\delta) = \alpha_i$, $i = 1, \cdots, k$ can improve, in a minimax sense, upon the completely randomized (inadmissible, cf. [2]) δ_0 with $\alpha_i(\delta_0) = \alpha_i$, $i = 1, \cdots, k + 1$, i.e., for all these δ, $\alpha_{k+1}(\delta) = \alpha_{k+1}$. But $p_i(r) = \alpha_i(R^r)$, $i = 1, \cdots, k + 1$ (by Lemma 3), and, therefore, as $r \to 0$,

$$p_{k+1}(r) \to \alpha_{k+1} = \inf_{r>0} p_{k+1}(r),$$

which completes the proof of the lemma.

Lemma 5. *For $0 < r \leqq 1$, the points τ_r which determine the admissible partitions R^r of Lemma 4 with $p_i(r) = \alpha_i$, $i = 1, \cdots, k$, lie in a compact subset T of E_k.*

Proof. Let $0 < \epsilon_i < \alpha_i$ such that $\epsilon_i < \frac{1}{2}$, $i = 1, \cdots, k + 1$. Then there are finite negative constants c_{ij} such that for each $i = 1, \cdots, k + 1$, since X is $N(\mu, I)$,

$$(6) \qquad P[\delta_{ij}(X) < c_{ij} \mid \mu = \mu_i] = \epsilon_i, \qquad j \neq i.$$

Define

$$T_{ij} = \{x \varepsilon E_k : \delta_{ij}(x) < c_{ij}\}, \qquad i \neq j.$$

Note that τ_1 must be in the complement $\bar{T}_{ij}$ of T_{ij} for each $i \neq j$, since otherwise, R_i^1 being a subset of T_{ij} for each $j \neq i$,

$$P[X \varepsilon R_i^1 \mid \mu = \mu_i] < P[X \varepsilon T_{ij} \mid \mu = \mu_i] = \epsilon_i < \alpha_i.$$

Hence τ_1 must lie in the intersection $T = \bigcap_{i \neq j} \bar{T}_{ij}$; this is the desired compact set. To see this note that each $\bar{T}_{ij}$ is a half-space determined by the hyperplane $H_{ij} : \delta_{ij}(x) = c_{ij}$ such that, by (6), $\mu_0 \varepsilon \bar{T}_{ij}$; each H_{ij} intersects every other hyperplane except $H_{ji} : \delta_{ji}(x) = c_{ji}$, and therefore the set T is in general the boundary and the interior of a polyhedron. Hence T is a compact set. Now if $0 < r < 1$, then again τ_r must lie in T, since, for each i, $i = 1, \cdots, k + 1$, if $\tau_r \varepsilon T_{ij}$ then by Lemma 3

$$P[\delta_{ij}(X) < c_{ij} \mid \mu = \mu_i(r)] < P[\delta_{ij}(X) < c_{ij} \mid \mu = \mu_i] = \epsilon_i < \alpha_i,$$

which contradicts either (3) (since $P[X \varepsilon R_i^r \mid \mu = \mu_i(r)] < P[\delta_{ij}(X) < c_{ij} \mid \mu = \mu_i(r)]$) or $p_{k+1}(r) > \alpha_{k+1}$ (by Lemma 4 (ii)).

2. Main results. We are now ready for the proof of the main result of

THEOREM 1. *Given any vector $\alpha = (\alpha_1, \cdots, \alpha_{k+1})$ of positive components with $\alpha_1 + \cdots + \alpha_{k+1} = 1$, there exists a unique partition $R(\tau(\alpha))$ similar to ω such that*

$$P[X \varepsilon R_i(\tau(\alpha)) \mid \mu = \mu_0] = \alpha_i, \qquad i = 1, \cdots, k+1.$$

PROOF. Consider a decreasing sequence $\{r_n\}$ converging to 0; then the corresponding sequence of points τ_{r_n} of Lemma 4 has at least one limit point τ^* (which lies in the compact set T of Lemma 5). Let $R(\tau^*)$ denote the similar partition to ω with vertex τ^*. It follows from Lemma 4 that

$$(7) \qquad \begin{aligned} P[X \varepsilon R_{k+1}(\tau^*) \mid \mu = \mu_0] &= \lim_{r \to 0} p_{k+1}(r) = \alpha_{k+1}, \\ P[X \varepsilon R_i(\tau^*) \mid \mu = \mu_0] &= p_i(r) = \alpha_i, \qquad i = 1, \cdots, k. \end{aligned}$$

Furthermore, there is no other partition similar to ω satisfying (7) since if $\tau' \neq \tau^*$ was another point defining a similar partition $R(\tau') = (R_1(\tau'), \cdots, R_{k+1}(\tau'))$, then at least one of the $R_i(\tau')$, $R_{i^*}(\tau')$ say, would be a subset of the $R_{i^*}(\tau^*)$ and $P[X \varepsilon R_{i^*}(\tau') \mid \mu = \mu_0] < \alpha_{i^*}$. Taking $R(\tau(\alpha)) = R(\tau^*)$ completes the proof.

From Lemma 1 and Theorem 1, we obtain immediately

COROLLARY 1. *Given any vector $\alpha = (\alpha_1, \cdots, \alpha_{k+1})$ of positive components such that $\sum_{i=1}^{k+1} \alpha_i = 1$, the partition $R(\tau(\alpha))$ which is similar to ω and satisfies*

$$P[X \varepsilon R_i(\tau(\alpha)) \mid \mu = \mu_0] = \alpha_i, \qquad i = 1, \cdots, k+1,$$

is an admissible partition for the topothetical problem of locating μ into one of $\omega_1, \cdots, \omega_{k+1}$ on the basis of X from $N(\mu, I)$, and it takes decision d_i correctly with probability at least α_i, $i = 1, \cdots, k+1$. The partition corresponding to $\alpha_1 = \alpha_2 = \cdots = \alpha_{k+1} = (k+1)^{-1}$ is an admissible minimax for the same problem.

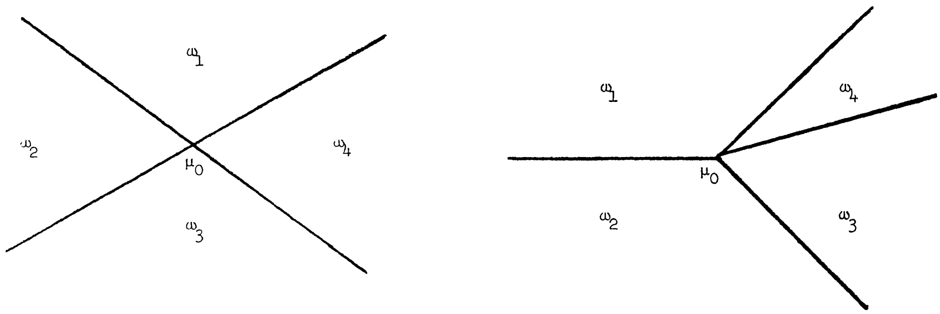
Also, combining the preceding discussion with the observation that there exists one-to-one correspondence between the set of similar partitions $R(\tau)$, and the corresponding probability vectors $\alpha(\tau)$ gives

COROLLARY 2. *Given any vector $\alpha = (\alpha_1, \cdots, \alpha_{k+1})$ as above and any partition R of E_k into $k+1$ convex polyhedral cones with the same vertex, there exists a unique point $\mu(\alpha, R)$ such that*

$$P[X \varepsilon R_i \mid \mu = \mu(\alpha, R)] = \alpha_i, \qquad i = 1, \cdots, k+1.$$

In addition, the partition R is admissible for the topothetical problem of locating the normal mean μ into one of the $k+1$ cones which constitute the similar partition to R with vertex $\mu(\alpha, R)$; the α_i's determine the minimum probabilities of correct decision.

3. Partitioning E_k into more than $k+1$ regions. So far we have considered the case of $k+1$ convex polyhedral conical regions in E_k. The question now



(a) (b)

FIG. 1. Examples of nonexistence of four equiprobable angular regions under the circular normal distribution centered at μ_0.

arises as to whether it is possible to partition E_k into more than $k + 1$ regions of the same type with preassigned probability contents. One can easily construct examples, such as the ones indicated in Figure 1 for $k = 2$, which show that, in general, this is no longer possible. This, in a way, reflects the fact that in a classification problem with more than $k + 1$ alternatives the class of admissible procedures is no more determined by regions of the type considered, though still each of the regions is bounded by hyperplanes. (A study of such regions and related topothetical problems will appear in a subsequent paper.) Thus, in the case of $k + 2$ alternatives, the specification of an admissible partition, roughly speaking, requires in general, besides the directions of the bounding hyperplanes, the specification of two points and not one (the translation vector τ) as in the case of $k + 1$ alternatives. Indeed, it should be observed that the limiting argument employed before rests very heavily on the conical shape of the admissible partitions.

Analogous counter-examples can easily be constructed for the case of unequal probabilities α_i, as well as for the case of more than $k + 2$ cones.

REMARK. The nonexistence of similar partitions to a given partition ω with specified probability contents under the k-variate normal distribution when the number m of conical components of ω are more than $k + 1$ should be expected in view of the fact that a similar partition to ω is completely specified by its vertex V, and hence there are k unknowns, the coordinates of V, whereas, there are $m - 1 > k$ independent equations which, in general, do not have a solution. For $m = k + 1$ however, the k independent equations corresponding to any probability vector $\alpha = (\alpha_1, \cdots, \alpha_{k+1})$, $\alpha_i > 0$, have, by Theorem 1, a unique solution $V = V_\alpha$. Nevertheless, the author has not been able to prove the result without the assumption of convexity of the $k + 1$ cones.

4. Some applications. Several multi-decision problems concerning normal population means may be reduced to the topothetical problem of locating a k-variate normal mean into one of $k + 1$ convex polyhedral cones. If an indifference region is properly chosen (cf. Lemma 2 and [2]) then a unique admissible minimax partition may be found.

Following are some multiple-decision problems lending themselves to our topothetical approach. The reader may certainly find more such examples.

EXAMPLE 1. *Selecting the Largest Mean.* On the basis of n independent observations $x_\alpha = (x_{1\alpha}, \cdots, x_{p\alpha})$, $\alpha = 1, \cdots, n$ on X distributed according to $N(\mu, \Sigma)$, choose the largest component of $\mu = (\mu_1, \cdots, \mu_p)$ assuming that the covariance matrix Σ is known. By sufficiency and invariance considerations the selection procedures may be based on a maximal invariant, e.g., $y = (\bar{x}_2 - \bar{x}_1, \cdots, \bar{x}_p - \bar{x}_1)$ where $\bar{x}_i = n^{-1} \sum_{i=1}^{n} x_{i\alpha}$, $i = 1, \cdots, p$. Clearly y is $N(\delta, \Sigma^*)$, where $\delta = (\mu_2 - \mu_1, \cdots, \mu_p - \mu_1)$ and Σ^* is a known $(p-1) \times (p-1)$ positive definite matrix. It is seen that the decision d_i, that μ_i is the largest, is appropriate when δ lies in a convex $(p-1)$-dimensional polyhedral cone in the $(p-1)$-space of δ. The same reduction holds when we have n_i observations on the ith component of X. In the statistical literature the case of $\Sigma = \sigma^2 I$ has been studied quite extensively both when σ^2 is known or unknown.

EXAMPLE 2. *A Slippage Problem.* This may be obtained as a special case of Example 1 if we assume that all the components of μ are equal except one which is larger (slips to the right by some positive amount $\Delta > 0$). Here we do not allow the possibility of all the μ_i being equal. The decision d_i that μ_i slipped corresponds to points δ on a ray through the origin. It follows that if Δ is bounded away from zero, i.e., $\Delta > \epsilon > 0$, then by Lemma 2 (b) there exists a unique invariant procedure which is minimax and which is admissible among invariant procedures.

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