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ANALYTIC EXPANSIONS OF MAX-PLUS LYAPUNOV EXPONENTS[1]

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We give an explicit analytic series expansion of the (max, plus)-Lyapunov exponent $\gamma(p)$ of a sequence of independent and identically distributed random matrices, generated via a Bernoulli scheme depending on a small parameter p. A key assumption is that one of the matrices has a unique normalized eigenvector. This allows us to obtain a representation of this exponent as the mean value of a certain random variable. We then use a discrete analogue of the so-called light-traffic perturbation formulas to derive the expansion. We show that it is analytic under a simple condition on p. This also provides a closed form expression for all derivatives of $\gamma(p)$ at $p = 0$ and approximations of $\gamma(p)$ of any order, together with an error estimate for finite order Taylor approximations. Several extensions of this are discussed, including expansions of multinomial schemes depending on small parameters $(p_1, \ldots, p_m)$ and expansions for exponents associated with iterates of a class of random operators which includes the class of nonexpansive and homogeneous operators. Several examples pertaining to computer and communication sciences are investigated: timed event graphs, resource sharing models and heap models.

Contents

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1. Introduction. It is well known that under mild conditions, the Perron–Frobenius eigenvalue of a nonnegative matrix, the parameters of which depend analytically on a parameter z, also depends analytically on this parameter. This result of the conventional algebra cannot be extended to the (max, plus) algebra, since the Perron–Frobenius eigenvalue, which is given by the maximal cycle mean formula [1], is then the maximum of a finite family of analytic functions of z, which is not analytic in z in general.

In the conventional algebra, there are also several well-known results on the analyticity of the Lyapunov exponent of i.i.d. matrices, the law of which depends on a parameter p [13, 14, 15]. The focus of the present paper is to investigate the analogue of this second type of question in the (max, plus) algebra [1].

Using a simple class of random matrices sampled from a finite set, we show that, under technical conditions to be specified later, not only the associated (max, plus)-Lyapunov exponent depends analytically on the probabilities used for the sampling, but also that a constructive approximation scheme can be given, based on a closed form representation of the coefficients of all orders of the analytic expansion and on error estimates for finite order Taylor approximations. For p small enough, the error bound decreases geometrically to 0. So for all given intervals, there exists a finite algorithm allowing one to decide whether $\gamma(p)$ is in this interval. This situation differs significantly with the result of [16] that conventional Lyapunov exponents are *not algorithmically approximable*.

Computing such a Taylor expansion of order n requires a number of arithmetical operations which is not polynomially bounded in n. Nevertheless, we will give several examples where this approach can be used to derive expansions of practical use.

The paper is structured as follows.

In Section 2, we briefly recall tools and results which we will need to state and analyze the problem.

In Section 3, we consider the series expansion of the (max, plus)-Lyapunov exponent of a sequence of i.i.d. random matrices, where each matrix is sampled among two possible values using a Bernoulli scheme with parameter p. The main theorem is first given under certain restrictive assumptions (H1), (H2), (H3), defined in Section 3.1 and is illustrated through various examples. All proofs are gathered in Section 6.

In Section 4, we present three extensions of the main theorem: the multinomial case is considered in Section 4.1, whereas Section 4.2 focuses on the weakening of Assumption (H2). The extension of this class of results to iterates of random operators [which are not necessarily linear operators in a semiring, as is the case for the (max, plus) setting considered above] is given in Section 4.3 with an example of the (min, max, plus) system in Section 4.3.1. A key property for such an operator extension is the *finite range coupling of some pattern* of the operators. Section 4.4 summarizes these three extensions into a generic theorem. An example of application to heap models is given in Section 4.5.

Section 5 focuses on the interpretation of the results in terms of perturbation-type formulas.

The proofs of the main theorems are concentrated in Section 6.

Section 7 gives further expansions covering some cases with continuous distributions. It also contains comments on the relationship holding between different expansions which may be proposed for the same exponent.

Finally, Section 8 focuses on the regenerative theory approach. This helps to understand the form of the coefficients of the expansion and also allows one to derive the same type of analytical expansion in certain particular cases.

2. Preliminaries on (max, plus)-Lyapunov exponents.

2.1. *Algebraic framework and basic spectral theorems.* Most of this paper bears on product of matrices in the so called (max, plus) algebra, namely over the semifield $\mathbb{R}_{\max} = \mathbb{R} \cup \{\varepsilon\}$ where $\varepsilon = -\infty$, endowed with an addition denoted $\oplus$, which is the max operation and with a product, denoted $\otimes$, which is the sum. The element ε is the neutral element of this semifield.

We shall denote $\mathbb{R}^d_{\max}$ the set of vectors of dimension d and $\mathbb{R}^{d\times d}_{\max}$ the set of square matrices of dimension $d \times d$ over this semifield. The set $\mathbb{R}^{d\times d}_{\max}$ is endowed with two operations, also denoted $\oplus$ and $\otimes$ and defined by

$$(A \otimes B)_{ij} = \bigoplus_{1\le k\le d} A_{ik} \otimes B_{kj}, \qquad (A \oplus B)_{ij} = A_{ij} \oplus B_{ij}.$$

$(\mathbb{R}_{\max}^{d\times d}, \oplus, \otimes)$ is a semiring. The nth power of matrix A, denoted $A^{\otimes n}$ or A^n, is to be understood in the (max, plus) sense, that is, $A^n = A \otimes \cdots \otimes A$, n times. Note that if $d = 1$, $a^{\otimes n} = na$.

The following notations and definitions will be used throughout the paper: for all $Z \in \mathbb{R}_{\max}^d$ and $M \in \mathbb{R}_{\max}^{d\times d}$,

$$\|Z\|_\infty = \bigoplus_{1\le i\le d} Z_i, \quad \|M\|_\infty = \bigoplus_{1\le i,\, j\le d} M_{ij}, \quad |||Z||| = \bigoplus_{\substack{1\le i\le d\\ Z_i>\varepsilon}} |Z_i|,$$

$$|||M||| = \bigoplus_{\substack{1\le i,\, j\le d\\ M_{ij}>\varepsilon}} |M_{ij}|, \quad \|Z\|_{\mathscr{D}} = \max_{1\le i\le d} Z_i - \min_{1\le i\le d} Z_i.$$

For $Z \in \mathbb{R}^d$, we shall denote $\overline{Z}$ the equivalence class of Z for the (colinearity) equivalence relation $Y \equiv Z$ iff $Y = Z \otimes \alpha$, that is, for all $i = 1, \dots, d$, $Y_i = Z_i + \alpha$, for some scalar $\alpha \neq \varepsilon$.

The graph of A is the directed graph with nodes $\{1, \dots, d\}$ and with an arc from i to j iff $A_{ij} \neq \varepsilon$. A path of A is a sequence $\{i = i_0, i_1, \dots, i_n = j\}$, $(n \ge 1)$ such that $A_{i_l, i_{l+1}} > \varepsilon$ for all $l \in \{0, \dots, n-1\}$. We write $i \sim j$ if there exists a path from i to j and a path from j to i. A matrix $A \in \mathbb{R}_{\max}^{d\times d}$ is irreducible if $i \sim j$, $\forall i, j \in \{1, \dots, d\} \times \{1, \dots, d\}$. Then the graph associated to A is said to be strongly connected.

A key result concerning irreducible matrices is the following theorem.

RESULT 1 (Cyclicity theorem for deterministic matrices [1]). *For each irreducible matrix A, there exist two positive integers c, σ and a real number γ, such that*

$$A^{\otimes[c+\sigma]} = \gamma^{\otimes\sigma} \otimes A^{\otimes c}.$$

(i) *The real number γ which satisfies this relation is unique and coincides with the* eigenvalue *of A, that we denote $\gamma(A)$.*

(ii) *The minimal value of c, that we denote $c(A)$, is called* coupling *time.*

(iii) *The minimal value of σ, that we denote $\sigma(A)$, is called* cyclicity.

It follows that for all $n \ge c(A)$,

$$A^{\otimes[n+\sigma(A)]} = \big(\gamma(A)\big)^{\otimes\sigma(A)} \otimes A^{\otimes n}.$$

In particular if A has cyclicity 1 and has a unique eigenvector class $\overline{V}$, then for all $n \ge c(A)$ and for all vectors X in $\mathbb{R}^d$,

$$A^{\otimes(n+1)} = \gamma(A) \otimes A^{\otimes n} \quad \textit{and} \quad A^{\otimes n} \otimes X \equiv V.$$

An irreducible matrix always has a single eigenvalue, but it does not always have a unique eigenvector class. A sufficient condition for this last uniqueness property to hold is that the critical matrix of A has a single maximal strongly connected subgraph.

For a matrix A, we define:

1. *Circuit*: a circuit is a path $\xi = \{i_0, i_1, \ldots, i_n\}$ such that $i_0 = i_n$. Its average weight is defined by $|\xi| = (A_{i_0, i_1} \otimes \cdots \otimes A_{i_{n-1}, i_n})/n$.
2. *Critical circuit*: a circuit ξ is critical if its average weight is maximal, that is, if $|\xi| = \gamma(A)$.
3. *Critical matrix*: the matrix obtained from A by replacing all entries of A not belonging to the critical circuit by ε is called critical matrix.

The combination of the two properties: (a) the critical matrix of A has a unique maximal strongly connected subgraph and (b) the cyclicity of the critical matrix is equal to 1 (here gcd of all circuit lengths in this subgraph), will be referred to as *scs1-cyc1* below.

2.2. *Stochastic setting and Lyapunov exponents.* Let some probability space be given on which all random variables introduced below are defined. A random element of $\mathbb{R}_{\max}^d$ or of $\mathbb{R}_{\max}^{d\times d}$ will be said to have *fixed support* if each of its entries is either a.s. equal to ε or a.s. nonequal to ε. In this case, it will be said to be integrable if in addition each entry nonequal to ε is integrable. Note that the definition of irreducibility can be extended directly to a random matrix with fixed support.

The general setting of the paper will be that of a given sequence of random matrices of $\mathbb{R}_{\max}^{d\times d}$, say $\{A(n)\}$, and of the sequence of random vectors $X_n \in \mathbb{R}_{\max}^d$ defined by the recurrence relation

$$X_{n+1} = A(n) \otimes X_n, \qquad n \geq 0 \tag{1}$$

and by the initial condition $X_0 \in \mathbb{R}^d$, which will be assumed to be constant in what follows.

Here are two general results on this type of sequences.

RESULT 2 (Lyapunov exponents via subadditivity [11, 7, 1, 12]). *Assume that $\{A(n)\}$ is a stationary and ergodic sequence of random matrices of $\mathbb{R}_{\max}^{d\times d}$, and that $A(0)$ has fixed support and is irreducible and integrable. Then the following limits exist regardless of the initial condition*:

$$\lim_{n\to+\infty} \frac{X_n}{n} = \lim_{n\to+\infty} \mathbb{E}\left(\frac{X_n}{n}\right) = \begin{bmatrix} \gamma \\ \vdots \\ \gamma \end{bmatrix} = \Gamma. \tag{2}$$

The constant γ is referred to as the (max, plus)-Lyapunov exponent *of the sequence of random matrices $\{A(n)\}$.*

The proof of this result is in two steps: using first the fact that for all $n > m > l$,

$$\|A(n) \otimes \cdots \otimes A(l)\|_\infty \le \|A(n) \otimes \cdots \otimes A(m+1)\|_\infty \otimes \|A(m) \otimes \cdots \otimes A(l)\|_\infty,$$

it follows from the subadditive ergodic theorem that a.s.,

$$\lim_{n \to +\infty} \frac{\|A(n) \otimes \cdots \otimes A(0)\|_\infty}{n} = \gamma,$$

for some constant γ. The second step consists in showing that all coordinates of X_n/n have the same limit γ, regardless of the initial condition, which follows from the irreducibility assumption.

RESULT 3 (Strong coupling [12], 6.8). *Assume that $\{A(n)\}$ is an i.i.d sequence of random matrices, independent of X_0, and that $A(0)$ takes its values in a finite set $\{A_l, l \in \mathscr{L}\}$ of irreducible matrices of $\mathbb{R}_{\max}^{d \times d}$, where each element of the set has a positive probability of occurrence.*

If there exists a scs1-cyc1 pattern *in $\{A_l, l \in \mathscr{L}\}$, namely, a product $A_{l_1} \otimes \cdots \otimes A_{l_q}$ of elements of this set, which is irreducible and scs1-cyc1, then $\{\overline{X}_n\}$ converges with* strong coupling *to a unique stationary sequence. In particular, there exists a unique random equivalence class $\overline{X}(\omega)$ such that for all deterministic initial conditions X_0*:

1. *The law of $\{\overline{X}_n\}$ converges in total variation to that of $\overline{X}$.*
2. *For a.s. all ω, there exists $N(\omega) < \infty$ s.t.*,

$$\forall\, n \ge N(\omega),\ \overline{A(0,\omega) \otimes A(-1,\omega) \otimes \cdots \otimes A(-n,\omega) \otimes X_0} = \overline{X(\omega)}. \tag{3}$$

This result, which is stated under much more general assumptions (stationary and ergodic) in [12], is essentially based on Borovkov's renovating events theorem (see [4], where the definition of strong coupling may also be found).

Result 3 allows for another representation of the Lyapunov exponent, which will be crucial in this paper: from (3), under the above assumptions,

$$\exists \lim_{n \to \infty} A(1) \otimes \cdots \otimes A(-n) \otimes X_0 - A(0) \otimes \cdots \otimes A(-n) \otimes X_0 = \Delta \quad \text{a.s.},$$

where

$$\Delta = A(1) \otimes X - X \tag{4}$$

is a finite random variable. Therefore, if $|A(1) \otimes Z - Z|$ is uniformly (in Z) bounded by an integrable random variable (this is a restrictive hypothesis which we shall partially relax later; see Remark 10 in Section 6.1.2), then

$$\begin{aligned}
\exists \lim_{n\to+\infty} & \mathbb{E}[X_{n+2} - X_{n+1}] \\
&= \lim_{n\to+\infty} \mathbb{E}[A(1)\otimes\cdots\otimes A(-n)\otimes X_0 - A(0)\otimes\cdots\otimes A(-n)\otimes X_0] \\
&= \mathbb{E}\Big[\lim_{n\to+\infty} A(1)\otimes\cdots\otimes A(-n)\otimes X_0 - A(0)\otimes\cdots\otimes A(-n)\otimes X_0\Big] \\
&= \mathbb{E}[\Delta],
\end{aligned}$$

where we used the dominated convergence theorem to get second equality. Using now Result 2 and a Cesaro averaging argument, we get that

$$\lim_{n\to+\infty} \mathbb{E}[X_{n+2} - X_{n+1}] = \lim_{n\to+\infty} \mathbb{E}\left[\frac{X_n}{n}\right] = \Gamma,$$

where Γ is the Lyapunov exponent of $\{A(n)\}$. Therefore under the above assumptions, we also have the following representation:

$$\Gamma = \mathbb{E}[\Delta] = \lim_{n\to+\infty} \mathbb{E}[X_{n+1} - X_n]. \tag{5}$$

3. Bernoulli case.

3.1. *Assumptions and main results for the Bernoulli case.* The setting of this section is the following: $\{A(n)\}$ in an i.i.d. sequence of matrices of $\mathbb{R}_{\max}^{d\times d}$, and $A(0)$ takes its value in the set $\{A, A'\}$: for all n,

$$A(n) = \begin{cases} A, & \text{with probability } 1-p, \\ A', & \text{with probability } p. \end{cases}$$

In this section, the assumptions on A and A' are the following:

(H1) The matrix A is irreducible.
(H2) The matrix A is scs1-cyc1.
(H3) The matrix A' has at least one entry different from ε on each row.

If in addition A' is irreducible, this setting is then a special case of that of Result 3. If A and A' have the same support, it is also a special case of that of Result 2. As we shall see below (Lemma 1 in Section 6), under (H1), (H2) and (H3), both Result 2 and Result 3 hold, so that the Lyapunov exponent

$$\Gamma = \Gamma(p) = \begin{bmatrix} \gamma(p) \\ \vdots \\ \gamma(p) \end{bmatrix}$$

of the above Bernoulli scheme is well defined via Result 2. The main result in this case is the following theorem.

THEOREM 1. (i) *Under assumptions* (H1), (H2) *and* (H3), $\gamma(p)$ *is analytic at point* 0, *with a radius of convergence larger than or equal to* $1/(2c)$, *where* c *denotes the coupling time of* A: $c = c(A)$. *The coefficients of the analytic* *expansion are given by the following formula, where V denotes an element of the unique eigenvector class of A*:

$$
\begin{aligned}
&\frac{1}{l!}\frac{d^l}{dp^l}[\Gamma(p)]_{p=0}\\
&\qquad=\begin{bmatrix}\pi(l)\\ \vdots\\ \pi(l)\end{bmatrix}=(-1)^l\left\{\binom{c}{l-1}V+\binom{c+1}{l}\Gamma(0)\right\}\\
&\qquad\quad+\sum_{k=1}^{l}\Bigg\{(-1)^{l-k}\sum_{j_1,\dots,j_{k-1}=0}\binom{2c+j_1+\cdots+j_{k-1}}{l-k}\\
&\qquad\qquad\times A^c\otimes A'\otimes A^{j_1}\otimes A'\cdots A^{j_{k-1}}\otimes A'\otimes V\Bigg\}.
\end{aligned}
\tag{6}
$$

(ii) *For all $p\in[0,\delta]$, with $\delta<1/(2c)$, the error term in the Taylor expansion of $\gamma(p)$ of order l is bounded from above by*:

$$\frac{D(2c\delta)^{l+1}[1+l(1-2c\delta)]}{(1-2c\delta)^2}$$

where $D=[2(|||A|||\vee|||A'|||)+||V||_{\mathscr{D}}](c+1)$.

In (6), we adopted the following conventions:

$$\binom{n}{p}=0\quad\text{if } p>n \text{ or } p<0$$

for $k=1$,

$$
\begin{aligned}
&\sum_{j_1,\dots,j_{k-1}=0}^{c-1}\binom{2c+j_1+\cdots+j_{k-1}}{l-k}A^c\otimes A'\otimes A^{j_1}\otimes A'\cdots A^{j_{k-1}}\otimes A'\otimes V\\
&\qquad=\binom{2c}{l-1}A^c\otimes A'\otimes V,
\end{aligned}
$$

$$
\begin{aligned}
&\binom{2c+j_1+\cdots+j_{k-1}}{l-k}A^c\otimes\cdots\otimes V\\
&\qquad\text{means }\binom{2c+j_1+\cdots+j_{k-1}}{l-k}(A^c\otimes\cdots\otimes V),
\end{aligned}
$$

which is different from

$$\left(\binom{2c+j_1+\cdots+j_{k-1}}{l-k}A^c\right)\otimes\cdots\otimes V.$$

The first three coefficients of the expansion of $\Gamma(p)$ are the following:

$$\begin{bmatrix} \pi(0) \\ \vdots \\ \pi(0) \end{bmatrix} = \begin{bmatrix} \gamma(0) \\ \vdots \\ \gamma(0) \end{bmatrix} = \Gamma(0),$$

$$\begin{bmatrix} \pi(1) \\ \vdots \\ \pi(1) \end{bmatrix} = A^c \otimes A' \otimes V - V - (c+1)\Gamma(0),$$

$$\begin{bmatrix} \pi(2) \\ \vdots \\ \pi(2) \end{bmatrix} = \sum_{j=0}^{c-1} \{A^c \otimes A' \otimes A^j \otimes A' \otimes V\}$$

$$-2cA^c \otimes A' \otimes V + cV + \binom{c+1}{2} \Gamma(0).$$

REMARK 1. An estimate of the computational cost to evaluate $\pi(l)$ using Equation 6 is $d^2 l c^l$. For more details see Section 6.1.4.

REMARK 2. Note that due to Result 1, the generic term in (6) can be rewritten as follows:

$$A^c \otimes A' \otimes A^{j_1} \otimes A' \cdots A^{j_{k-1}} \otimes A' \otimes V = \lambda(j_1, \ldots, j_{k-1}) \otimes V,$$

where $\lambda(j_1, \ldots, j_{k-1})$ is a scalar. We known that (6) is not modified if we replace V by $V \otimes \alpha$ for some scalar α. Since $A \otimes (X \otimes \alpha) = (A \otimes X) \otimes \alpha$, this property implies that we can rewrite (6) as follows:

$$\frac{1}{l!} \frac{d^l}{dp^l} [\Gamma(p)]_{p=0} = \sum_{k=1}^{l} \Bigg\{ (-1)^{l-k} \sum_{j_1, \ldots, j_{k-1}=0}^{c-1} \binom{2c + j_1 + \cdots + j_{k-1}}{l-k} \times \lambda(j_1, \ldots, j_{k-1}) \Bigg\} + (-1)^l \binom{c+1}{l} \gamma(0).$$

REMARK 3. Note also that in (6), it is enough to consider the indices k larger than or equal to α, with α the integer part of $[(l-2)/c]$. It is easy to see that for $k < \alpha$, the binomial coefficients in the sum are equal to zero.

The proof of Theorem 1 is given in Section 6.1.

3.2. *Examples.*

3.2.1. *A simple case of closed cyclic Jackson network.* We consider a closed Jackson network with two single server FIFO stations (see Figure 1). Assume

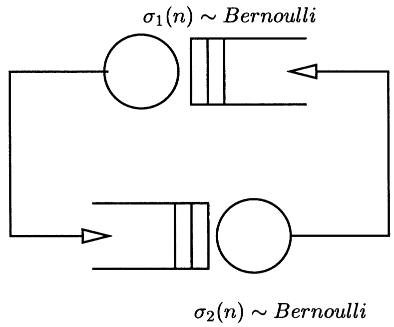


FIG. 1. *A cyclic Jackson network with two stations.*

that there are exactly two customers and that there is initially one customer in each station. All service times are independent, with a Bernoulli distribution,

$$\begin{aligned} \forall n,\ \sigma_i(n) &= \sigma_i \quad \text{with probability } 1-p, \\ &= \sigma_i' \quad \text{with probability } p. \end{aligned}$$

The evolution of such a model can be captured via the following (max, plus) recurrence:

$$X_{n+1} = A(n) \otimes X_n,$$

$$A(n) = \begin{pmatrix} \sigma_1(n) & \sigma_1(n) \\ \sigma_2(n) & \sigma_2(n) \end{pmatrix}.$$

Here, we take for state vector $X_n = (X_n^1, X_n^2)^t$, where X_n^i is the epoch of the nth departure from station i, and we take for initial condition $X_0 = (0,0)^t$, which corresponds to the case when each station starts its very first service when the evolution begins.

We can evaluate the Lyapunov exponent characterizing its stationary throughput using Theorem 1. Assume $\sigma_1 > \sigma_2$ (this is not a restrictive assumption as, in a cyclic network, the choice of the first coordinate is arbitrary). Then we have $V = (\sigma_1, \sigma_2)^t$ and $c = 1$.

We denote $\gamma = \sigma_1 \vee \sigma_2$ and $\gamma' = \sigma_1' \vee \sigma_2'$. Formula (6) gives

$$\pi(l) = \sum_{k=1}^{l} \left\{ (-1)^{l-k} \binom{2}{l-k} A \otimes (A')^k \otimes V \right\} + (-1)^l \left\{ \binom{1}{l-1} V + \binom{2}{l} \Gamma \right\}.$$

A direct evaluation shows that

$$\begin{aligned} \pi(0) &= \gamma, \\ \pi(1) &= \gamma' - \gamma, \\ \pi(l) &= 0 \quad \text{for } l > 1. \end{aligned}$$

Then we have

$$\gamma(p) = \gamma + (\gamma' - \gamma)p = \gamma' p + \gamma(1-p). \tag{7}$$

REMARK 4. One may wonder why (7) holds and when this occurs. Whenever A and A' have a common right or left eigenvector [here $(0, 0)$ is a common left eigenvector], the usual law of large numbers yields a formula like (7).

3.2.2. *Network with breakdowns.* Consider a network with two stations, where the first one is a single server station as above, and the second one has two servers. The network has three customers (see the Petri net of Figure 2). This network can be described as a (max, plus)-linear system with matrix A given by the formula $A = \mathbb{P}_0^* \otimes \mathbb{P}_1$ (see [1]) with

$$\mathbb{P}_0 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \qquad \mathbb{P}_1 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix},$$

$$\mathbb{P}_0^* = \bigotimes_{n \geq 0} \mathbb{P}_0^n = \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix},$$

so that

$$A = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}.$$

Let Y_n^i denote the epoch of the nth firing of transition i in the Petri net of Figure 2, and let $Y_n = (Y_n^i)$. It is easy to check that if one takes the initial condition $Y_0 = (0, 0, 0, 0)^t$, then $\{Y_n\}$ is the solution of the evolution equation

$$Y_{n+1} = A \otimes Y_n, \qquad n \geq 1$$

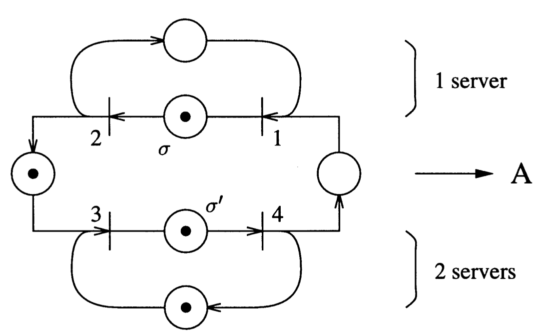


FIG. 2. *A network with three servers modeled by A.*

and is coordinatewise nondecreasing. Notice the further use that this recurrence reads

$$
\begin{aligned}
Y^1_{n+1} &= \sigma \otimes Y^1_n \oplus \sigma' \otimes Y^3_n,\\
Y^2_{n+1} &= \sigma \otimes Y^1_n,\\
Y^3_{n+1} &= Y^2_n \oplus Y^4_n,\\
Y^4_{n+1} &= \sigma' \otimes Y^3_n.
\end{aligned}
$$

Consider now another network with one server less in station 2 (Figure 3). By similar arguments,

$$
\mathbb{P}'_0 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \qquad \mathbb{P}'_1 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix},
$$

$$
\mathbb{P}'^*_0 = \bigoplus_{n \geq 0} \mathbb{P}'^n_0 = \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & 0 \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix},
$$

that is,

$$
A' = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \sigma' & \varepsilon \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}.
$$

The same observations as above can be made, and in this case,

$$
\begin{aligned}
Y'^1_{n+1} &= \sigma \otimes Y'^1_n \oplus \sigma' \otimes Y'^3_n,\\
Y'^2_{n+1} &= \sigma \otimes Y'^1_n,\\
Y'^3_{n+1} &= Y'^2_n \oplus Y'^3_n \otimes \sigma' = Y'^2_n \oplus Y'^4_{n+1},\\
Y'^4_{n+1} &= \sigma' \otimes Y'^3_n.
\end{aligned}
$$

Let us now consider a third network with an initial condition as above, namely, one customer starting its service in station 1 and two customers in station 2, one starting its service and the other one in the buffer, and with

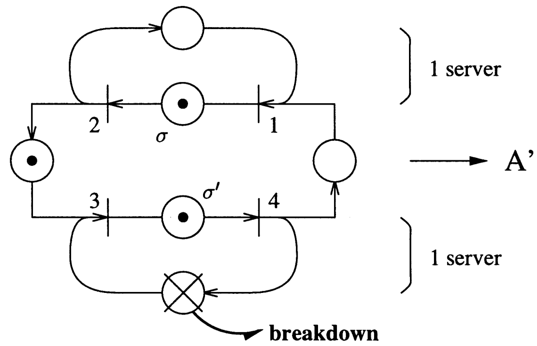


FIG. 3. *Breakdown case modeled by A'.*

the following types of breakdowns: when the nth customer enters station 2, $n \geq 1$,

1. Either there is a breakdown (which takes place with probability p), and this customer can only start its service there after n departures have taken place from this station (as in the one server case);
2. Or there is no breakdown (which takes place with probability $1 - p$) and he only has to wait for $n - 1$ departures to have taken place (as in the two server case).

To describe the evolution of a such system, we put

$$X_{n+1} = A(n) \otimes X_n,$$

where X_n^i denote the nth epochs when a customer leaves the transition i [i.e., X_n^1 (resp. X_n^3) is the nth epochs when a customer starts its service in station 1 (resp. 2) and X_n^2 (resp. X_n^4) is the epoch when a customer leaves station 1 (resp. 2)].

Then we can use the above framework with $A(n)$ equal to A or A' with probability $(1-p)$ and p, respectively, to describe the network with breakdown (for justifications and details, see the Appendix). We check that for A, $c = 4$, $\gamma(0) = \sigma$ and $V = (2\sigma, 2\sigma, \sigma, \sigma')^t$.

1. For $\sigma' < 2\sigma$, there are two regimes: if $2\sigma' < 3\sigma, \gamma(p) = \sigma$ or if $2\sigma' > 3\sigma$, we find by direct computations the following series expansion:

$$\gamma(p) = \sigma + (2\sigma' - 3\sigma)p + (5\sigma - 3\sigma')p^2 + (4\sigma' - 6\sigma)p^3 + (7\sigma - 5\sigma')p^4$$
$$+ (9\sigma' - 13\sigma)p^5 + (26\sigma - 17\sigma')p^6 + \cdots.$$

2. For $\sigma' > 2\sigma$, A, becomes 2-periodic.

3.2.3. *Window flow control.* Now we proceed in the same way as the previous example except that instead of breaking down servers, we cut down the number of customers allowed in the system. This is what happens in the

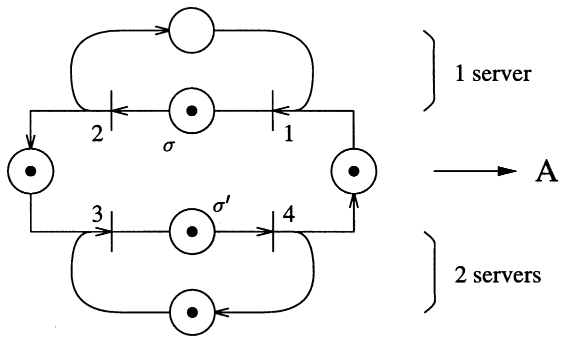


FIG. 4. *A network with four customers modeled by A.*

TCP/IP protocol, where the window size is reduced in case of overload. We consider here the case where there are four or two customers in the network.

Let A denote the matrix of the system with four customers and A' that of the system with 2 (cf. Figures 4 and 5). We have

$$A = \begin{pmatrix} \sigma & \varepsilon & \varepsilon & 0 \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}, \qquad A' = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}.$$

We will only consider the case $\sigma' \in]\sigma, 2\sigma[$. We find that $c = 7$, $\gamma(0) = \sigma$ and $V = (2\sigma, 2\sigma, \sigma, \sigma')^t$.

Direct computations give the following series expansion:

$$\gamma(p) = \sigma + (-\sigma + \sigma')p^2 + (\sigma - \sigma')p^3 + (-\sigma + \sigma')p^4 + \cdots.$$

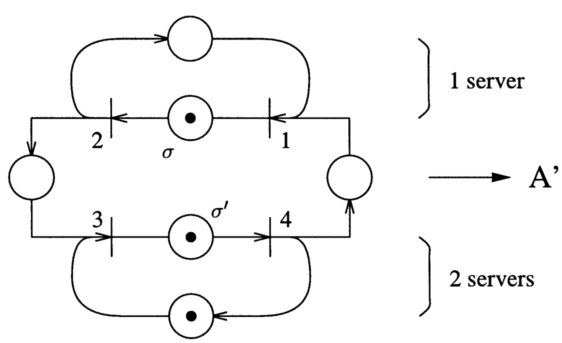


FIG. 5. *Two customers only allowed case modeled by A'.*

4. Extensions. In this section, we give three extensions of the Bernoulli case.

4.1. *Multinomial case.* We first extend Theorem 1 to the multinomial case. That is, instead of having one perturbation possibility by A', we allow m types of perturbations through m matrices $A'_1, \dots, A'_m$.

4.1.1. *Notations and assumptions.* Let $\{A(n)\}$ be an i.i.d. sequence of matrices of $\mathbb{R}_{\max}^{d,d}$, following a discrete distribution,

$$A(n) = \begin{cases} A, & \text{with probability } 1 - p_1 - p_2 - \cdots - p_m, \\ A'_1, & \text{with probability } p_1, \\ \vdots & \vdots \\ A'_m, & \text{with probability } p_m. \end{cases}$$

The sequence $\{X_n\}$ is defined by the same linear recurrence equation as in (1). This will be referred to as $(m+1)$-nomial scheme in what follows. In this section, the assumptions are the following:

(H1)′ The matrix A is irreducible.
(H2)′ The matrix A is scs1-cyc1.
(H3)′ Each of the m matrices A'_i has at least one entry different from ε on each row.

By the same arguments as in the binomial case, under these assumptions, both Result 2 and Result 3 still hold, and in particular the Lyapunov exponent Γ is well defined.

For stating the multinomial theorem, we will need the following notations, where all indices in capital letters are vector indices:

$$P = (p_1, \dots, p_m), |P| = \max_{i=1,\dots,m} |p_i|,$$

$$K = (k_1, \dots, k_m).$$

If $B = \{A(n)\}$, $|B| = K$ means that in the sequence B, Card $\{i\colon A_i = A'_i\} = k_i$, for $1 \le i \le m$.

$$|K| = \sum_{i=1}^{m} k_i,$$

$$K! = \prod_{i=1}^{m} k_i!,$$

$$\binom{n}{L} = \frac{n!}{L!(n-|L|)!}, \qquad \binom{L}{K} = \frac{L!}{K!(L-K)!},$$

$K \le L$ means that $\forall i, k_i \le l_i$.

THEOREM 2. *Under assumptions* (H1)′, (H2)′ *and* (H3)′, *the Lyapunov exponent of the* $m+1$-*nomial scheme*, $\Gamma(P)$, *is analytic at point* 0, *with a radius of* *convergence (w.r.t $|P|$) larger than or equal to $1/(2cm)$, where c is the coupling time of A. The coefficients of the expansion are given by the following formula, where V is the eigenvector of A:*

$$
\begin{aligned}
&\frac{1}{L!}\frac{d^L}{dP^L}[\Gamma(P)]_{P=0} \\
&\quad = (-1)^{|L|}\left\{\sum_{|K|=1}\left\{\binom{c}{L-K}V\right\} + \binom{c+1}{L}\Gamma(0)\right\} \\
&\qquad + \sum_{\substack{1\le|K|\\ K\le L}}\left\{(-1)^{|L|-|K|}\sum_{j_1\cdots j_{|K|-1}=0}^{c-1}\ \sum_{\substack{C_n,\ n=1,\dots,|K|\\ \text{with Card}\{n:\ C_n=A'_i\}=k_i}}\binom{2c+j_1+\cdots+j_{|K|-1}}{L-K}\right. \\
&\qquad\qquad \left.\times A^c\otimes C_1\otimes A^{j_1}\otimes C_2\otimes\cdots\otimes A^{j_{|K|-1}}\otimes C_{|K|}\otimes V.\right\}
\end{aligned}
\tag{8}
$$

For all $|P|\in[0,\delta]$, with $\delta<1/(2cm)$, the error term in the Taylor expansion of $\Gamma(P)$ of order l is bounded from above by

$$\frac{D(2cm\delta)^{l+1}[1+l(1-2cm\delta)]}{(1-2cm\delta)^2},$$

where $D = \big[2(|||A|||\vee|||A'_1|||\vee\ldots\vee|||A'_m|||)+\|V\|_D\big](c+1)$.

The proof is given in Section 6.2.

4.1.2. *Examples.* Here is an example of a 3-nomial scheme, that is, of an i.i.d. sequence $\{A(n)\}$ sampled from three values A, A'_1 and A'_2, with respective probabilities p_0, p_1, and p_2.

The three matrices are chosen as follows:

$$A=\begin{pmatrix}1&\varepsilon&0\\2&0&1\\\varepsilon&0&0\end{pmatrix},\qquad A'_1=\begin{pmatrix}0&\varepsilon&2\\1&0&2\\\varepsilon&0&3\end{pmatrix},\qquad A'_2=\begin{pmatrix}2&\varepsilon&0\\1&0&3\\\varepsilon&1&0\end{pmatrix}.$$

A is scs1-cyc1, $c=2$, $V=(0,1,0)^t$ and $\gamma(0)=1$. We can then compute the coefficient of the Taylor series expansion from Formula (8), which gives

$$
\begin{aligned}
\gamma(p_1,p_2) = 1 &+ p_1 + p_2 \\
&+ p_1^2 \\
&+ 0 \\
&+ p_1^2p_2^2 - p_1^3p_2 \\
&- p_1^3p_2^2 \\
&+ p_1^2p_2^4 - 3p_1^3p_2^3 + p_1^4p_2^2 \\
&- 2p_1^3p_2^4 + 2p_1^4p_2^3 \\
&- 5p_1^3p_2^5 + 6p_1^4p_2^4 - p_1^5p_2^3
\end{aligned}
$$

$$+ 7p_1^4p_2^5 - 3p_1^5p_2^4$$
$$- 10p_1^5p_2^5 + o(\|(p_1, p_2)\|_\infty^{10}).$$

In this example, the random sequence has a fixed structure but this is not necessary for this type of computation. This case was chosen in such a way that when starting from the initial condition V, the Markov chain $\overline{X}_n$ evolves on a finite number of states, which allows us to obtain the exact value of the Lyapunov exponent,

$$\gamma(p_1, p_2) = 1 + p_1 + p_2 + \frac{p_1^2(1 - p_2 + p_1p_2 - p_1p_2^2)}{1 - p_2 - p_2^2 + p_2^3 + 2p_1p_2 - p_1p_2^2 - p_1p_2^3 + p_1^2p_2^2}.$$

Of course it is generally not the case that such a direct computation can be made. The interest of a series expansion stems from the fact that it also holds when the above finiteness property is not satisfied (a sufficient condition of this property can be given solving the well-known *Burnside problem*; see [8]).

4.2. *Weakening of the* (H2) *assumption.* All assumptions are as in the binomial case, except for (H2) which is replaced by the weaker assumption that there is a pattern of $\{A, A'\}$ of length q which is scs1-cyc1 and irreducible (see definition in Result 3).

Let $\widetilde{A}$ denote the pattern, $\widetilde{A}(n)$ be defined by the relation

$$\widetilde{A}(n) = A((n+1)q - 1) \otimes \cdots \otimes A(nq)$$

and let $\widetilde{X}_n$ be defined by the recurrence relation

$$\widetilde{X}_{n+1} = \widetilde{A}(n) \otimes \widetilde{X}_n.$$

The sequence $\{\widetilde{A}(n)\}$ is i.i.d. and each matrix in this sequence can take at most 2^q values. If all these matrices have at least one entry different from ε on each row, we can check that

$$\widetilde{\Gamma}(p) = q\Gamma(p),$$

where $\widetilde{\Gamma}(p)$ is the Lyapunov exponent of the 2^q-nomial scheme.

REMARK 5. One cannot generally get the analyticity of Γ w.r.t. p at 0 from the analyticity of $\widetilde{\Gamma}$ w.r.t. the new $2^q - 1$ parameters. If the pattern contains both A and A', it is clear that its probability cannot be close to 1 whatever the value of p.

4.3. *The operator case.* An operator $A\colon \mathbb{R}^d \to \mathbb{R}^d$ is 1-homogeneous if for all vectors $X \in \mathbb{R}^d$ and all scalars α, if $Y = X + \alpha\mathbf{1}$ is the vector with coordinates $Y_i = X_i + \alpha$, then $A(Y) = A(X) + \alpha\mathbf{1}$, namely $A(Y)_i = A(X)_i + \alpha$ for all i.

Consider the following setting: let $A\colon \mathbb{R}^d \to \mathbb{R}^d$ and $A'\colon \mathbb{R}^d \to \mathbb{R}^d$ be two deterministic 1-homogeneous operators. Let $A(n)$ be an i.i.d. sequence of operators defined the same way as in the (max, plus)-binomial case, namely $A(n)$ is equal to A with probability p and to A' with probability $(1-p)$. Let

$$X_n = A(n-1) \circ \cdots \circ A(0)(X_0),$$

where X_0 is some deterministic vector. Let $f\colon \mathbb{R}^d \to \mathbb{R}$, be such that for all X and Y as above, $f(Y) = \alpha + f(X)$.

The results of the binomial setting can be generalized to such an operator setting under the condition that the properties (P1)–(P3) below are satisfied.

(P1) (Existence of the Lyapunov exponent).

$$\lim_{n\to+\infty} \frac{\mathbb{E}[f(X_n)]}{n} = \gamma_f(p),$$

where $\gamma_f(p) \in \mathbb{R}$ and $X_0 = V$.

(P2) (Uniform coupling property). A^n, the nth iterate of A is such that

$$\exists V, c \text{ s.t. } \forall n \geq c \qquad \forall Y \in \mathbb{R}^d,\ A^n(Y) \equiv V,$$

where $\equiv$ is the colinearity equivalence relation.

(P3) (Growth rate condition). If $X_0 = V, \forall\, l, |f(X_l)| \leq g(A, A', V, l)$, for some nonnegative function g such that

$$\frac{g(A, A', V, l+1)}{g(A, A', V, l)} \overset{l\to+\infty}{\longrightarrow} \eta \in \mathbb{R}^+.$$

A class of operators for which these properties have been studied is that of topical operators, which includes the class of (min, max, plus) functions. For the deterministic theory of such operators, see [10] and [6]; for the random case, see [17], where an analogue of Result 2 can be found, and [3], which contains an analogue of Result 3.

THEOREM 3. *Under* (P1), (P2) *and* (P3), $\gamma_f(p)$ *is analytic at point* 0; *the radius of convergence is larger than or equal to* $1/(2c\eta)$ *and the coefficients of the analytic expansion are given by the following formula*:

$$\begin{aligned}\pi_f(l) = (-1)^l &\left\{ \binom{c}{l-1} f(V) + \binom{c+1}{l} \gamma_f(0) \right\} \\ &+ \sum_{k=1}^{l} \left\{ (-1)^{l-k} \sum_{j_1,\ldots,j_{k-1}=0}^{c-1} \binom{2c + j_1 + \cdots + j_{k-1}}{l-k} \right. \\ &\qquad \left. \times f\big(A^c \circ A' \circ A^{j_1} \circ A' \cdots A^{j_{k-1}} \circ A'(V)\big) \right\}\end{aligned} \tag{9}$$

The proof is given in Section 6.3.

REMARK 6. In the binomial (max, plus) framework described in the previous sections, (P2) is satisfied when A^c is of rank one. As was shown above, for this, it is enough to have A irreducible and scs1-cyc1. However, condition (P2) includes cases of reducible matrices. In this case, Theorem 3 allows one to the evaluate all components of the first-order limits such as $\lim_n((X_n)_i/n)$ and, in particular,

$$\lim_{n\to+\infty} \frac{\max_{1\le i\le d}(X_n)_i}{n} = \gamma^{\text{top}}(p), \quad \text{and} \quad \lim_{n\to+\infty} \frac{\min_{1\le i\le d}(X_n)_i}{n} = \gamma^{\text{bot}}(p),$$

when taking $f(X_n) = \max_i(X_n)_i$ [resp. $\min_i(X_n)_i$].

4.3.1. *Example: random (min, max, plus) operators.* Here we give an example of random topical operators belonging to the (min, max, plus) class of functions (see [10]).

For operator A, we take

$$A(x_1, x_2, x_3) = \begin{bmatrix} (x_1 \vee (x_2+1) \vee (x_3+1)) \wedge ((x_1+1) \vee (x_2+1) \vee (x_3+1)) \\ ((x_1+1) \vee (x_2+2) \vee (x_3+1)) \wedge (x_1 \vee (x_2+1) \vee x_3) \\ ((x_1+1) \vee x_2 \vee (x_3+2)) \wedge (x_1 \vee (x_2+1) \vee (x_3+2)) \end{bmatrix}.$$

For this operator, one can show as in [6] that an analogue of Result 1 holds, with $c = 4$, $V = (1, 0, 2)^t$ and $\gamma = 2$.

For A' we take operator A of the last (max, plus) example, that is,

$$A'(x_1, x_2, x_3) = \begin{bmatrix} (x_1+1) \vee x_3 \\ (x_1+2) \vee x_2 \vee (x_3+1) \\ x_2 \vee x_3 \end{bmatrix}.$$

From (9), we get

$$\gamma(p) = 2 - 2p + p^2 + 2p^3 - p^4 - 4p^5 + 7p^7 + 3p^8 + o(p^8),$$

a formula which can be ckecked by simulation.

4.4. *General extension theorem.* The three extensions we have presented above are all compatible; we can consider a multinomial case with a pattern of length larger than one and satisfying (H2)$'$, or consider a multinomial scheme in a more general operator framework, or such a framework with a general pattern, etc. In the general theorem below, we handle these three extensions and we also replace (P2) by a weaker condition (P2)$'$.

Let $\mathscr{L} = \{A, A'_1, \dots, A'_m\}$ be a finite set of operators from $\mathbb{R}^d$ to $\mathbb{R}^d$ and let $\{A(n)\}$ be a sequence of i.i.d. $\mathscr{L}$-valued operators, where $A(n)$ is equal to A'_l with probability p_l and to A with probability $(1 - \sum p_l)$. Let

$$X_n = A(n-1) \circ \cdots \circ A(0)(X_0),$$

where X_0 is some vector.

For all $f\colon \mathbb{R}^d \to \mathbb{R}$, $X_0 \in \mathbb{R}^d$ and all sequences or operators $B = (O(1), O(0), O(-1), \ldots)$, from $\mathbb{R}^d$ to $\mathbb{R}^d$, let

$$\Psi_{n,X_0,B} = f \circ O(1) \circ O(0) \circ \cdots \circ O(-n)(X_0) \\ - f \circ O(0) \circ \cdots \circ O(-n)(X_0). \tag{10}$$

Consider the following assumptions.

(P2)′ (Memory loss property). There exists a vector $V \in \mathbb{R}^d$ and a sequence $M[\mathscr{L}] = (C_1^0, C_2^0, \ldots, C_q^0)$ of q elements of $\mathscr{L}$, such that for all $C_1, C_0, \ldots, C_{-n}, C_{-n-q-1}, C_{-n-q-2}, \ldots$ in $\mathscr{L}$, and all $p \ge n+q+1$,

$$\Psi_{p,\, X_0,\, (C_1, C_0, \ldots, C_{-n}, M[\mathscr{L}], C_{-n-q-1}, C_{-n-q-2}, \ldots)} = \Psi_{n,\, V,\, (C_1, \ldots, C_n, \ldots)}.$$

REMARK 7. Assumption (P2)′ is a generalization of (P2) to the case of operators which are not necessarily homogeneous: if the elements of $\mathscr{L}$ and f are homogeneous, and if the pattern $C_1^0 \circ \cdots \circ C_q^0$ satisfies the uniform coupling property (P2) with $c = 1$, then (P2)′ is satisfied.

Note that, under the memory loss assumption, γ_f does not depend on the initial condition (cf. Remark 13).

We can embed this problem in a $(m+1)^q$-nomial scheme similar to that of Section 4.2 as follows. Let

$$\widetilde{A} = C_1^0 \circ C_2^0 \circ \cdots \circ C_q^0$$

and let $\tilde{\mathscr{L}}$ be the set of all compositions of q operators of $\mathscr{L}$, such that the composition is different from $\widetilde{A}$. This set has at most $\tilde{m} = (m+1)^q$ elements. If we define

$$\widetilde{A}(n) = A((n+1)q - 1) \circ \cdots \circ A(nq),$$

then the sequence $\{\widetilde{A}(n)\}$ is i.i.d. and the law of $\widetilde{A}(n)$ is multinomial on the finite set $\tilde{\mathscr{L}} \bigcup \{\widetilde{A}\}$. Let $\tilde{p}_i$, for i ranging from 1 to $\tilde{m} = (m+1)^q - 1$, denote the probabilities of the elements of $\tilde{\mathscr{L}}$ and let

$$\widetilde{P} = (\tilde{p}_1, \ldots, \tilde{p}_m).$$

THEOREM 4. *Under* (P1), (P2)′ *and* (P3),

$$\tilde{\gamma}_f(\widetilde{P}) = q\gamma_f(\widetilde{P})$$

and the function $\gamma_f(\widetilde{P})$ *is analytic at least in the open ball of radius* $1/(2m\eta)$*; the coefficients of its series expansion are given by*

$$\tilde{\pi}_f(L) = (-1)^{|L|}\Biggl\{ \sum_{|K|=1} \biggl\{ \binom{1}{L-K} f(V) \biggr\} + \binom{2}{L} \tilde{\gamma}_f(0) \Biggr\}$$

$$+ \sum_{\substack{1\le |K| \\ K \le L}} \Biggl\{ (-1)^{|L|-|K|} \sum_{\substack{\widetilde{C}_n \in \widetilde{\mathscr{I}},\, n=1,\dots,|K| \\ \mathrm{Card}\{n:\, \widetilde{C}_n = \widetilde{A}'_i\} = k_i \\ i=1 \text{ to } (m+1)^q - 1}} \binom{2}{L-K} \tag{11}$$

$$\times f\Bigl(\widetilde{A} \circ \widetilde{C}_1 \circ \cdots \circ \widetilde{C}_{|K|}(V)\Bigr) \Biggr\}.$$

REMARK 8. Note that in (11), we only need to sum over the set $\{\max(1, |L|-2) \le |K|, K \le L\}$.

4.5. *Example: task resource model.* We consider the following task resource model described in [9]. We recall some notation:

1. $\mathscr{A}$ is a finite set of tasks.
2. $\mathscr{R}$ is a finite set of resources.
3. R: $\mathscr{A} \to \mathscr{P}(R)$ gives the subset of resources required by a task.
4. h: $\mathscr{A} \times \mathscr{R} \to \mathbb{R}^+ \cup \{-\infty\}$ gives the execution time of a task.

We assume: $\mathscr{A} = \{a, a_1, a_2\}, \mathscr{R} = \{r_1, r_2\}, R(a) = \{r_1, r_2\}, R(a_1) = \{r_1\}, R(a_2) = \{r_2\}, h \equiv 1$. That is, the matrices associated with this model are (cf. Figure 6)

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A'_1 = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}, \qquad A'_2 = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$$

(see [9]). We assume that the sequence of tasks $a(n) \in \mathscr{A}$ is i.i.d. with

$$\begin{aligned} p_1 &= P(a(n) = a_1), \\ p_2 &= P(a(n) = a_2), \\ 1 - p_1 - p_2 &= P(a(n) = a). \end{aligned}$$

In this case, the Lyapunov exponent $\gamma(p)$ was calculated explicitly in [5]:

$$\gamma(p_1, p_2) = \frac{1}{2} \frac{p_1 + p_2 - 4p_1p_2 + (2 - p_1 - p_2)\sqrt{1 - 4p_1p_2}}{\sqrt{1 - 4p_1p_2}}. \tag{12}$$

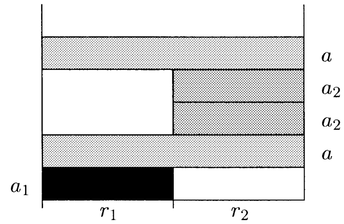


FIG. 6. *Two resources model.*

By application of (11) for $q = 1$, we get the formula

$$\pi(L) = \sum_{\substack{1 \le |K| \\ K \le L}} \left\{ (-1)^{|L|-|K|} \sum_{\substack{A' = A'_i \\ \mathrm{Card}\{n: A_n = A'_i\} = k_i}} \binom{2}{L-K} A \otimes (A')^{|K|} \otimes V \right\}$$

$$+ (-1)^{|L|} \left\{ \sum_{|K|=1} \left\{ \binom{1}{L-K} V \right\} + \binom{2}{L} \Gamma(0) \right\},$$

with $\gamma(0) = 1$ and any vector V, for instance $V = (0, 0, 0)^t$. Then after some elementary calculations,

$$\gamma(p_1, p_2) = 1 - 2 \sum_{n=1}^{+\infty} \binom{2n-2}{n-1} (p_1 p_2)^n$$

$$+ \sum_{n=0}^{+\infty} \binom{2n-1}{n} \left[p_1 (p_1 p_2)^n + p_2 (p_1 p_2)^n \right],$$

which is equal to (12) for $p_1 + p_2 < 1$.

In case of generalized heap models (Tetris type [9]), the formulas of Theorem 4 hold if the memory loss property (P2)′ is satisfied, that is, if there is a heap which can be associated to a task requiring all resources: the corresponding matrix would be of rank one. Here, a guarantees this property.

5. Perturbation representation of the coefficients of the expansion. In this section we show how (11) can be interpreted as a perturbation formula. This representation will allow us to get a new formulation of (11) which is both more compact and more easy to understand. The setting is that of Section 4.4: without loss of generality we take $q = 1$ and $C_1^0 = A$.

For any $K \in \mathbb{R}^m$, we define

$$S^f(K) = \sum_{\substack{C_n \in \{A'_1, \ldots, A'_m\},\, n = 1, \ldots, |K| \\ \mathrm{Card}\{n:\ C_n = A'_i\} = k_i \\ i = 1 \text{ to } m}} f\big(A \circ\ C_1 \circ \cdots \circ C_{|K|}(V)\big),$$

with

$$S^f\big((0, \ldots, 0)\big) = f(A(V)).$$

Let

$$\delta_i S^f(K) = S^f(K - e_i),$$

where e_i is the vector with all components equal to zero except the ith one which is equal to 1, and by convention $S^f(K) = 0$ if at least one component of K is negative.

Finally, we define

$$\Delta S^f(K) = S^f(K) - \sum_{i=1}^{m} \delta_i S^f(K).$$

Then we get

$$\Delta^2 S^f(K) = \Delta(\Delta S^f(K)) = S^f(K) - 2\sum_{i=1}^{m} \delta_i S^f(K) + \sum_{i,j=1}^{m} \delta_i \delta_j S^f(K).$$

We now come back to (11): we can rewrite this, when $|L| > 2$, as

$$\begin{aligned}
\pi_f(L) = (-1)^{|L|} \Bigg\{ \sum_{|K|=1} \left\{ \binom{1}{L-K} f(V) \right\} + \binom{2}{L} \tilde{\Psi}(M_\varnothing) \Bigg\} \\
+ \sum_{\substack{|L|-2 \le |K| \\ K \le L}} \Bigg\{ (-1)^{|L|-|K|} \sum_{\substack{\text{Card } \{n:\ C_n = A'_i\} = k_i \\ C_n \in \{A'_1, \ldots, A'_m\},\ n=1,\ldots,|K| \\ i=1 \text{ to } m}} \binom{2}{L-K} \\
\times f\big(A \circ C_1 \circ \cdots \circ C_{|K|}(V)\big) \Bigg\} \\
= \sum_{\substack{C_n \in \{A'_1, \ldots, A'_m\},\ n=1,\ldots,|L| \\ \text{Card}\{n:\ C_n = A'_i\} = l_i \\ i=1 \text{ to } m}} f\big(A \circ C_1 \circ \cdots \circ C_{|L|}(V)\big) \\
- 2 \sum_{j=1}^{m} \sum_{\substack{C_n \in \{A'_1, \ldots, A'_m\},\ n=1,\ldots,|L|-1 \\ \text{Card}\{n:\ C_n = A'_i\} = l_i - \delta_{ij} \\ i=1 \text{ to } m}} f\big(A \circ C_1 \circ \cdots \circ C_{|L|-1}(V)\big)
\end{aligned}$$

$$+ 2 \sum_{j \neq h=1}^{m} \sum_{\substack{C_n \in \{A'_1, \ldots, A'_m\},\ n=1, \ldots, |L|-2 \\ \mathrm{Card}\{n:\ C_n = A'_i\} = l_i - \delta_{ij} - \delta_{ih} \\ i=1 \text{ to } m}} f\big(A \circ C_1 \circ \cdots \circ C_{|L|-2}(V)\big)$$

$$+ \sum_{j}^{m} \sum_{\substack{C_n \in \{A'_1, \ldots, A'_m\},\ n=1, \ldots, |L|-2 \\ \mathrm{Card}\{n:\ C_n = A'_i\} = l_i - 2\delta_{ij} \\ i=1 \text{ to } m}} f\big(A \circ C_1 \circ \cdots \circ C_{|L|-2}(V)\big)$$

with $\delta_{jh} = 0$ if $j \neq h$ and $\delta_{jj} = 1$. It is then easy to verify that

$$\pi_f(L) = \Delta^2 S^f(L). \tag{13}$$

This relation is verified for any L if we put $\Delta^2 S^f((0, \ldots, 0)) = \Delta f((A(V)) = f(A(V)) - f(V) = \gamma_f(0)$.

6. Proofs.

6.1. *Bernoulli case.* This section focuses on the proof of Theorem 1. This proof is based on a light traffic type expansion in the spirit of that of [2]: the expansion is first derived in the finite memory case; the infinite memory expansion is then obtained via some direct analytical convergence arguments.

Its first part is proved in three steps (6.1.1, 6.1.2, 6.1.3). The notation and assumptions are those of Section 3.1.

LEMMA 1. *Under* (H1), (H2) *and* (H3), *the conclusions of Results* 2 *and* 3 *hold*.

PROOF. (i) Extension of Result 2. The first step of the proof is the same. Let $\gamma^{\mathrm{top}} = \lim_{n\to\infty}(\max_{1\le i\le d}(X_n)_i/n)$, $\gamma^{\mathrm{bot}} = \lim_{n\to\infty}(\min_{1\le i\le d}(X_n)_i/n)$. We now prove that all coordinates of X_n/n again have the same a.s. limit. Under (H1) and (H2), there exists n_0 such that for all (i, j), $(A^{n_0})_{ji} > -\infty$.

Then if $a = \max_{i,j} |(A^{n_0})_{ji}|$ for all (i, j),

$$(X_{n+n_0})_j \ge -a + (X_n)_i$$

whenever $A(n + n_0 - 1) \otimes \cdots \otimes A(n) = A^{n_0}$. In particular,

$$\frac{\min_{1\le i\le d}(X_{n+n_0})_i}{n} \ge -\frac{a}{n} + \frac{\max_{1\le i\le d}(X_n)_i}{n}$$

for an infinite number of integers a.s. and since both $\min_{1\le i\le d}(X_n)_i/n$ and $\max_{1\le i\le d}(X_n)_i/n$ tend to limits a.s., these limits satisfy the inequality $\gamma^{\mathrm{bot}} \ge \gamma^{\mathrm{top}}$. Therefore $\gamma^{\mathrm{top}} = \gamma^{\mathrm{bot}}$.

(ii) Extension of Result 3. The proof is the same as in [12], (6.8, 8).

Thanks to (3), we can see $\overline{X}$ (Result 3) as a $\mathbb{R}^d$-valued functional of the sequence

$$B = \{A_i\}_{i\le 1}.$$

We will also use the following truncation of B:

$$B^n = \{\widetilde{A}_i\}_{i\geq 1} \quad \text{with } \begin{cases} \widetilde{A}_i = A_i, & \text{if } i \geq n, \\ \widetilde{A}_i = A, & \text{if } i < n. \end{cases}$$

We will denote $\mathscr{M}$ the set of all possible values of B and $\mathscr{M}_n$ that of all possible values of B^n. Finally, we will denote $\Psi: \mathscr{M} \to \mathbb{R}^d$ the mapping

$$\Psi(B) = A(1) \otimes X - X = \Delta, \tag{14}$$

where X is any vector in the equivalence class of $\overline{X}$ [see (4)].

If V is an eigenvector of A, then

$$\begin{aligned} \Psi(B^n) &= A_1 \otimes A_0 \cdots A_{-n} \otimes V - A_0 \cdots A_{-n} \otimes V \\ &= A_1 \otimes X_{n+1} \circ \theta^{-n} - X_{n+1} \circ \theta^{-n} \quad \text{if } X_0 = V, \\ &= X_{n+2} \circ \theta^{-n} - X_{n+1} \circ \theta^{-n} \quad \text{if } X_0 = V, \end{aligned} \tag{15}$$

where θ is the basic shift of the sequence $\{A_n\}$ ($A_n = A_0 \circ \theta^n$ for all n). Note that the result is not modified if we replace V by $V \otimes \alpha$ for any scalar α).

In view of Result 3, for all $B \in \mathscr{M}$,

$$\Psi(B^n) \to_n \Psi(B) = \Delta \quad \text{a.s.}$$

REMARK 9. In (15), since V is an eigenvector of an irreducible matrix, for all $i \in \{1, \ldots, d\}$, $V_i > \varepsilon$. Hence taking $\alpha = |||V|||$, we can assume $V \geq 0$, that is, each component of V is positive.

In the same way, replacing A and A' by $\widetilde{A} = A \otimes \alpha$ and $\widetilde{A}' = \widetilde{A} \otimes \alpha$ with $\alpha = |||A||| \oplus |||A'|||$, we get that $\tilde{\Psi} = \Psi \otimes \alpha$. From this, we obtain the initial Lyapunov exponent by the relation $\gamma(p) = \tilde{\gamma}(p) - \alpha$. We conclude that we can assume V, A and A' positive (namely, all entries of these matrices which are not equal to ε are nonnegative). Then for all n, X_n is positive too.

Throughout the section, we will use the following notation:

1. In order to simplify notations, we will replace the product operation $\otimes$ in (max, plus) by "$\cdot$" and the conventional product operation "$\times$" will be omitted.
2. $|B| = k$ means that in the sequence B, $\text{Card}\{i: A_i = A'\} = k$.
3. $M_{i_1 \cdots i_k}$ is a sequence B where $A_i = A'$ for $i = i_1 \cdots i_k$ and only for these indices.
4. $\mathscr{A}_n^p$ denotes the number of arrangements of p elements among n that is,

$$A_n^p = \frac{n!}{(n-p)!}.$$

5. $i_1 \neq \cdots \neq i_k$ means that $i_1, \ldots, i_k$ are pairwise distinct.

6.1.1. *Evaluation of $E\Psi$-truncated case.*

LEMMA 2. *We have*

$$\mathbb{E}[\Psi(B^n)] = \sum_{l=0}^{n+2} \pi^n(l), \tag{16}$$

with

$$\pi^n(l) = \frac{1}{l!} \sum_{\substack{i_1 \neq \cdots \neq i_l \\ -n \leq i_j \leq 1}} \left\{ \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} \Psi(M_{i_1 \cdots i_k}) \right\} \tag{17}$$

where $M_\varnothing$ is the sequence where $A_i = A$ for all i.

PROOF. We obtain an expansion of $E\Psi$ by the following conditioning:

$$\begin{aligned} \mathbb{E}[\Psi(B^n)] &= \sum_{k=0}^{n+2} \mathbb{E}[\Psi(B^n) \mid |B^n| = k] P(|B^n| = k) \\ &= \sum_{k=0}^{n+2} \binom{n+2}{k} p^k (1-p)^{n+2-k} \left(\frac{1}{\binom{n+2}{k}} \sum_{\substack{M \in \mathscr{M}^n \\ |M| = k}} \Psi(M) \right), \end{aligned} \tag{18}$$

because $P(|B^n| = k) = \binom{n+2}{k} p^k (1-p)^{n+2-k}$ and B^n conditioned by $|B^n| = k$ follows a uniform distribution.

By expanding $(1-p)^{n+2-k}$ and regrouping the coefficients of p^k, we get the equation

$$\mathbb{E}[\Psi(B^n)] = \sum_{l=0}^{n+2} \left\{ \sum_{k=0}^{l} \binom{n+2-l+k}{k} (-1)^k \left(\sum_{\substack{M \in \mathscr{M}^n \\ |M| = l-k}} \Psi(M) \right) \right\} p^l.$$

Note that the coefficient $\pi^n(l)$ of $p^l, 0 \leq l \leq n+2$, in the last expression can be rewritten as follows:

$$\pi^n(l) = \sum_{k=0}^{l} (-l)^{l-k} \binom{n+2-k}{l-k} \left(\sum_{\substack{M \in \mathscr{M}^n \\ |M| = k}} \Psi(M) \right).$$

Remark that the two following relations hold:

$$\sum_{|M|=k} \Psi(M) = \frac{1}{k!} \sum_{\substack{i_1 \neq \cdots \neq i_k \\ -n \leq i_j \leq 1}} \Psi(M_{i_1 \cdots i_k}),$$

$$\mathscr{A}_{n-k}^{l-k} \sum_{\substack{i_1 \neq \cdots \neq i_k \\ 1 \leq i_j \leq n}} \Psi(M_{i_1 \cdots i_k}) = \sum_{\substack{i_1 \neq \cdots \neq i_k \\ 1 \leq i_j \leq n}} \Psi(M_{i_1 \cdots i_k}), \qquad k \leq l \leq n.$$

The second relation comes from the fact that $l-k$ indices have to be chosen within the $n-k$ free indices in the r.h.s. sum.

Using these relations, we get

$$\pi^n(l) = \sum_{k=0}^{l} \binom{n+2-k}{l-k} \frac{(-1)^{l-k}}{k!} \left\{ \sum_{\substack{i_1 \neq \cdots \neq i_k \\ -n \leq i_j \leq 1}} \Psi(M_{i_1 \cdots i_k}) \right\}$$

$$= \sum_{k=0}^{l} (-1)^{l-k} \frac{1}{k!} \binom{n+2-k}{l-k} \frac{(n+2-l)!}{(n+2-k)!} \left\{ \sum_{\substack{i_1 \neq \cdots \neq i_l \\ -n \leq i_j \leq 1}} \Psi(M_{i_1 \cdots i_k}) \right\}$$

$$= \sum_{k=0}^{l} (-1)^{l-k} \frac{1}{l!} \binom{l}{k} \left\{ \sum_{\substack{i_1 \neq \cdots \neq i_l \\ -n \leq i_j \leq 1}} \Psi(M_{i_1 \cdots i_k}) \right\}.$$

The proof is concluded by interchanging the summations. □

6.1.2. *Convergence.* Convergence of the coefficients. We prove that, for each l, $\pi^n(l) \to \pi(l)$, when $n \to +\infty$. In fact this limit exists and is reached in a finite time. This result will be shown to be a direct consequence of the following lemma.

LEMMA 3. *Under the foregoing assumptions*:

(i) *If the* l *indices* $i_1, i_2, \ldots, i_l$ *in* (17) *are all such that* $i_j < -c+1, \forall j \in \{1, \ldots, l\}$, *then*

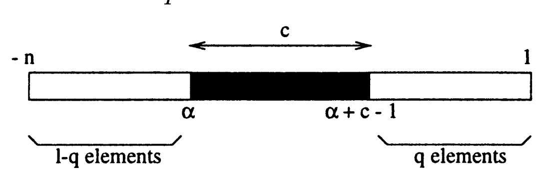
$$\sum_{\substack{i_1 \neq \cdots \neq i_l \\ -n \leq i_j \leq -c}} \left\{ \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} \Psi(M_{i_1 \cdots i_k}) \right\} = 0.$$

(ii) *Let* $\mathscr{N} = \{-n, \ldots, 1\}$ *and let* $C(\alpha) = \{\alpha, \alpha+1, \ldots, \alpha+c-1\}$, *where* α *is an integer such that* $\alpha + c - 1 < 1$ *and* $-n < \alpha$. *Then we have*

$$\sum_{\substack{i_1 \neq \cdots \neq i_l \\ i_j \in \mathscr{N} - \mathscr{C}(\alpha)}} \left\{ \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} \Psi(M_{i_1 \cdots i_k}) \right\} = 0.$$

PROOF. Property 1 follows from the fact the if $i_j < -c+1$ for all j, then $\Psi(M_{i_1 \cdots i_k})$ is equal to $\Gamma(0)$ and of the fact that $\sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} = 0$.

For property 2, consider the case where there are q indices $\alpha_1, \ldots, \alpha_q$, all larger than $\alpha + c - 1$ and $l - q$ indices smaller than α.



In the expression

$$W = \sum_{\substack{i_1 \neq \cdots \neq i_l \\ i_j \in \mathcal{N} - \mathcal{C}(\alpha)}} \left\{ \binom{l}{l} \Psi(M_{i_1 \dots i_l}) - \binom{l}{l-1} \Psi(M_{i_1 \dots i_{l-1}}) + \cdots + (-1)^l \binom{l}{0} \Psi(M_\theta) \right\}, \tag{19}$$

we start by regrouping the terms which contribute to the same value of Ψ. Because of the gap of length c, the $l-q$ indices on the left of α do not affect the value of Ψ. Therefore, once the l indices $\alpha_1 \cdots \alpha_1$ and the integer q are given, the different values of Ψ are

$$\begin{array}{ll} \Psi(M_{\alpha_1 \alpha_2, \dots, \alpha_q}) & \rightsquigarrow 1 \text{ possibility} \\ \Psi(M_{\alpha_1, \dots, \alpha_{q-1}}), \Psi(M_{\alpha_1, \dots, \alpha_{q-2}, \alpha_q}), \text{etc.} & \rightsquigarrow q = \binom{q}{1} \text{ possibilities} \\ \Psi(M_{\alpha_1, \dots, \alpha_{q-2}}), \text{etc.} & \rightsquigarrow \binom{q}{2} \text{ possibilities} \\ \vdots & \vdots \\ \Psi(M_\varnothing) & \rightsquigarrow 1 \text{ possibility.} \end{array}$$

Let us represent W as the sum

$$W = \sum_{\alpha_{\alpha_1, \dots, \alpha_l}} \sum_{q=0}^{l} W^q_{\alpha_1, \dots, \alpha_l},$$

where $W^q_{\alpha_1, \dots, \alpha_l}$ collects all terms with q indices bigger than $\alpha + c - 1$, chosen in the set $\{\alpha_1, \dots, \alpha_l\}$. This can be rewritten as

$$W^q_{\alpha_1, \dots, \alpha_l} = \sum_{k=0}^{q} \beta_k \left\{ \underbrace{\sum_{\alpha(1) < \cdots < \alpha(k) \in \{\alpha_1, \dots, \alpha_q\}}}_{\binom{q}{k} \text{ terms,}} \Psi(M_{\alpha(1) \cdots \alpha(k)}) \right\},$$

since all terms of type $\Psi(M_{\alpha(1)\cdots\alpha(k)})$ depending on k indices have a common factor β_k by symmetry.

To get β_q we count how many times $\Psi(M_{\alpha_1, \alpha_2, \dots, \alpha_q})$ appears in $W^q_{\alpha_1, \dots, \alpha_q}$. For the first term of (19), we have to count the number of ways of arranging the q elements $\{\alpha_1, \dots, \alpha_q\}$ among the l positions. For the second one, we have to count the number of ways of arranging the q elements $\{\alpha_1, \dots, \alpha_q\}$ among the $l - 1$st first positions and so on; in each case we have to multiply this by the number of permutations of the $l - q$ indices on the left of α, so that,

$$\beta_q = (l-q)! \left\{ \binom{l}{l} \mathscr{A}^q_l - \binom{l}{l-1} \mathscr{A}^q_{l-1} + \cdots + (-1)^{l-q} \binom{l}{q} \mathscr{A}^q_q \right\}.$$

Consider now the general case β_{q-j}. First we must remove at least j indices from the q first ones. This means that the first term giving β_{q-j} comes from the jth term of (19), that is, $\Psi(M_{i_1, \dots, i_{l-j}})$: here we have to count the number of ways of arranging the $q-j$ indices $\alpha(1),\dots,\alpha(q-j)$ among the $l-j$ indices showing up in Ψ, and to multiply this by the number of arrangements of the j indices $i_{l-j+1,\dots,i_l}$ among the j removed indices. For the $(l-j-l+k)$th term from $\Psi(M_{i_1\dots,i_k})$, we have to count the number of ways of arranging $q-j$ elements among k indices, times the number of ways of arranging j indices among the $l-k$ removed indices and so on. Therefore,

$$\frac{1}{(l-q)!}\beta_{q-j} = \sum_{k=q-j}^{l-j}(-1)^{l-k}\binom{l}{k}\mathscr{A}_k^{q-j}\mathscr{A}_{l-k}^{j} = \sum_{k=0}^{l}(-1)^{l-k}\binom{l}{k}\mathscr{A}_k^{q-j}\mathscr{A}_{l-k}^{j},$$

with the convention $\mathscr{A}_n^p = 0$ if $p > n$. Now the lemma is almost proved, because $\mathscr{A}_k^{q-j}\ \mathscr{A}_{l-k}^{j}$ is a polynomial in k, the degree of which is less than q, that is less than $l-1$. We just need the following result to conclude the proof:

$$\forall\ j\in\{1,\dots,l-1\},\qquad \sum_{k=0}^{l}(-1)^k k^j\binom{l}{k} = 0,$$

and this can be easily proved by induction when differentiating j times the function $(1-x)^l$ and evaluating it at point 1. □

CONSEQUENCE 1. *This lemma enables us to write $\pi(l)$ as*

$$\pi(l) = \frac{1}{l!}\sum_{\substack{i_1\neq\cdots\neq i_l\\ |i_m-i_n|\le c\\ \text{for all successive indices } (i_m,i_n)}}\ \sum_{k=0}^{l}(-1)^{l-k}\binom{l}{k}\Psi(M_{i_1\dots i_k}), \tag{20}$$

with $\Psi(M_{i_1\dots i_k}) = \Psi(M_\varnothing) = \Gamma(0)$ if $k=0$. Remark that the first index should be in $[-c+1,1]$ due to (i) *of Lemma* 3. *Consequently, it is sufficient to look for indices in $[-(lc-1),1]$ and this implies that*

$$\pi_n(l) = \pi(l)\qquad \forall\ n\ge lc-1. \tag{21}$$

Convergence of the series.

LEMMA 4. *The series representation of $\mathbb{E}[\Psi(B^n)]$, namely,*

$$S_n = \sum \pi^n(l)p^l,$$

is uniformly convergent when n goes to infinity. In addition, $S=\lim_n S_n$ is such that

$$S = \sum \pi(l)p^l = \Gamma(p). \tag{22}$$

PROOF. We first discuss the conditions under which the series S is convergent. For this, we give a simple bound on $\Psi(M_{i_1\dots i_k})$. For all $i \in \{1, \dots, d\}$, $Z \in \mathbb{R}^d$,

$$\begin{aligned}|(A(1)\cdot Z - Z)_i| &= \left|\bigoplus_{1\le j\le d}\big(A(1)_{ij}\cdot Z_j\big) - Z_i\right|\\ &= \left|\bigoplus_{\substack{1\le j\le d\\ A(1)_{ij}\ne\varepsilon}}\big(A(1)_{ij}\cdot Z_j\big) - Z_i\right|\\ &= |A(1)_{ijo} + Z_{jo} - Z_i| \quad \text{for some } j_o\\ &\le |A(1)_{ijo}| + |Z_{jo} - Z_i|\\ &\le F + \|Z\|_{\mathscr{D}},\end{aligned}$$

with $F = |||A||| \vee |||A'|||$. In these inequalities, we have used the assumption that both A and A' have at least one non-ε element on each row.

Similarly, for all i, j, all $Z \in \mathbb{R}^d$ and all n,

$$\big|(A(n)\cdot Z)_i - (A(n)\cdot Z)_j\big| = \big|A(n)_{ii_0} + Z_{i_o} - A(n)_{jj_0} - Z_{j_o}\big| \le 2F + \|Z\|_{\mathscr{D}}$$

(sharper bounds can be derived whenever A and A' are positive, in the sense of Remark 9). Hence

$$\|A(n)\cdot Z\|_{\mathscr{D}} \le 2F + \|Z\|_{\mathscr{D}}$$

and by induction

$$\forall k, \forall i_j, \forall l \ge 1 \quad |\Psi(M_{i_1\dots i_k})| \le (cl+1)D' \le Dl,$$

with

$$D = \big[2(|||A||| \vee |||A'|||) + \|V\|_{\mathscr{D}}\big](c+1).$$

Using this, it follows from (20) that $|\pi(l)|$ is bounded from above by

$$\frac{l!}{l!} l c^l D \sum_{k=0}^{l} \binom{l}{k} = Dl(2c)^l. \tag{23}$$

This bound also holds for $\pi^n(l)$, for all n. Therefore, we have dominated convergence for S_n on every compact of the complex plane $\{|p| \le \delta\}$ with $\delta < (2c)^{-1}$, so that the convergence is uniform on every $[0, \delta]$, and therefore

$$S = \lim_{n\to+\infty} S_n = \sum_l \pi(l) p^l$$

is analytic in p in the open disc of radius $(2c)^{-1}$.

In addition, by Cesaro averaging,

$$S = \lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}[\Psi(B^i)]}{n}$$

and since

$$\frac{\sum_{i=1}^{n} \mathbb{E}[\Psi(B^i)]}{n} = \frac{\mathbb{E}[X_{n+2}] - \mathbb{E}[X_2]}{n} \overset{n\to+\infty}{\longrightarrow} \Gamma(p),$$

(where the last limit follows from Result 2, which holds here due to Lemma 1), the last assertion of the lemma follows immediately. □

REMARK 10. Here are two sufficient conditions where, in addition to (22), we have

$$\Gamma(p) = \mathbb{E}[\Delta] \tag{24}$$

[Δ is the random variable defined in (4)].

(i) If A and A' have all their entries different from ε, then we can show that $\|X_n\|_{\mathscr{D}}$ is bounded, which implies the boundedness of $\|A_1 \cdot X_n - X_n\|_\infty$. By dominated convergence, we can then conclude as indicated at the end of section 2.2 [see the proof of (5)].

The case when there exists a positive integer q such that all matrices obtained by products of q matrices of $\{A, A'\}$ have all their entries different of ε can be handled in the same way.

A more general sufficient condition can be obtained from the finitely generated torsion semigroup property related to the so-called Burnside problem [8].

(ii) When p is small enough, we can bound $\Psi(B^n)$ uniformly by an integrable random variable, using the same kind of arguments as above. Indeed, let $l(A')$ be the last index $n, n \in \{1, 0, -1, \dots\}$ such that $A(n) = A'$ and such that the sequence $1, 0, -1, \dots, l(A')$ has no subsequence of more than c consecutive A. Then

$$\exists\, D \text{ such that } \forall\, n, |\Psi(B^n)| \le D l(A').$$

It is easy to show that $l(A')$ is integrable. So we obtain by dominated convergence that $\Psi(B^n)$ tends to $\Psi(B) = \Delta$ in L_1, which concludes the proof of (24).

REMARK 11. The exact domain of analyticity is generally larger than the disc of radius $(2c)^{-1}$. We found no numerical evidence indicating the existence of a singularity at this point. This question will be examined in a future paper.

6.1.3. *Simplification of the coefficients.* The aim of this section is to simplify the expression in (20). Indeed we can easily see that many terms in (20) are redundant. Let us give an example through a simple case.

First, remark that there is another way to write our expressions using the notation

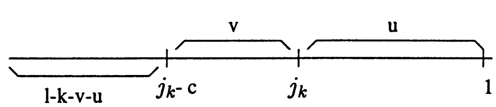
$$A(i_1, \dots, i_k) = A^{i_1} \cdot A' \cdot A^{i_2} \cdot A' \cdots A' \cdot A^{i_k} \cdot A'.$$

Then

$$
\begin{aligned}
\frac{1}{l!}\sum_{i_1\neq\cdots\neq i_l}\Psi(M_{i_1\cdots i_l}) &= \sum_{i_1>\cdots>i_l}\Psi(M_{i_1\cdots i_l})\\
&= \sum_{i_1=-c+2}^{1}\sum_{i_2=i_1-c}^{i_1-1}\cdots\sum_{i_l=i_{l-1}-c}^{i_{l-1}-1}\Psi(M_{i_1\cdots i_l})\\
&= \sum_{\cdots}\cdots\sum_{\cdots}\bigl\{A^{1-i_1}\cdot A'\cdot A^{i_1-i_2-1}\cdot A'\cdots A'\\
&\qquad\qquad\cdot A^{i_{l-1}-i_l-1}\cdot A'\cdot V - A^{-i_1}\cdot A'\cdots A'\cdot V\bigr\}\\
&\quad\text{with the convention } A^{-1}\cdot A' = I_d \text{ identity matrix}\\
&= \sum_{i_1=0}^{c}\sum_{i_2=0}^{c-1}\cdots\sum_{i_l=0}^{c-1}\bigl\{A^{i_1}\cdot A'\cdot A^{i_2}\cdot A'\cdots A'\cdot A^{i_l}\cdot A'\cdot V\\
&\qquad\qquad - A^{-i_1-1}\cdot A'\cdots A'\cdot V\bigr\}\\
&= \sum_{i_1=0}^{c}\sum_{i_2=0}^{c-1}\cdots\sum_{i_l=0}^{c-1}A^{i_1}\cdot A'\cdot A^{i_2}\cdot A'\cdots A'\cdot A^{i_l}\cdot A'\cdot V\\
&\quad - \sum_{i_1=-1}^{c-1}\sum_{i_2=0}^{c-1}\cdots\sum_{i_l=0}^{c-1}A^{i_1}\cdot A'\cdot A^{i_2}\cdot A'\cdots A'\cdot A^{i_l}\cdot A'\cdot V\\
&= \sum_{i_2=0}^{c-1}\cdot s\sum_{i_l=0}^{c-1}\bigl\{A^{c}\cdot A'\cdot A^{i_2}\cdot A'\cdot sA'\cdot A^{i_l}\cdot A'\cdot V - A^{i_2}\cdot sV\bigr\}\\
&= \sum_{i_1=0}^{c-1}\cdot s\sum_{i_{l-1}=0}^{c-1}\{A^{c}\cdot A'\cdot A(i_1\cdot si_{l-1})\cdot V - A(i_1\cdot si_{l-1})\cdot V\}.
\end{aligned}
$$

Now we would like to operate in the same way for the other terms. But the difficulty is that the general term $\Psi(M_{i_1\cdots i_k})$ is not invariant by permutation of $\{i_1\cdots i_l\}$. This will be taken care of by the two following lemmas.

LEMMA 5. *Let $j_k < \cdots < j_1$ be such that $|j_n - j_{n+1}| \le c$ for $n = 1$ to $k-1$ [this guarantees $\Psi(M_{j_1\cdots j_k})$ is really depending on k terms). Assume there are v indices of $j_{k+1},\dots,j_l$ in $]j_k - c, j_k[$ and u indices of $j_{k+1},\dots,j_l$ between j_k and 1. We assume the l indices $j_1,\dots,j_k$ and $j_{k+1},\dots,j_l$ fixed. Then in the sum* (20), *all the factors of terms $\Psi(M_{j_1\cdots j_k})$ where u+v indices have been suppressed in $]j_k - c, 1[$, sum up to zero if $l-k-v-u$ is not equal to zero.*



PROOF. To get $\Psi(M_{j_1\cdots j_k})$ we have to suppress at least $v+u$ elements between $j_k - c$ and 1. Therefore we have to look at the contributions of the following terms:

$$(-1)^{v+u}\binom{l}{l-(v+u)}\Psi(M_{j_1\dots j_{l-(v+u)}}),$$

$$(-1)^{v+u+1}\binom{l}{l-(v+u+1)}\Psi(M_{j_1\dots j_{l-(v+u+1)}}),$$

$$\vdots$$

$$(-1)^{l-k}\binom{l}{k}\Psi(M_{j_1\dots j_k}),$$

From the first term we get

$$\frac{1}{l!}(-1)^{v+u}\binom{l}{l-(v+u)}(v+u)!(l-(v+u))! = (-1)^{v+u}$$

because the l indices are fixed and we can only permute those giving the same value, that is, $v+u$ and $l-(v+u)$.

From the second term,

$$\frac{1}{l!}(-1)^{v+u+1}\binom{l}{l-(v+u+1)}(v+u+1)!(l-(v+u+1))!\binom{l-k-v-u}{1}$$
$$= (-1)^{v+u+1}\binom{l-k-v-u}{1}$$

because the $(v+u+1)$th index to remove can be chosen among the $l-k-v-u$ indices on the left.

From the $(n+1)$th term,

$$(-1)^{v+u+n}\binom{l-k-v-u}{n}$$

because we can remove n indices chosen among the $l-k-v-u$ indices on the left.

So by summation,

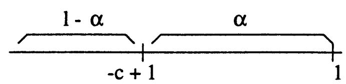
$$(-1)^{v+u}\binom{l-k-v-u}{0} + \cdots + (-1)^{l-k}\binom{l-k-v-u}{l-k-v-u} = 0$$

and the lemma is proved. □

CONSEQUENCE 2. *The sum differs from zero only when $l-k-v-u=0$. Therefore, in order to get the factor of $\Psi(M_{j_1\dots j_k})$, we just need to consider cases where only indices between $j_k - c$ and 1 are suppressed. And for this operation, we choose $l-k$ indices among $(1-(j_k-c)+1)-k$ so that the factor is*

$$\frac{1}{l!}(-1)^{l-k}\binom{l}{k}k!(l-k)!\binom{2-j_k+c-k}{l-k} = (-1)^{l-k}\binom{2-j_k+c-k}{l-k}.$$

LEMMA 6 (5 bis). *In the sum* (20) *all the factors giving* $\Psi(M_\varnothing)$ *when suppressing* α *indices in* $[-c+1, 1]$ *sum up to zero if* α *is not equal to* l.



PROOF. Just do as in the previous lemma to obtain

$$(-1)^{\alpha}\binom{l-\alpha}{0} + \cdots + (-1)^{l}\binom{l-\alpha}{l-\alpha} = 0. \qquad \square$$

CONSEQUENCE 3. *The factor of* $\Psi(M_\varnothing) = \Gamma(0)$ *in* $\pi(l)$ *comes from the case* $l - \alpha = 0$. *Hence it is equal to* $(-1)^l\binom{c+1}{l}$ *if* $l \le c+1$ *and zero if not.*

Now we can rewrite $\pi(l)$ as follows:

$$\pi(l) = \sum_{k=1}^{l}\left\{\sum_{i_1=1}^{-c+1}\sum_{i_2=i_1-1}^{i_1-c}\cdots\sum_{i_k=i_{k-1}-1}^{i_{k-1}-c}(-1)^{l-k}\binom{2-i_k+c-k}{l-k}\Psi(M_{i_1\cdots i_k})\right\} \tag{25}$$
$$+(-1)^l\binom{c+1}{l}\Gamma(0),$$

with the convention $\binom{n}{p} = 0$ if $p > n$.

Take now as new variables

$$\begin{aligned} j_1 &= 1 - i_1,\\ j_2 &= i_1 - i_2 - 1, \qquad j_1 + j_2 + \cdots + j_k = -i_k + 2 - k,\\ &\ \vdots\\ j_l &= i_{l-1} - i_l - 1. \end{aligned}$$

Then

$$\begin{aligned}\pi(l) = \sum_{k=1}^{l}\sum_{j_1=0}^{c}\sum_{j_2,\dots j_k=0}^{c-1}&(-1)^{l-k}\binom{c+j_1+\cdots+j_k}{l-k}\\ &\times\left[A(j_1, j_2\cdots j_k)\cdot V - A(j_1-1\cdots j_k)\cdot V\right]\\ &+(-1)^l\binom{c+1}{l}\Gamma(0).\end{aligned}$$

For the k fixed indices, we have, omitting V,

$$\sum_{\cdots}\binom{c+j_1+\cdots+j_k}{l-k}[A(j_1\cdots j_k) - A(j_1-1\cdots j_k)]$$
$$= \sum_{\substack{j_1=0\\ \cdots}}^{c}\binom{c+j_1+\cdots+j_k}{l-k}A(j_1\cdots j_k)$$

$$- \sum_{\substack{j_1=-1\\ \cdots}}^{c-1} \underbrace{\binom{c+j_1+\cdots+j_k+1}{l-k}}_{\binom{c+j_1\cdots j_k}{l-k}+\binom{c+j_1\cdots j_k}{l-k-1}} A(j_1\cdots j_k)$$

$$= \left\{ \sum_{\substack{j_1=0\\ \cdots}}^{c} \binom{c+j_1\cdots j_k}{l-k} A(j_1\cdots j_k) - \sum_{\substack{j_1=-1\\ \cdots}}^{c-1} \binom{c+j_1\cdots j_k}{l-k} A(j_1\cdots j_k) \right\}$$

$$- \sum_{\substack{j_1=0\\ \cdots}}^{c-1} \binom{c+j_1\cdots j_k}{l-k-1} A(j_1\cdots j_k) - \sum_{\substack{j_2=0\\ \cdots}}^{c-1} \binom{c-1+j_2\cdots j_k}{l-k-1} A(j_1\cdots j_k).$$

Simplifying the two first sums by summing w.r.t. j_1, we get

$$= \sum_{\substack{j_1=0\\ \cdots}}^{c-1} \left\{ \binom{2c+j_1\cdots j_{k-1}}{l-k} A^c\cdot A'\cdot A(j_1\cdots j_{k-1}) \right.$$

$$\left. \underline{-\binom{c-1+j_1\cdots j_{k-1}}{l-k} A(j_1\cdots j_{k-1})} \right\}$$

$$- \sum_{\substack{j_1=0\\ \cdots}}^{c-1} \binom{c+j_1\cdots j_k}{l-k-1} A(j_1\cdots j_k) - \sum_{\substack{j_1=0\\ \cdots}}^{c-1} \underline{\binom{c-1+j_1\cdots j_{k-1}}{l-k-1} A(j_1\cdots j_{k-1})}$$

Using the relation $\binom{n}{p}+\binom{n}{+1}=\binom{n+1}{p+1}$ in the underlined terms, we get

$$= \sum_{\substack{j_1=0\\ \cdots}}^{c-1} \left\{ \binom{2c+j_1\cdots j_{k-1}}{l-k} A^c\cdot A'\cdot A(j_1\cdots j_{k-1}) \right.$$

$$\left. - \binom{c+j_1\cdots j_{k-1}}{l-k} A(j_1\cdots j_{k-1}) \right\} - \sum_{\substack{j_1=0\\ \cdots}}^{c-1} \binom{c+j_1\cdots j_k}{l-k-1} A(j_1\cdots j_k)$$

$$= \sum_{\substack{j_1=0\\ \cdots}}^{c-1} \binom{2c+j_1\cdots j_{k-1}}{l-k} A^c\cdot A'\cdot A(j_1\cdots j_{k-1})$$

$$- \underbrace{\sum_{\substack{j_1=0\\ \cdots}}^{c-1} \binom{c+j_1\cdots j_k}{l-k-1} A(j_1\cdots j_k)}_{\Sigma_k} - \underbrace{\sum_{\substack{j_1=0\\ \cdots}}^{c-1} \binom{c+j_1\cdots j_{k-1}}{l-k} A(j_1\cdots j_{k-1})}_{\Sigma_{k-1}}.$$

Injecting this in the first equation, the terms $\sum_k$ cancel each other except for the last one for $k = 1$, which is equal to $\binom{c}{l-1}V$, so that

$$\pi(l) = \sum_{k=1}^{l} \left\{ (-1)^{l-k} \sum_{\substack{j_1=0\cdots c-1,\ldots \\ j_{k-1}=0\cdots c-1}} \binom{2c + j_1 + \cdots + j_{k-1}}{l-k} A^c \cdot A' \cdot A(j_1 \cdots j_{k-1}) \cdot V \right\}$$
$$- (-1)^{l-1}\binom{c}{l-1}V + (-1)^l \binom{c+1}{l}\Gamma(0).$$

This proves the first part of Theorem 1. $\square$

6.1.4. *Complexity and error term.* In this section, we give an estimate of the computational cost of evaluating the Lyapunov exponent by (6), when using a Taylor approximation of order l [and when supposing that the coupling time c, the eigenvector V, the eigenvalue $\gamma(0)$ and all binomial coefficients are given]. We also compare this to what would be obtained by (20). The estimates are given up to a multiplicative constant.

To evaluate $\pi(l)$ by applying (20), or equivalently (17) with $n = cl$, we need, for $\Psi(M_{i_1\cdots i_k})$, $2d^2(2 - \max(i_1 \cdots i_k))$ operations (summations and multiplications) since we multiply $(2 - \max(i_1 \cdots i_k))$ matrices of size $d \times d$ by a vector. Hence we need at least $2d^2 k$ operations.

We write, for $\sum_{i_1 \neq \cdots \neq i_l,\, i_j \geq -lc} \Psi(M_{i_1\cdots i_k})$,

$$k!(l-k)! \underbrace{\sum_{i_1 > \cdots > i_k}}_{\binom{lc}{k}\text{ terms}} \underbrace{\sum_{i_{k+1} > \cdots > i_l}}_{\binom{lc-k}{l-k}\text{ terms}} \underbrace{\Psi(M_{i_1\cdots i_k})}_{2d^2 k}$$

Then we need $\binom{lc}{k} \times \binom{lc-k}{l-k} \times 2d^2 k$ operations. This implies that the total number of operations is at least $d^2 l 2^l \binom{lc}{l}$.

In order to evaluate $\pi(l)$ using (6), for the generic term, we need

$$\begin{aligned} 2d^2 \sum_{j_1 \cdots j_{k-1}=0}^{c-1} (k + j_1 + \cdots + j_{k-1}) &= \left(kc^{k-1} + (k-1)\frac{c(c-1)}{2} c^{k-2} \right) 2d^2 \\ &= c^{k-1}\left[k + (k-1)\frac{c-1}{2} \right] 2d^2 \\ &\sim kc^k d^2. \end{aligned}$$

This implies that the total number of operations is equivalent to $d^2 l c^l$.

As for the error term in a Taylor approximation or order l, since we have a geometrically dominated convergence in the analyticity region from (23), the simplest bound on this error is

$$D\delta \left(\frac{(2c\delta)^{l+1}}{1 - 2c\delta} \right)' = \frac{D(2c\delta)^{l+1}[1 + l(1 - 2c\delta)]}{(1 - 2c\delta)^2} \quad \text{for } p \in [0, \delta],\ \delta < \frac{1}{2c}.$$

This proves the second part of Theorem 1. $\square$

6.2. *Multinomial case.* The proof in the multinomial case is an extension of the above calculations.

6.2.1. *Evaluation of* Ψ. By conditioning as in the binomial case, we obtain

$$\mathbb{E}[\Psi(B^n)] = \sum_{|K|\le n+2} \mathbb{E}\big[\Psi(B^n)\big||B^n| = K\big]P(|B^n| = K)$$
$$= \sum_{|K|\le n+2} p_1^{k_1}\cdots p_m^{k_m}(1-p_1-\cdots-p_m)^{n+2-|K|}\Bigg(\sum_{\substack{M\\|M|=K}}\Psi(M)\Bigg)$$

with

$$(1-p_1-\cdots-p_m)^{n+2-|K|} = \sum_{|L|\le n+2-|K|}(-1)^{|L|}p_1^{l_1}\cdots p_m^{l_m} \times \frac{(n+2-|K|)!}{l_1!\cdots l_d!(n+2-|K|-|L|)!},$$

so that

$$\mathbb{E}[\Psi(B^n)] = \sum_{|K|\le n+2}\prod_{1\le i\le m}p_i^{k_i}\sum_{|L|\le n+2-|K|}(-1)^{|L|} \times \prod_{1\le i\le m}p_i^{l_i}\binom{n+2-|K|}{L}\Bigg(\sum_{\substack{M\\|M|=K}}\Psi(M)\Bigg)$$
$$= \sum_{|K|\le n+2}\sum_{|L|+|K|\le n+2}(-1)^{|L|} \times \prod_{1\le i\le m}p_i^{l_i+k_i}\binom{n+2-|K|}{L}\Bigg(\sum_{\substack{M\\|M|=K}}\Psi(M)\Bigg).$$

Taking as new variables $l_i := l_i + k_i$,

$$\mathbb{E}[\Psi(B^n)] = \sum_{|K|\le n+2}\sum_{\substack{K\le L\\|L|\le n+2}}(-1)^{|L|-|K|}\prod_{1\le i\le m}p_i^{l_i}\binom{n+2-|K|}{L-K}\Bigg(\sum_{\substack{M\\|M|=K}}\Psi(M)\Bigg)$$
$$= \sum_{|L|\le n+2}\Bigg\{\sum_{K\le L}(-1)^{|L|-|K|}\binom{n+2-|K|}{L-K}\Bigg(\sum_{\substack{M\\|M|=K}}\Psi(M)\Bigg)\Bigg\}\prod_{1\le i\le m}p_i^{l_i}.$$

We shall denote $\pi^n(L)$ be the coefficient of $\prod_{1\le i\le m}p_i^{l_i}$ in the last sum. Below, $q_j^i \neq$ means that the variables q_j^i are all different, and $M_{q_1^1\cdots q_{k_1}^1 q_1^2\cdots q_{k_2}^2\cdots q_1^m\cdots q_{k_m}^m}$

denotes the infinite sequence with all its elements equal to A, but for those of indices $q_1^i, \dots, q_{k_1}^i$ which are all equal to A_i', $\forall\, i = 1, \dots, m$. We have

$$\sum_{|M|=K} \Psi(M) = \frac{1}{K!} \sum_{\substack{q_j^i \neq \\ -n \leq q_j^i \leq 1}} \Psi(M_{q_1^1 \cdots q_{k_1}^1 q_1^2 \cdots q_{k_2}^2 \cdots q_1^m \cdots q_{k_m}^m}),$$

$$\mathscr{A}_{n-|K|}^{|L|-|K|} \sum_{\substack{q_j^i \neq \\ 1 \leq q_j^i \leq n \\ 1 \leq j \leq k_j}} \Psi(M_{q_1^1 \cdots q_{k_m}^m}) = \sum_{\substack{q_j^i \neq \\ 1 \leq q_j^i \leq n \\ 1 \leq j \leq l_j}} \Psi(M_{q_1^1 \cdots q_{k_m}^m}).$$

Applying this and simplifying, we get

$$\pi^n(L) = \frac{1}{L!} \sum_{\substack{q_j^i \neq \\ -n \leq q_j^i \leq 1 \\ 1 \leq j \leq l_j}} \left\{ \sum_{K \leq L} (-1)^{|L|-|K|} \binom{L}{K} \Psi(M_{q_1^1 \cdots q_{k_m}^m}) \right\}. \tag{26}$$

6.2.2. *Convergence.* We proceed exactly in the same way as for Lemmas 3 and 5. Reasoning term by term, one can check that the coefficient β_{Q-J} becomes

$$\beta_{Q-J} = (L-Q)! \left\{ \sum_{K \leq L} (-1)^{|L|-|K|} \binom{L}{K} \prod_{1 \leq i \leq m} \mathscr{A}_{k_i}^{q_i - j_i} \mathscr{A}_{l_i - k_i}^{j_i} \right\}.$$

Here $\mathscr{A}_{k_i}^{q_i - j_i} \mathscr{A}_{l_i - k_i}^{j_i}$ is a multivariate polynomial in k, the degree of which in k_i is less than q_i, that is, less than $l_i - 1$. And we have again

$$\sum_{K \leq L} (-1)^{|K|} \binom{L}{K} k_1^{\omega_1} \cdots k_m^{\omega_m} = 0,$$

if for all i $\omega_i \leq l_i - 1$. This can be proved by differentiating ω_i times the function $\prod_{1 \leq i \leq m} (1 - x_i)^{l_i}$ w.r.t. x_i and evaluating it at the point $(1, \dots, 1)$.

The variable $\pi^n(L)$ is bounded from above (26) by $D|L|(2c)^{|L|}(|L|!/L!)$ which can be bounded itself by $Dl(2cm)^l$ when summing on $|L| = l$ since $m^n = \sum_{|K|=n} (n!/K!)$, so that the convergence region contains the ball of radius $(2cm)^{-1}$ w.r.t. norm $\|\cdot\|_\infty$ on $\mathbb{R}^m$ defined by $\|P\|_\infty = \max_{1 \leq i \leq m} |p_i|$.

For the extension of Lemma 5, the coefficients we obtain are

$$\sum_{L-(V+U) \leq I \leq K} (-1)^{|L|-|I|} \binom{L - I - V - U}{L - K - V - U} = 0.$$

The other steps of the proof of Theorem 2 are very similar to those of the binomial case.

6.2.3. *Complexity.* To evaluate $\pi(L)$ by applying (8), we need a first summation with

$$2d^2 \sum_{j_1 \cdots j_{|K|-1}=0}^{c-1} (|K| + j_1 + \cdots + j_{|K|-1}) \sim |K| c^{|K|} d^2$$

terms.

The sum $\sum_{C_n,\, n=1,\ldots,|K| \text{ with } C_n=A'_i \text{ for some } i \text{ Card}\{n:\, C_n=A'_i\}=k_i}$, multiplies this by $|K|!/K!$.

The sum $\sum_{1\le|K|,\, K\le L}(|K|!/K!)$, multiplies this by $m^{|L|}$ (because $\sum_{1\le|K|,\, K\le L}$ $(|K|!/K!) \le \sum_{|K|\le|L|} (|K|!/K!)$ that is of the order $m^{|L|}$).

Hence, for expansion of order $|L|$ we get the estimate

$$d^2|L|(cm)^{|L|}. \tag{27}$$

6.3. *The operator case.* We first consider the *finite horizon* expansion for which we define (with notation similar to that of Section 6.1)

$$\Psi(B^n) = f \circ A(1) \circ A(0) \circ \cdots \circ A(-n)(V) - f \circ A(0) \circ \cdots \circ A(-n)(V).$$

Conditioning w.r.t the choices made for the random variables $A(l),\ 1 \ge l \ge -n$ leads to a direct analogue of the expansion of Lemma 2,

$$\mathbb{E}[\Psi(B^n)] = \sum_{l=0}^{n+2} \pi_f^n(l) p^l.$$

LEMMA 7. *Under* (P2), *for all* l, *we have convergence of the coefficients*

$$\lim_{n\to\infty} \pi_f^n(l) = \pi_f(l),$$

where the limit $\pi_f(l)$ *is reached in finite time and is given by*

$$\pi_f(l) = \sum_{k=1}^{l} \left\{ \sum_{i+1=1}^{-c+1} \sum_{i_2=i_1-1}^{i_1-c} \cdots \sum_{i_k=i_{k-1}-1}^{i_{k-1}-c} (-1)^{l-k} \binom{2-i_k+c-k}{l-k} \Psi(M_{i_1\cdots i_k}) \right\}$$
$$+(-1)^l \binom{c+1}{l} \Psi(M_\varnothing),$$

where $M_{i_1\cdots i_k} = (A(1), A(2), \ldots, A(i_k))$ *is the sequence with* $A(i) = A$ *for all* i, *but for* $i_1, \ldots, i_k$, *where* $A(i) = A'$.

For the proof, one easily checks that (P2) is a sufficient condition to obtain Lemmas 3, 5, 6.

LEMMA 8. *Under* (P2), $\pi_f(l)$ *is given by* (9).

For the proof, just repeat the last calculations of Section 6.1.3, replacing all terms $A(j_1, \ldots, j_k) \otimes V$ by $f(\Xi(j_1, \ldots, j_k)(V))$, where

$$\Xi(j_1, \ldots, j_k) = A^{j_1} \circ A' \circ \cdots \circ A^{j_k} \circ A'.$$

PROOF OF THEOREM 3. From (P3), on $[0, 1/2c\eta[$,

$$\lim_{n\to+\infty} \mathbb{E}[\Psi(B^n)] = \sum_l \pi_f(l)p^l$$

and this convergence is dominated and hence uniform on every compact of $[0, 1/2c\eta[$.

Property (P3) also implies that $\mathbb{E}[f(X_n)]$ is finite for $n \geq 0$. Now using (P1) and a Cesaro averaging argument,

$$\frac{\sum_{i=1}^n \mathbb{E}[\Psi(B^i)]}{n} = \frac{\mathbb{E}[f(X_{n+2})] - \mathbb{E}[f(X_2)]}{n} \overset{n\to+\infty}{\longrightarrow} \gamma_f(p).$$

6.4. *The general case.* The proofs are similar to those of the Bernoulli case for $c = 1$. The main property leading to (20) from the first relation (18) is indeed only based on the fact that the function Ψ *loses memory of the past* regardless of the initial condition.

7. Other expansions and relationship between expansions.

7.1. *Yet another binomial expansion.* There is an immediate extension of Theorem 1 when each occurrence of matrix A' is replaced by that of an independent random matrix $A'(\omega)$, which admits a discrete or continuous distribution. More precisely, A_n is equal to A with probability $1 - p$ and to B_n with probability p, where the sequence $\{B_n\}$ is i.i.d. and independent of the sequence used to chose between A and the alternative. Then, if we note E' the expectation w.r.t. this distribution, (6) can be reformulated as follows:

$$\begin{aligned}\begin{bmatrix}\pi(l)\\ \vdots \\ \pi(l)\end{bmatrix} &= (-1)^l\left\{\binom{c}{l-1}V + \binom{c+1}{l}\Gamma(0)\right\}\\ &\quad + \sum_{k=1}^{l}\Bigg\{(-1)^{l-k}\sum_{j_1,\dots,j_{k-1}=0}^{c-1}\binom{2c+j_1+\cdots+j_{k-1}}{l-k}\\ &\qquad\qquad \times E'[A^c\otimes A'\otimes A^{j_1}\otimes A'\cdots A^{j_{k-1}}\otimes A'\otimes V]\Bigg\}.\end{aligned} \tag{28}$$

Similar extensions can of course be contemplated for all the extensions considered in Sections 4.1–4.4.

7.2. *Binomial versus multinomial.* An interesting special case of what is considered in the previous subsection is when the law of B_n is discrete and with finite support. Assume, for instance, that we are in the setting of Section 4.1, namely, $A'(\omega)$ is equal to A'_j with probability p_j, $j = 1, \dots, m$, and each of the matrices in questions has at least one non-ε element on each row. In this case, we have both an expansion in the parameter $p = p_1 + p_2 + \cdots + p_m$

which is that of (28) above, and an expansion in the multiparameter $P' = (pp_1, \dots, pp_m)$ which is that of Theorem 2.

7.3. *Symmetrical versus asymmetrical.* In the multinomial case, we obtained an analytic expansion of the Lyapunov exponent w.r.t. the parameters $(p_1, \dots, p_m)$ of the law, which are such that $p_0 = 1 - p_1 - \cdots - p_m$ represents the probability of an event which should be frequent enough to entail the memory loss property [e.g., in the (max, plus)-algebra case the sampling of the scs1-cycl matrix A]. This is an asymmetrical expansion in that it is not made w.r.t all parameters $(p_0, \dots, p_m)$.

Within the general setting of Section 4.4, symmetrical expansions can also be derived in complement to those obtained so far in the following way. For all integers $l \geq 0$, we have the following representation:

$$\begin{aligned}
\mathbb{E}[X_l] &= \sum_{k_0+\cdots+k_m=l} \left\{ \sum_{\substack{|\{C_1,\dots,C_l\}|=(k_0,\dots,k_m \\ C_i \in \mathscr{L}}} C_1 \circ \cdots \circ C_l(X_0) \right\} p_0^{k_0} \cdots p_m^{k_m} \\
&= \sum_{|\overline{K}|=l} \left\{ \sum_{\substack{|C|=\overline{K} \\ C_i \in \mathscr{L}}} C_1 \circ \cdots \circ C_l(X_0) \right\} \overline{P}^{\overline{K}},
\end{aligned}$$

where $\overline{K} = (k_0, k_1, \dots, k_m)$ and $\overline{P} = (p_0, \dots, p_m)$. So for $l \geq 2$,

$$\begin{aligned}
&\mathbb{E}[X_l - X_{l-1}] - \mathbb{E}[X_{l-1} - X_{l-2}] \\
&\quad = \sum_{|\overline{K}|=l} \left\{ \sum_{\substack{|C|=\overline{K} \\ C_i \in \mathscr{L} \\ i=1\cdots l}} C_1 \circ \cdots \circ C_l(X_0) \right. \\
&\qquad\qquad \left. -2C_1 \circ \cdots \circ C_{l-1}(X_0) + C_1 \circ \cdots \circ C_{l-2}(X_0) \right\} \overline{P}^{\overline{K}}.
\end{aligned}$$

So, if we define

$$\pi(\overline{L}) - \Delta^2 S(\overline{L}),$$

where

$$S(\overline{L}) = \sum_{\substack{|C|=(l_0,\dots l_m) \\ C_i \in \mathscr{L}}} \left\{ C_1 \circ \cdots \circ C_{|L|}(X_0) \right\},$$

we obtain

$$\mathbb{E}[X_{l+1} - X_l] - \mathbb{E}[X_l - X_{l-1}] = \sum_{|\overline{L}|=l} \pi(\overline{L}) \overline{P}^{\overline{L}}.$$

This also gives the terms of order l in the symmetrical expansion of

$$\mathbb{E}[X_{n+1} - X_n] = \sum_{j=0}^{n+1} \{\mathbb{E}[X_j - X_{j-1}] - \mathbb{E}[X_{j-1} - X_{j-2}]\}$$

with $X_{-1} = X_{-2} = 0$.

We can then use the convergence of $\mathbb{E}[X_{n+1} - X_n]$ to Γ to derive the coefficients of the symmetrical expansion of Γ.

Let us comment on how this result is related to the asymmetrical expansion with $c = 1$. Consider the multinomial (max, plus)-setting of Section 4.1; we associate with this a $(m + 2)$-nomial scheme with parameters $(p, p_0 - p/(m+1), \dots, p_m - p/(m+1))$, where p is a real number such that this vector is a probability law. In this scheme, matrix A_i' is sampled with probability $p_i' = p_i - p/(m+1)$, for all $i = 0, \dots, m$ (we take $A_0' = A$) and with probability p, the matrix 0 which has all its entries equal to 0 is sampled. We can then apply the asymmetrical expansion w.r.t. $(p, p_0', \dots, p_m')$, when taking as scs1 cyc1 matrix the matrix $\mathbf{0}$, which has for unique eigenvector the vector $\mathbf{e}$ with all its coordinates equal to 0, and for which $c = 1$. It is then easy to check that the coefficient of $(p_0')^{l_0} \cdots (p_m')^{l_m}$ in the asymmetrical expansion for this $(m + 2)$-nomial scheme is equal to the coefficient of $(p_0)^{l_0} \cdots (p_m)^{l_m}$ in the above symmetrical expansion.

8. The regenerative theory approach. The aim of this subsection is to investigate another potential way of obtaining the main result, based on regenerative theory.

As in Section 6, we use here the simplified notation " $\cdot$ " instead of $\otimes$.

We start with the following basic observation, where the sequence of interest (and the notations) is that of Section 3.

THEOREM 5. *Assume* (H1), (H2) *and* (H3) *hold. We define*

$$T_1 = \inf\{n \ge c : A(n-1) = A(n-2) = \cdots = A(n-c) = A\},$$
$$T_k = \inf\{n \ge T_{k-1} + c : A(n-1) = A(n-2) = A(n-c) = A\} \qquad \forall\, k \ge 2.$$

For all $n \ge 0$, *let* $\Lambda_n = X_{n+1} - X_n$, $S_n = T_{n+1} - T_n$ (*with* $T_0 = 0$), $Z_n = X_{T_{n+1}} - X_{T_n}$. *Then* $\{\Lambda_n\}_{n\ge 0}$ *is a regenerative process with regeneration times* $\{T_n\}$, *that is*, $\{S_n, Z_n\}_{n\ge 1}$ *forms an i.i.d. sequence.*

Moreover for all $p \in]0, 1[$,

$$\Gamma(p) = \frac{\mathbb{E}[X_{T_2} - X_{T_1}]}{\mathbb{E}[S_1]}, \qquad \mathbb{E}[S_1] = \frac{1 - (1-p)^c}{p(1-p)^c}, \tag{29}$$

which are independent of the initial condition X_0.

One can write also

$$\Gamma(p) = \frac{\mathbb{E}_{X_0 = V}[X_{T_1} - V]}{\mathbb{E}[T_1]}. \tag{30}$$

REMARK 12. The relation (29) is valid for $p = 0$ or $p = 1$ by taking the proper limits.

PROOF. Let $F_n = \sigma\{A(i),\ i \le n\}$. For all $i \ge 0$, the random variable T_i is a finite F_n-stopping times. In addition, $\{\Lambda_n\}$ is a F_n-Markov chain in $\mathbb{R}^d$. That the random variables $\{S_n, Z_n\}_{n\ge 1}$ are i.i.d. follows from the strong Markov property for discrete time Markov chains.

From the strong law of large numbers,

$$\frac{1}{m} \sum_{0\le k\le m-1} \Lambda_k \xrightarrow{\text{a.s}} \frac{1}{\mathbb{E}[S_1]} \mathbb{E}[X_{T_2} - X_{T_1}]$$

since, on the other hand,

$$\frac{1}{m} \sum_{0\le k\le m-1} \Lambda_k = \frac{X_m}{m} \xrightarrow{\text{a.s.}} \Gamma(p).$$

For proving (29), we use the relations

$$\begin{aligned}
&P(T_1 = k) = 0 \quad \text{for } k < c,\\
&P(T_1 = c) = (1-p)^c,\\
&P(T_1 = k) = p(1-p)^c P(T_1 > k-c-1) \quad \text{for } k > c.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{k\ge 0} P(T_1 = k) &= (1-p)^c + \sum_{k>c} p(1-p)^c P(T_1 > k-c-1)\\
&= (1-p)^c + p(1-p)^c \left(\sum_{k\ge 0} P(T_1 > k) \right).
\end{aligned}$$

So

$$1 = (1-p)^c + p(1-p)^c \mathbb{E}[T_1].$$

This concludes the proof. $\square$

REMARK 13. Equation (30) still holds more generally under the memory loss property (P2)$'$.

Let $\mathscr{C}_n$ be the set of all sequences of n elements of the set $\{0,1\}$ such that there are never more than $c-1$ consecutive 0's. Let $p_1 = p,\ A_1 = A',\ p_0 = 1-p$ and $A_0 = A$. From (29), it is easy to check that $\gamma(p)$ admits a representation of the form

$$\begin{aligned}
\frac{1}{\mathbb{E}[T_1]} \sum_{n=c}^{\infty} P(T_1 = n) &\sum_{i_0, i_1, i_1, i_{n-c-1} \in \mathscr{C}_{n-c}} p_{i_0} \cdots p_{i_{n-c-1}}\\
&\qquad \times \left[A^c \cdot A_{i_{n-c-1}} \cdots A_{i_0} \cdots A_{i_0} V - V \right],
\end{aligned}$$

where one recognizes terms as in (6). In order to rederive the complete result of Theorem 1 via this representation, we now need a direct evaluation of the numerator of the first equation of (29). We have

$$\begin{aligned}\mathbb{E}[X_{T_1}] &= \sum_{n\geq 0}\mathbb{E}[X_{T_1}/T_1=n]P[T_1=n]\\ &= \mathbb{E}[X_{T_1}/T_1=c]P[T_1=c]+\sum_{n\geq c+1}\mathbb{E}[X_{T_1}/T_1=n]P[T_1=n]\\ &= (1-p)^c A^c\cdot X_0+\sum_{n\geq 0}\mathbb{E}[X_{T_1}/T_1=n+c+1]P[T_1=n+c+1],\end{aligned}$$

with

$$\begin{aligned}\mathbb{E}[X_{T_1}/T_1=n+c+1] = \sum_{\substack{i_1+\cdots+i_{l+1}+l=n\\ i_j\leq c-1}} &\binom{n}{l}p^l(1-p)^{n-l}A^c\cdot A'\\ &\cdot A^{i_1}\cdots A'\cdot A^{i_{l+1}}\cdot X_0.\end{aligned}$$

Substituting this expression in the upper equality,

$$\begin{aligned}\mathbb{E}[X_{T_1}] = {} &(1-p)^c A^c\cdot X_0\\ &+\sum_{n\geq 0}\sum_{\substack{i_1+\cdots+i_{l+1}+l=n\\ i_j\leq c-1}}\binom{n}{l}A^c\cdot A'\cdot A^{i_1}\cdots A'\cdot A^{i_{l+1}}\\ &\cdot X_0 p^l(1-p)^{n-l}P[T_1=n+c+1].\end{aligned}\tag{31}$$

In order to obtain the law of T_1, we use generating functions as follows:

$$\begin{aligned}\mathbb{E}[z^{T_1}] &= \sum_{k=0}^{+\infty}P(T_1=k)z^k\\ &= (1-p)^c z^c + p(1-p)^c\sum_{k\geq c+1}P(T_1>k-c-1)z^k\\ &= (1-p)^c z^c + p(1-p)^c\sum_{k\geq 0}P(T_1>k)z^{c+1+k}\\ &= (1-p)^c z^c + p(1-p)^c z^{c+1}\sum_{k\geq 0}P(T_1>k)z^k\\ &= (1-p)^c z^c + p(1-p)^c z^{c+1}\sum_{k\geq 1}P(T_1=k)\underbrace{(z^0+\cdots+z^{k-1})}_{\frac{z^k-1}{z-1}}\\ &= (1-p)^c z^c + p(1-p)^c\frac{z^{c+1}}{z-1}\left(\mathbb{E}[z^{T_1}]-1\right).\end{aligned}$$

Thus,

$$\Phi(z)=\mathbb{E}[z^{T_1}]=\frac{(1-p)^c z^c\left(1+p^z/(1-z)\right)}{1+p(1-p)^c\left(z^{c+1}/1-z\right)}.$$

Now we have

$$P(T_1 = n) = \frac{1}{n!}\Phi^{(n)}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(e^{jw})e^{-jnw}\,dw. \tag{32}$$

Let us see first what $\Phi^{(n)}(0)$ looks like:

$$\Phi(z) = \frac{(1-p)^c z^c(1-z) + p(1-p)^c z^{c+1}}{(1-z) + p(1-p)^c z^{c+1}} = a\frac{(p-1)z^{c+1} + z^c}{1-(z - paz^{c+1})},$$

with $a = (1-p)^c$.

We can now expand this as a power series using the relation

$$\frac{1}{1-(z-paz^{c+1})} = \sum_{k=0}^{+\infty} z^k(1-paz^c)^k,$$

that is,

$$\frac{1}{1-(z-paz^{c+1})} = \sum_{k=0}^{+\infty}\sum_{i=0}^{k}\binom{k}{i}(-pa)^i z^{ci+k}.$$

So,

$$\begin{aligned}
\Phi(z) &= a\sum_{k=0}^{+\infty}\sum_{i=0}^{k}\binom{k}{i}(p-1)(-pa)^i z^{c(i+1)+k+1} + a\sum_{k=0}^{+\infty}\sum_{i=0}^{k}\binom{k}{i}(-pa)^i z^{c(i+1)+k} \\
&= a\sum_{k>0}\sum_{i=0}^{k-1}\binom{k-1}{i}(p-1)(-pa)^i z^{c(i+1)+k} + \text{idem.} \\
&= a\sum_{k>0}\sum_{i=0}^{k}\left(\binom{k}{i} + (p-1)\binom{k-1}{i}\right)(-pa)^i z^{c(i+1)+k} + az^c \\
&= a\sum_{k>0}\sum_{i=0}^{k}\left(\binom{k-1}{i-1} + p\binom{k-1}{i}\right)(-pa)^i z^{c(i+1)+k} + az^c.
\end{aligned}$$

To get the coefficient of $z^{c(m+1)+r}$ in the last expression, we have to take

$$\begin{aligned}
&i = m, \quad k = r, \\
&i = m-1, \quad k = c+r, \\
&\vdots \quad \vdots \\
&i = 0, \quad k = cm+r.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Phi(z) = a\sum_{\substack{n=c(m+1)+r\\ n>c}} \Bigg\{&\left(\binom{r-1}{m-1} + p\binom{r-1}{m}\right)(-pa)^m + \cdots \\
&+ \left(\binom{cm+r-1}{-1} + p\binom{cm+r-1}{0}\right)(-pa)^0\Bigg\} z^{c(m+1)+r} + az^c
\end{aligned}$$

and for $n \geq c+1$,

$$\frac{1}{n!}\Phi^{(n)}(0) = a\left\{\sum_{i=0}^{m}\left(\binom{(m-i)c+r-1}{i-1} + p\binom{(m-i)c+r-1}{i}\right)(-pa)^i\right\}.$$

Since in (31) n is bounded for a given l, we deduce from this expression that there is a finite number of terms in $\pi(l)$ indeed. Unfortunately, it seems difficult to derive an explicit expression from there. From the expression

$$\begin{aligned}\mathbb{E}[X_{T_1}] &= (1-p)^c A^c \cdot X_0 \\ &\quad + \sum_{n\geq 0} \sum_{\substack{i_1+\cdots+i_{l+1}+l=n \\ i_j \leq c-1}} \sum_{i=0}^{[(n+1)/c]} (-1)^i \binom{n}{l} \\ &\qquad \times \left(\binom{n-ic}{i-1} + p\binom{n-ic}{i}\right) \\ &\qquad \times p^{l+1}(1-p)^{n-l+c(i+1)} A^c \cdot A' \cdot A^{i_1} \cdots A' \cdot A^{i_{l+1}} \cdot X_0\end{aligned}$$

one can indeed find the result of Theorem 1 with this method in the particular case $c = 1$. The general cases ($c > 1$) seem much harder.

Starting with the second formula in (32), the difficulties are not fewer. Put $f(z) = \Phi(z)z^{-(n+1)}$ with $z = e^{\jmath w}$. Then $P(T_1 = n) = (1/2\pi\jmath)\int_{\partial\Omega} f(z)\,dz$, where $\partial\Omega$ is the unit circle covered from $-\pi$ to π and, from the theory of holomorphic functions,

$$\int_{\partial\Omega} f(z)\,dz = 2\pi\jmath \sum_k \operatorname{Res} f(z_k),$$

where z_k are the poles of f, namely 0 (if $n > c$) and those included in Ω among the $c+1$ complex roots of $paz^{c+1} - z + 1 = 0$. Here difficulties stem from the determination of z_k.

There is a third way to reach $P(T_1 = n)$. Put $p_n = P(T_1 = n)$. For $n \geq c+1$, we have

$$\begin{aligned} p_n &= p(1-p)^c P(T_1 > n-c-1) \\ &= p(1-p)^c[p_{n-c} + p_{n-c+1} + \cdots], \\ p_{n+1} &= p(1-p)^c[p_{n-c+1} + p_{n-c+2} + \cdots]. \end{aligned}$$

So,

$$p_{n+1} - p_n = -p(1-p)^c p_{n-c}.$$

The characteristic equation is

$$z^{c+1} - z^c + p(1-p)^c = 0,$$

which has $c+1$ simple complex roots for $p < c/(c+1)$ (as can be checked by differentiating once). Therefore,

$$p_n = \sum_{i=1}^{c+1} \alpha_i (z_i)^n,$$

with $c+1$ initial conditions $\{p_{c+1} = \cdots = p_{2c} = p(1-p)^c$ and $p_{2c+1} = p(1-p)^c[1-(1-p)^c]\}$. The problem here is that we only know one explicit root, namely $1-p$.

9. Conclusion. The paper primarily bears on (max, plus)-Lyapunov exponents obtained by sampling matrices from a finite set. More general stochastic assumptions can also be considered along the same lines (cf. Section 7), as well as generalizations to iterates of random operators. Two results which were derived should be stressed in contrast to the properties of Lyapunov exponents in the conventional algebra:

1. Closed form formulas can be obtained for the coefficients of Taylor approximations of all orders, provided one of the matrices in this set has a unique normalized eigenvector.
2. If the probability of occurrence of this specific matrix is large enough, the Lyapunov exponent is analytic in the parameters $(p_1, \ldots, p_m)$, which give the probabilities of the other matrices in the set; as a result, the exponent is also computationally approximable, in that a geometric error bound can be derived on its approximation by finite order Taylor expansions. In fact, even if we did know how to compute a confidence interval $I(n)$ for the estimator X_n/n of $\gamma(p)$, based on a simulation, there would, of course, only be a high probability, say 95%, that $\gamma(p)$ is in the interval $[X_n/n - I(n), X_n/n + I(n)]$. In contrast, when it can be used, the expansion gives certitude that $\gamma(p)$ is within a certain interval determined by an expansion of finite order and the associated error bound.

Further research will bear on better estimates of the radius of convergence and of the error bounds for the proposed computation method.

APPENDIX

Let us denote A_n^1, B_n^1 and D_n^1 the nth epochs when a customer enters station 1, starts its service there, and leaves station 1, respectively, and β_n^1 and δ_n^1 the epoch when the customer which enters at time A_n^1 begins its service in station 1, and leaves station 1, respectively. For the same reasons as above, whenever the initial condition is with one customer starting its service in station 1, then for all $n \geq 0$,

$$B_{n+1}^1 = D_{n+1}^1 \oplus A_{n+1}^1,$$
$$D_{n+1}^1 = \sigma \cdot B_n^1,$$

with an initial condition $B_0^1 = 0$, and with A_0^1 and D_0^1 undetermined yet. Notice that as above, $B_n^1 = \beta_n^1$ and $D_n^1 = \delta_{n-1}^1$.

In the same way, let A_n^2, B_n^2, D_n^2, β_n^2 and δ_n^2 be the corresponding quantities for station 2 (i.e., δ_n^2 is the departure time of the customer entering station 2 at A_n^2) and let b_n be the event that there is a breakdown for the customer entering at time A_n^2. Then in view of the way breakdowns take place, for all $n \geq 1$,

$$\beta_n^2 = \begin{cases} A_n^2 \oplus D_n^2, & \text{on } \bar{b}_n, \\ A_n^2 \oplus D_{n+1}^2, & \text{on } b_n. \end{cases}$$

$$\delta_n^2 = \sigma' \cdot \beta_n^2,$$

with $A_0^2 = 0$ and $D_0^2 = 0$. Let us now prove that for all n, $\beta_{n+1}^2 \geq \beta_n^2$. Indeed, using the fact that each of the sequences $\{A_n^2\}$ and $\{D_n^2\}$ is nondecreasing, we obtain that on b_{n+1},

$$\beta_{n+1}^2 = A_{n+1}^2 \oplus D_{n+2}^2 \geq A_n^2 \oplus D_{n+1}^2 \geq \beta_n^2.$$

On $\bar{b}_{n+1} \cap \bar{b}_n$,

$$\beta_{n+1}^2 = A_{n+1}^2 \oplus D_{n+1}^2 \geq A_{n+1}^2 \oplus D_n^2 = \beta_n^2$$

and finally, on $\bar{b}_{n+1} \cap b_n$,

$$\beta_{n+1}^2 = A_{n+1}^2 \oplus D_{n+1}^2 \geq A_n^2 \oplus D_{n+1}^2 = \beta_n^2.$$

Therefore, for all n, $\beta_n^2 = B_{n+1}^2$ and $\delta_n^2 = D_{n+2}^2$. We conclude from this that the following equalities must hold:

$$D_n^2 = A_n^1 \triangleq X_n^4,$$
$$D_n^1 = A_n^2 \triangleq X_n^2,$$

and if we take $X_n^3 \triangleq B_n^2$ and $X_n^1 \triangleq B_n^1$, we finally get

$$X_{n+1}^1 = X_{n+1}^2 \oplus X_{n+1}^4 = \sigma \cdot X_n^1 \oplus \sigma' X_n^3,$$
$$X_{n+1}^2 = \sigma \cdot X_n^1,$$
$$X_{n+1}^3 = \begin{cases} X_n^2 \oplus X_n^4, & \text{on } \bar{b}_n, \\ X_n^2 \oplus X_{n+1}^4 = X_n^2 \oplus \sigma' \cdot X_n^3 & \text{on } b_n, \end{cases}$$
$$X_{n+1}^4 = \sigma' \cdot X_n^3,$$

with initial condition $X_0 = (0, 0, 0, 0)^t$.

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