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SAMPLE PATH VARIATIONS OF HOMOGENEOUS PROCESSES

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1. Introduction. Let $X(t) = X(t, \omega)$ denote the variable at epoch t of a homogeneous process (process with stationary independent increments) normalized so that $X(0) = 0$ and almost all sample paths are right continuous and possess left limits at all epochs t. This paper is concerned with limits of sums

$$(1.1) \qquad Z(f, \mathfrak{S}) = \sum_{\{t_k \varepsilon \mathfrak{S}\}} f(X(t_{k+1}) - X(t_k))$$

where f is a certain non-negative function on the real line and $\mathfrak{S} = \{t_0, \cdots, t_n\}$ is a partition of $[0, t]$, the limit being taken along a sequence of partitions $\mathfrak{S}$ as the mesh of $\mathfrak{S}$ tends to zero.

For some results concerning this type of transformation of a process, see Blumenthal & Getoor [1], Bochner [2], Cogburn and Tucker [3] and Fristedt [5]. Rather than considering a very special class of functions f as in the above papers, we consider rather general functions f acting on a rather special class of processes X. The most important class of processes treated is the class of strictly stable processes.

2. Powerfully continuous processes. Let μ_t denote the distribution measure of $X(t)$, and let M denote the Levy measure for the weakly continuous convolution semigroup μ_t. We know (see, e.g., Feller [4]) that

$$(2.1) \qquad \int_{R\sim\{0\}} f\, d(t^{-1}\mu_t) \to \int_{R\sim\{0\}} f\, dM$$

for every bounded continuous function f on R such that $f(x) = O(x^2)$ near 0. We shall set $M\{0\} = 0$, even though M is usually not defined at 0, to avoid some unpleasantness. The set of possible Levy measures M is characterized by the conditions

$$(2.2) \qquad M(\Lambda) < \infty \text{ if } \Lambda \text{ is bounded away from 0, and}$$

$$(2.3) \qquad \int_{-1}^{1} x^2 M(dx) < \infty.$$

In case $X(t)$ is an increasing process, the μ_t are concentrated in $[0, \infty)$, and so therefore is M. The set of possible Levy measures for increasing processes is characterized by the conditions (2.2) and

$$(2.4) \qquad \int_0^1 xM(dx) < \infty.$$

DEFINITION. The weakly continuous convolution semigroup μ_t with Levy measure M is called powerfully continuous on the open set S in case there exists

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an absolute constant c such that for all $t > 0$ and all open subintervals B of S,

$$\mu_t(B) \leqq ctM(B). \tag{2.5}$$

We always let c denote an absolute constant, not necessarily the same at each occurrence.

LEMMA 2.1. *Let μ_t be powerfully continuous on S, and let M be the Levy measure for μ_t. Then, for every Borel set $\Lambda \subset S$,*

$$\mu_t(\Lambda) \leqq ctM(\Lambda). \tag{2.6}$$

PROOF. If Λ is open, it is a countable union of open intervals, and (2.6) follows immediately from (2.5). If $\Lambda \subset S$ is a Borel set with $0 < M(\Lambda) < \infty$, there exists an open set G such that $\Lambda \subset G \subset S$ and $M(G \cap \Lambda^c) < \frac{1}{2}M(\Lambda)$, so that $\mu_t(\Lambda) \leqq \mu_t(G) \leqq ctM(G) \leqq 2cM(\Lambda)$. In case $M(\Lambda) = \infty$, there is nothing to prove, and if $M(\Lambda) = 0$, we can find an open set G, $\Lambda \subset G \subset S$ such that $M(G)$ is arbitrarily small, and we see then that $\sup_t t^{-1}\mu_t(\Lambda)$ can be made arbitrarily small, hence zero. ∎

LEMMA 2.2. *Suppose M has infinite total mass, and that μ_t is powerfully continuous on the open set S and has Levy measure M. Then M has no atoms in S.*

PROOF. If $a \in S$ and $a = 0$, we are done since $M\{0\} = 0$, by convention. If $a \in S$ and $a \neq 0$, there exists a decreasing sequence $\{J_n\}$ of open subintervals of S, each bounded away from 0, such that $\cap J_n = \{a\}$, and no endpoint of any J_n is an atom of M. We know that

$$t^{-1}\mu_t\{a\} = t^{-1}\mu_t(J_1) - \sum_{n\geqq 1} t^{-1}\mu_t(J_n \cap J^c_{n+1}) \to M(J_1) - \sum_{n\geqq 1} M(J_n \cap J^c_{n+1})$$

as $t \to 0$, using the dominated convergence theorem for infinite sums. The last expression is exactly $M\{a\}$, and since M has infinite total mass, a well known result of Hartman and Wintner [6] implies that $\mu_t\{a\} = 0$ for all t, so $M\{a\} = 0$. ∎

LEMMA 2.3. *For any Borel function $f \geqq 0$ with support in S, if M has infinite total mass, then*

$$\int_S f\, d\mu_t \leqq ct \int_S f\, dM, \quad \text{and} \tag{2.7}$$

$$\int_S f\, d(t^{-1}\mu_t) \to \int_S f\, dM \quad \text{as } t \to 0, \qquad \text{if } \int_S f\, dM < \infty. \tag{2.8}$$

PROOF. The inequality (2.7) is true for indicator functions, their positive linear combinations, and increasing limits of these functions, and so is true for all positive Borel functions. For (2.8), we know its truth if $f = 1_\Lambda$, Λ an interval in S with $M(\Lambda) < \infty$, because of Lemma 2.2. To prove (2.8) for an arbitrary indicator function $f = 1_\Lambda$, $\Lambda \subset S$, we use the regularity of M to approximate 1_Λ by 1_{Λ^0}, where Λ^0 is a finite union of open intervals in S, letting $M(\Lambda \triangle \Lambda^0)$ be as small as desired. The convergence (2.8) holds for 1_{Λ^0}, and $\int |1_\Lambda - 1_{\Lambda^0}|\, dM = M(\Lambda \triangle \Lambda^0)$, whilst $\int |1_\Lambda - 1_{\Lambda^0}|\, d(t^{-1}\mu_t) \leqq cM(\Lambda \triangle \Lambda^0)$, so that an easy estimate proves (2.8) for arbitrary indicator functions.

To pass to the most general function f with support in S we define a function

q by the rules

$$(2.9) \qquad \text{(i)} \quad q = 1 \quad \text{on} \quad (-\infty, -1] \cup [1, \infty),$$
$$\text{(ii)} \quad q = 2^{-2k} \quad \text{on} \quad (-2^{-k}, -2^{-(k+1)}] \cup [2^{-(k+1)}, 2^{-k}).$$

Clearly, $\min(x^2, 1) \leqq q(x) \leqq \min(4x^2, 1)$, so $\int q\, dM < \infty$. For any non-negative function f on R, we define $f^{(n)}$ by

$$(2.10) \qquad \text{(i)} \quad \text{on } (-\infty, -1] \cup [1, \infty), f^{(n)} = j \cdot 2^{-n} \text{ if } j \cdot 2^{-n} \leqq f < (j+1) \cdot 2^{-n}$$
$$\text{(ii)} \quad \text{on } (-2^{-k}, -2^{-(k+1)}] \cup [2^{-(k+1)}, 2^{-k}),\ f^{(n)} = j \cdot 2^{-(n+2k)} \quad \text{if}$$
$$j \cdot 2^{-(n+2k)} \leqq f < (j+1) 2^{-(n+2k)}.$$

Then $0 \leqq f - f^{(n)} < 2^{-n} q$, and $f^{(n)}$ is an infinite positive linear combination of indicator functions. The usual dominated convergence argument for infinite sums shows that (2.8) holds when $f = f^{(n)}$, and a standard estimate shows that (2.8) is true in general. ▯

If M has finite total mass, and the process has no Gaussian component, the process is in fact compound Poisson, and the paths are a.s. step functions with a finite number of jumps. In this case, our problem is rather trivial. We henceforth always assume that the total mass of M is infinite, so that the results of Lemma 2.2 and Lemma 2.3 hold. Also, one easily sees that the paths $X(t)$ have no constancy intervals, almost surely.

3. Examples of powerfully continuous processes. To see that we are dealing with a somewhat interesting class of processes, we shall show that our theory applies to strictly stable non-Gaussian processes, and to processes which can be subordinated by increasing stable processes.

THEOREM 3.1. *If μ_t can be subordinated by a subordinator process which is powerfully continuous on $(0, \infty)$, then μ_t is powerfully continuous on $(-\infty, \infty)$.*

PROOF. If $Y(t) = X(T(t))$, where $T(t)$ is a powerfully continuous increasing process with semigroup ν_t and Levy measure N, $X(s)$ is a homogeneous process independent of T, and having semigroup λ_t with Levy measure L, the process $Y(t)$ has semigroup μ_t and Levy measure M given by

$$(3.1) \qquad \mu_t(\Lambda) = \int_0^\infty \lambda_s(\Lambda) \nu_t(ds)$$

so that $$t^{-1} \mu_t(\Lambda) = \int_0^\infty \lambda_s(\Lambda) t^{-1} \nu_t(ds) \leqq c \int_0^\infty \lambda_s(\Lambda) N(ds).$$

However, if Λ is an open interval bounded away from 0, $\lambda_s(\Lambda)$ is bounded and $O(s)$ near 0, so by Lemma 2.3

$$\lim_{t \to 0} t^{-1} \mu_t(\Lambda) = \int_0^\infty \lambda_s(\Lambda) N(ds) = M(\Lambda),$$

so that μ_t is powerfully continuous. ▯

With notation as in the proof of Theorem 3.1, if $E(e^{iuY(t)}) = e^{-t\varphi(u)}$, $E(e^{ivX(t)}) = e^{-t\psi(v)}$ and $E(e^{-sT(t)}) = e^{-t\sigma(s)}$, then (see Feller [4], p. 427)

$$(3.2) \qquad \varphi = \sigma \circ \psi.$$

In dealing with stable processes, we shall use the results and notation of Feller [4], p. 548. We denote by $p(x;\, \alpha,\, \gamma)$ the strictly stable density

(3.3) $$p(x;\, \alpha,\, \gamma) = 1/\pi \operatorname{Re} \textstyle\int_0^\infty \exp\{-ixy - y^\alpha e^{i\pi\gamma/2}\}\, dy$$

where

(3.4) $$|\gamma| \leqq \alpha \qquad \text{if } 0 < \alpha < 1$$
$$|\gamma| \leqq 2 - \alpha \quad \text{if } 1 < \alpha < 2.$$

In case $\alpha = 1$, the only strictly stable densities are translations of the Cauchy density, and we let

(3.5) $$p(x;\, 1) = \pi^{-1}(1 + x^2)^{-1}$$

denote this density. Note that the characteristic function of $p(x;\alpha,\gamma)$ is $\exp\{-|y|^\alpha e^{i\pi\gamma \operatorname{sgn} y/2}\}$ and the characteristic function of $p(x - \theta;\, 1)$ is $\exp\{i\theta y - |y|\}$. One can show that the density (3.3) is one-sided if and only if $|\gamma| = \alpha$ and $0 < \alpha < 1$. If $1 < \alpha < 2$ and $|\gamma| = 2 - \alpha$, the Levy measure is concentrated on one side of the origin, but $p(x;\, \alpha,\, \gamma)$ is not one-sided.

The Levy measure for a strictly stable process with index α is $k(x)|x|^{-(1+\alpha)}\, dx$, where $k(x)$ depends only on the sign of x.

THEOREM 3.2. *Let μ_t be strictly stable with index $\alpha < 2$. Then μ_t is powerfully continuous on $(-\infty,\, \infty)$ unless $1 < \alpha < 2$ and $|\gamma| = 2 - \alpha$. In this latter case, μ_t is powerfully continuous on $(-\infty, 0)$ if $\gamma = 2 - \alpha$, on $(0,\, \infty)$ if $\gamma = -(2 - \alpha)$. The Gaussian process is not powerfully continuous on any interval.*

PROOF. We firstly treat the case of a stable subordinator with index α, $\frac{1}{2} < \alpha < 1$. By a trivial change of time scale, we may assume that the density f_t of μ_t is determined by the rule

(3.6) $$f_1(x) = p(x;\, \alpha,\, -\alpha)$$

together with the strict stability condition

(3.7) $$f_t(x) = t^{-1/\alpha} f_1(x \cdot t^{-1/\alpha}).$$

To prove that μ_t is powerfully continuous, it suffices to prove that

(3.8) $$x^{1+\alpha} f_t(x) \cdot t^{-1} \leqq c, \qquad t > 0 \quad \text{and} \quad x > 0.$$

Setting $s = xt^{-1/\alpha}$, we see that (3.8) is equivalent to

(3.9) $$s^{1+\alpha} p(s;\, \alpha,\, \gamma) \leqq c, \qquad s > 0\ (\gamma = -\alpha).$$

By Zolotarev's Lemma (Lemma 2 on p. 549 of [4]),

$$s^{1+\alpha} p(s;\, \alpha,\, -\alpha) = p(s^{-\alpha};\, \alpha^{-1},\, -2 + 1/\alpha)$$

Noting that the parameter $\gamma = -2 + 1/\alpha$ indeed satisfies (3.4) for α^{-1}, we deduce that the term on the right is uniformly bounded, since it is a Fourier transform of an integrable function ((3.3)).

According to (3.2), when we subordinate a strictly stable process with parameters (α, γ) by a stable subordinator with index β, the process obtained is strictly stable with parameters $(\alpha\beta, \gamma\beta)$. Considering the range of the mappings $(\alpha, \gamma) \to (\alpha\beta, \gamma\beta)$ for $\frac{1}{2} < \beta < 1$, and using Theorem 3.1, we see that every strictly stable process with parameters (α, γ), $\alpha \neq 1$, where if $1 < \alpha < 2$, $|\gamma| < 2 - \alpha$, is powerfully continuous on $(-\infty, \infty)$. The case of the Cauchy process with a translation can be settled easily using elementary calculus.

If $1 < \alpha < 2$ and $\gamma = -(2 - \alpha)$, then M is concentrated on $(0, \infty)$ and $M(dx) = kx^{-(1+\alpha)}\,dx$. We may suppose that μ_t has density f_t given by $f_t(x) = p(xt^{-1/\alpha}; \alpha, \gamma)t^{-1/\alpha}$. We must prove (3.8) for $t > 0$ and $x > 0$. As before, this is equivalent to (3.9) with $\gamma = -(2 - \alpha)$, and Zolotarev's Lemma shows in this case that

$$s^{1+\alpha}p(s; \alpha, \gamma) = p(s^{-1/\alpha}; \alpha^{-1}, -2/\alpha),$$

and the term on the right is certainly uniformly bounded, being a density with an integrable characteristic function.

To deal with the case $\gamma = 2 - \alpha$, note simply that $p(-x; \alpha, \gamma) = p(x; \alpha, -\gamma)$, and use the above result. □

4. Variations of the sample paths. For the remainder of the paper, $X(t)$ is assumed to be a homogeneous process almost all of whose sample paths are right continuous and possess left limits for all $t > 0$. We assume that μ_t is the distribution semigroup of $X(t)$, that the Levy measure, M, of μ_t has infinite total mass, and that the process is powerfully continuous on an open set S. We shall denote by $J(t) = J(t, \omega)$ the jump of $X(t, \omega)$ at t; i.e. $J(t) = X(t) - X(t-)$.

The functions f with which we operate will always be assumed to satisfy $f \geqq 0$,

$$M\{x \mid f(x) > c\} < \infty, \text{ and} \tag{4.1}$$

$$\int_{\{x \mid f(x) \leqq c\}} f\,dM < \infty \tag{4.2}$$

for some $c > 0$. It is easy to see that (4.1) and (4.2) are then satisfied for all $c > 0$. Note that (4.1) and (4.2) are exactly the conditions required to assure that $M \circ f^{-1}$ satisfies (2.2) and (2.4), and so is a Levy measure for an increasing homogeneous process. To avoid a little notation at a later point, we make the harmless assumption

$$f(0) = 0. \tag{4.3}$$

We shall take as known the result that for any Borel set Λ such that $M(\Lambda) < \infty$, the number of jumps of size Λ up to epoch t is a Poisson process with parameter $M(\Lambda)$.

THEOREM 4.1. *Let $f \geqq 0$ satisfy* (4.1), (4.2) *and* (4.3). *Then* $\sum_{s \leqq t} f(J_s)$ *converges for all $t > 0$, almost surely, and defines a subordinator with Levy measure* $M \circ f^{-1}$.

PROOF. For almost all ω, J_s is non-zero for only countably many s. Also, if we define ${}^m f(x) = f(x)1_{\{f(x) > m\}}$, $m = 1, 2, \cdots$. Then $M\{x \mid f(x) \neq {}^m f(x)\} < \infty$

for all m, by (4.1), and $\{x \mid f(x) \neq {}^m f(x)\}$ decreases to the empty set, so $M\{x \mid f(x) \neq {}^m f(x)\} \to 0$. For any Borel set Λ, let $N(t, \Lambda)$ denote the number of jumps of X of size Λ up to epoch t. The sums $\sum_{s \leqq t} f(J_s)$ and $\sum_{s \leqq t} {}^m f(J_s)$ differ only on $\{N(t, (m, \infty)) \neq 0\}$ and this latter set decreases as $m \to \infty$, and $P\{N(t, (m, \infty)) \neq 0\} = 1 - e^{-tM(m,\infty)} \to 0$ as $m \to \infty$. To prove a.s. convergence for a fixed $t > 0$, it may therefore be assumed that f is bounded, say $0 \leqq f \leqq K$. Let $\Gamma_n = f^{-1}(K \cdot 2^{-n}, K \cdot 2^{-(n-1)})$, $n \geqq 1$. Then

$$\begin{aligned} E \textstyle\sum_{s \leqq t} f(J_s) &= E \textstyle\sum_{n=1}^{\infty} \sum_{s \leqq t, J_s \varepsilon \Gamma_n} f(J_s) = \sum_{n=1}^{\infty} E \sum_{s \leqq t, J_s \varepsilon \Gamma_n} f(J_s) \\ &\leqq \textstyle\sum_{n=1}^{\infty} E \cdot K \cdot 2^{-(n-1)} N(t, \Gamma_n) = t \sum_{n=1}^{\infty} K 2^{-(n-1)} M(\Gamma_n) \\ &\leqq 2t \textstyle\int_0^K x M \circ f^{-1}(dx) = 2t \int_0^{\infty} f(x) M(dx) < \infty. \end{aligned}$$

The sum $\sum_{s \leqq t} f(J_s)$ clearly represents a subordinator and the number of jumps of size Λ is clearly the same as the number of J_s in $f^{-1}(\Lambda)$, which has expectation $tM \circ f^{-1}(\Lambda)$, so the process $\sum_{s \leqq t} f(J_s)$ has Levy measure $M \circ f^{-1}$. □

We denote $\sum_{s \leqq t} f(J_s)$ by ${}^f X(t)$. Our main result, Theorem 4.2, says that we can obtain ${}^f X(t)$ in a slightly weaker way by taking limits of $Z(f, \mathfrak{S})$ as described in the introduction.

Theorem 4.2. *With hypotheses on $X(t)$, μ_t and M as above, if f is a non-negative Borel function with support in S and satisfying* (4.1) *and* (4.2), *and if $\{\mathfrak{S}_n\}$ is a sequence of partitions of* $[0, t]$ *with* mesh $\mathfrak{S}_n \to 0$ *as $n \to \infty$, then $Z(f, \mathfrak{S}_n) \to {}^f X(t)$ in probability as $n \to \infty$. If $\int f\, dM < \infty$, convergence takes place in L^1 norm.*

To prove this theorem, we begin with the simplest possible function f, and work up to complete generality.

Lemma 4.1. *If $\mathfrak{S} = \{t_0, \cdots, t_n\}$ is a partition of* $[0, t]$, *then $EZ(1_\Lambda, \mathfrak{S}) = \sum_k \mu_{(t_{k+1}-t_k)}(\Lambda) \to tM(\Lambda)$ as* mesh $\mathfrak{S} \to 0$. *If f is any non-negative Borel function with support in S such that $\int f\, dM < \infty$, then*

$$EZ(f, \mathfrak{S}) \leqq ct \int f\, dM. \tag{4.4}$$

Proof. The first equality is clear, and

$$\begin{aligned} &\left|\textstyle\sum_k \mu_{(t_{k+1}-t_k)}(\Lambda) - tM(\Lambda)\right| \\ &\qquad \leqq \{\sup_k |(t_{k+1} - t_k)^{-1} \mu_{(t_{k+1}-t_k)}(\Lambda) - M(\Lambda)|\} \cdot \textstyle\sum_k (t_{k+1} - t_k) \to 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as mesh } \mathfrak{S} \to 0. \end{aligned}$$

The inequality (4.4) is certainly true if f is the indicator function of a set $\Lambda \subset S$ with finite M-measure. It is therefore true for non-negative linear combinations of indicators and their monotone limits, and consequently must hold generally if $f \geqq 0$, supp $f \subset S$, and $\int f\, dM < \infty$. □

Lemma 4.2. *Suppose $\{W_n\}$ is a sequence of non-negative random variables such that*

$$\liminf_n W_n \geqq W, \quad \textit{and} \tag{4.5}$$

$$\lim_n E(W_n) = E(W). \tag{4.6}$$

Then W_n converges to W in L^1.

PROOF. Let $V_n = \inf\{W_k \mid k \geqq n\}$, and let $\lim_n V_n = V = \liminf_n W_n \geqq W$. By Fatou's Lemma, $E(V) \leqq E(W)$, so $V = W$. Since V_n increases, $V_n \to W$ in L^1. Now, $W_n \geqq V_n$, so

$$E|W_n - V_n| = E(W_n - V_n) = E(W_n - W) + E(V - V_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $W_n \to W$ in L^1. $\square$

LEMMA 4.3. *If $f = 1_G$, where G is an open interval bounded away from 0 and strictly interior to S, and if $\mathfrak{S}_n$ is a sequence of partitions of $[0, t]$, then $Z(f, \mathfrak{S}_n) \to N(t, G)$ in L^1 as $n \to \infty$, if* mesh $\mathfrak{S}_n \to 0$.

PROOF. Let H be an open interval bounded away from 0 and containing G strictly in its interior. The number of jumps in H is almost surely finite, and for any sample path with a finite number of jumps in H, suppose the jumps occur at $s_1, \cdots, s_k$. Since M has no atoms at the endpoints of G, we may assume that no jumps have the magnitude of the endpoints of G. We may therefore construct intervals I_i about the s_i so that $X(s) - X(r) \in G$ if $r < s_i \leqq s$, $r \in I_i$, $s \in I_i$ and $J(s_i) \in G$. Then, when the mesh of $\mathfrak{S}$ is sufficiently small, $\mathfrak{S}$ includes points in each I_i straddling the s_i, implying that $Z(1_G, \mathfrak{S}_n) \geqq N(t, G)$ when n is large. Hence $\liminf_r Z(1_G, \mathfrak{S}_n) \geqq N(t, G)$, and by Lemma 4.1, $EZ(1_G, \mathfrak{S}_n) \to EN(t, G)$. Hence, by Lemma 4.2, $Z(1_G, \mathfrak{S}_n) \to N(t, G)$ in L^1. $\square$

LEMMA 4.4. *If $f = 1_\Lambda$, where $\Lambda \subset S$ is a Borel set with $M(\Lambda) < \infty$, and if* mesh $\mathfrak{S}_n \to 0$, *then $Z(f, \mathfrak{S}_n) \to N(t, \Lambda)$ in L^1 as $n \to \infty$.*

PROOF. Let $\epsilon > 0$ be given; choose Λ^0, a finite union of open intervals bounded away from 0 and strictly interior to S such that $M(\Lambda \triangle \Lambda^0) < t^{-1}\epsilon/3$. Choose n_0 so large that $E|Z(1_{\Lambda^0}, \mathfrak{S}_n) - N(t, \Lambda^0)| < \epsilon/3$ when $n \geqq n_0$. Now, $E|N(t, \Lambda) - N(t, \Lambda^0)| \leqq EN(t, \Lambda \triangle \Lambda^0) = tM(\Lambda \triangle \Lambda^0) < \epsilon/3$, so we obtain, when $n \geqq n_0$, $E|Z(f, \mathfrak{S}_n) - N(t, \Lambda)| \leqq 2\epsilon/3 + E|Z(1_\Lambda, \mathfrak{S}_n) - Z(1_{\Lambda^0}, \mathfrak{S}_n)| < 2\epsilon/3 + EZ(1_{\Lambda \triangle \Lambda^0}, \mathfrak{S}_n) < \epsilon$, because of Lemma 4.1. $\square$

LEMMA 4.5. *Let $f = \sum_1^\infty \alpha_j 1_{\Lambda_j}$, where the $\Lambda_j \subset S$ are disjoint Borel sets, $\alpha_j > 0$,*

$$\sum_{\{\alpha_j > c\}} M(\Lambda_j) < \infty, \quad \textit{and} \tag{4.7}$$

$$\sum_{\{\alpha_j \leqq c\}} \alpha_j M(\Lambda_j) < \infty \tag{4.8}$$

for some $c > 0$. Then, if $\{\mathfrak{S}_n\}$ is a sequence of partitions of $[0, t]$ with mesh $\mathfrak{S}_n \to 0$, *then $Z(f, \mathfrak{S}_n)$ converges in probability to ${}^fX(t) = \sum_1^\infty \alpha_j N(t, \Lambda_j)$. If $\int f\,dM < \infty$, the convergence takes place in L^1.*

PROOF. Suppose $\int f\,dM < \infty$. Let $\epsilon > 0$ be given; choose m so large that $\sum_{m+1}^\infty \alpha_j M(\Lambda_j) < c^{-1}t^{-1}\epsilon/3$. Let $f_m = \sum_1^m \alpha_j 1_{\Lambda_j}$ and $g_m = f - f_m$. By Lemma 4.4, $E|Z(f_m, \mathfrak{S}_n) - \sum_1^m \alpha_j N(t, \Lambda_j)| < \epsilon/3$ when n is large. Now,

$$E|Z(f, \mathfrak{S}_n) - Z(f_m, \mathfrak{S}_n)| = EZ(g_m, \mathfrak{S}_n) \leqq ct \int g_m\,dM < \epsilon/3,$$

and $E|{}^fX(t) - {}^{f_m}X(t)| = E\sum_{m+1}^\infty \alpha_j N(t, \Lambda_j) = \sum_{m+1}^\infty \alpha_j tM(\Lambda_j) < c^{-1}\epsilon/3$. Hence, $E|Z(f, \mathfrak{S}_n) - {}^fX(t)| \to 0$ as $n \to \infty$.

If $\int f\,dM = \infty$, let $h = \min(f, K)$, and let $A = \{x \mid f(x) \neq h(x)\} = \{x \mid f(x) > K\}$. Then $M(A) \to 0$ as $K \to \infty$. The sums $Z(f, \mathfrak{S}_r)$ and $Z(h, \mathfrak{S}_n)$

differ only on the paths where $X(t_k) - X(t_{k-1}) \varepsilon A$ for some k, and the probability of this event is

$$1 - P\{X(t_k) - X(t_{k-1}) \varepsilon A^c \forall k\}$$
$$= 1 - \prod_k \mu_{(t_k - t_{k-1})}(A^c) = 1 - \prod_k [1 - \mu_{(t_k-t_{k-1})}(A)]$$
$$\leqq 1 - \prod_k [1 - c(t_k - t_{k-1})M(A)].$$

Now, $1 - s \geqq e^{-\gamma s}$, $0 \leqq s \leqq \delta$, for some γ (depending on δ), so the last expression above is

$$\leqq 1 - \prod_k e^{-\gamma c(t_k - t_{k-1})M(A)}$$
$$= 1 - e^{-\gamma c t M(A)},$$

if mesh $\mathfrak{S} < \delta c^{-1} M(A)^{-1}$ so that when $M(A)$ is very small, the sums $Z(f, \mathfrak{S}_n)$ and $Z(h, \mathfrak{S}_n)$ differ on a set of arbitrarily small measure, when mesh $\mathfrak{S}_n$ is small. Since $Z(h, \mathfrak{S}_n)$ converges in L^1 to ${}^hX(t)$, and ${}^hX(t)$ converges in probability to ${}^fX(t)$ as $k \to \infty$, it is easy to see that $Z(f, \mathfrak{S}_n)$ converges in probability to ${}^fX(t)$. $\square$

Proof of Theorem 4.2. Recall the function q of (2.9) and define $f^{(m)}$ as in (2.10). Evidently, $f^{(m)}$ is a function of the type described in Lemma 4.5, and

$$0 \leqq f - f^{(m)} \leqq 2^{-m} q. \tag{4.9}$$

Hence

$$0 \leqq Z(f, \mathfrak{S}_n) - Z(f^{(m)}, \mathfrak{S}_n) \leqq 2^{-m} Z(q, \mathfrak{S}_n), \tag{4.10}$$

and by Lemma 4.1,

$$E(2^{-m} Z(q, \mathfrak{S}_n)) \leqq c \cdot 2^{-m} \int q \, dM. \tag{4.11}$$

If $\int f \, dM < \infty$, $\int f^{(m)} \, dM < \infty$, and

$$E|Z(f, \mathfrak{S}_n) - {}^fX(t)| \leqq EZ(f - f^{(m)}, \mathfrak{S}_n)$$
$$+ E|Z(f^{(m)}, \mathfrak{S}_n) - {}^{f^{(m)}}X(t)| + E|{}^fX(t) - {}^{f^{(m)}}X(t)|$$

and by Lemma 4.1, and inequalities (4.10) and (4.11),

$$EZ(f - f^{(m)}, \mathfrak{S}_n) \leqq c 2^{-m} t \int q \, dM. \tag{4.12}$$

It is also easy to see that

$$E|{}^fX(t) - {}^{f^{(m)}}X(t)| \leqq t \cdot 2^{-m} \int q \, dM. \tag{4.13}$$

By Lemma 4.5, $E|Z(f^{(m)}, \mathfrak{S}_n) - {}^{f^{(m)}}X(t)| \to 0$ as $n \to \infty$. Using this fact together with (4.12) and (4.13) we see that $E|Z(f, \mathfrak{S}_n) - {}^fX(t)| \to 0$ as $t \to 0$.

In case $\int f \, dM = \infty$, we have

$$P\{|Z(f, \mathfrak{S}_n) - {}^fX(t)| > \delta\}$$
$$\leqq P\{Z(f - f^{(m)}, \mathfrak{S}_n) > \delta/3\} + P\{|Z(f^{(m)}, \mathfrak{S}_n) - {}^{f^{(m)}}X(t)| > \delta/3\}$$
$$+ P\{{}^fX(t) - {}^{f^{(m)}}X(t) > \delta/3\}.$$

The first and last terms may be made as small as desired by taking m sufficiently large, using (4.12) and (4.13). The proof is completed by applying Lemma 4.5 to the middle term. $\square$

As an example, if $X(t)$ is strictly stable with parameters (α, γ), and $f(x) = |x|^{\beta}$, except when $1 < \alpha < 2$ and $|\gamma| = 2 - \alpha$, when we take $f(x) = |x|^{\beta}$ if the sign of x is opposite to the sign of γ, and $f(x) = 0$ otherwise, we see that $\int_{\{f \leqq 1\}} f\, dM < \infty$ if and only if $\beta > \alpha$. The measure $M \circ f^{-1}(dx) = cx^{-1+\alpha/\beta}\, dx$, $x > 0$, so the resulting limit process is the stable subordinator with index α/β. In case $\alpha < \beta \leqq 1$, $f(x)$ is subadditive and one can see (Blumenthal and Getoor [1]) that the limit holds almost surely.

Our main theorem should be extendable to processes in R^d, with a simple extension of the notion of powerfully continuous processes. With the obvious extension, a symmetric stable process in R^d can be obtained from Brownian motion using a stable subordinator, and so should be powerfully continuous on R^d by the analogue of Theorem 3.1.

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