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Stochastic representations of derivatives of solutions of one-dimensional parabolic variational inequalities with Neumann boundary conditions

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Abstract. In this paper we explicit the derivative of the flows of one-dimensional reflected diffusion processes. We then get stochastic representations for derivatives of viscosity solutions of one-dimensional semilinear parabolic partial differential equations and parabolic variational inequalities with Neumann boundary conditions.

Résumé. Dans cet article, nous explicitons la dérivée du flot d'un processus de diffusion réfléchi. Nous obtenons des représentations stochastiques des dérivées des solutions de viscosité d'équations aux dérivées partielles paraboliques semi-linéaires. Nous en déduisons des représentations stochastiques des dérivées des solutions de viscosité d'inégalités variationnelles paraboliques avec conditions au bord de Neumann.

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1. Introduction

Consider the parabolic variational inequality in the whole Euclidean space

$$\begin{cases} \min\bigl\{V(t,x)-L(t,x); -\frac{\partial V}{\partial t}(t,x)-\mathcal{A}V(t,x) \\ \quad -f\bigl(t,x,V(t,x),(\nabla V\sigma)(t,x)\bigr)\bigr\}=0, & (t,x)\in[0,T)\times\mathbb{R}^d, \\ V(T,x)=g(x), & x\in\mathbb{R}^d, \end{cases} \tag{1}$$

where $\mathcal{A}$ is the infinitesimal generator of a diffusion process. The numerical resolution of such a problem requires to introduce a boundary and artificial boundary conditions in order to allow the discretization of a PDE problem posed in a bounded domain. We thus localize the preceding variational inequality. If nonhomogeneous Neumann boundary conditions are chosen, one then has to solve

$$\begin{cases} \min\bigl\{v(t,x)-L(t,x); -\frac{\partial v}{\partial t}(t,x)-\mathcal{A}v(t,x) \\ \quad -f\bigl(t,x,v(t,x),(\nabla v\sigma)(t,x)\bigr)\bigr\}=0, & (t,x)\in[0,T)\times\mathcal{O}, \\ v(T,x)=g(x), & x\in\overline{\mathcal{O}}, \\ \bigl(\nabla v(t,x)+h(t,x);\eta(x)\bigr)=0, & (t,x)\in[0,T)\times\partial\mathcal{O}, \end{cases} \tag{2}$$

where, for all x in $\partial\mathcal{O}$, $\eta(x)$ denotes the inward unit normal vector at point x. From a numerical analysis point of view, one needs to estimate $|V(t,x)-v(t,x)|$. Berthelot, Bossy and Talay [3] have tackled this issue by using a stochastic approach based on Backward Stochastic Differential Equations (BSDE). Given the reflected forward Stochastic Differential Equation (SDE)

$$\begin{cases} X_s^{t,x} = x + \int_t^s b\big(X_\theta^{t,x}\big)\,\mathrm{d}\theta + \int_t^s \sigma\big(X_\theta^{t,x}\big)\,\mathrm{d}W_\theta + K_s^{t,x}, & 0\le t\le s\le T,\\ K_s^{t,x} = \int_t^s \eta\big(X_\theta^{t,x}\big)\,\mathrm{d}|K|_\theta^{t,x} & \text{with } |K|_s^{t,x} = \int_t^s \mathbb{I}_{\{X_\theta^{t,x}\in\partial\mathcal{O}\}}\,\mathrm{d}|K|_\theta^{t,x}, \end{cases} \tag{3}$$

they have proven the following estimate: under smoothness conditions on the coefficients and on $\partial\mathcal{O}$, there exists $C>0$ such that, for all $0\le t\le T$ and $x\in\mathcal{O}$,

$$\big|V(t,x)-v(t,x)\big| \le C\Big\{\mathbb{E}\sup_{t\le s\le T}\big|\big(\nabla V\big(s,X_s^{t,x}\big)+h\big(s,X_s^{t,x}\big);\eta\big(X_s^{t,x}\big)\big)\big|^4\mathbb{I}_{\{X_s^{t,x}\in\partial\mathcal{O}\}}\Big\}^{1/4}.$$

Motivated by applications in Finance, where the space derivative of $v(t,x)$ allows one to construct hedging strategies of American options, we aim in this paper to estimate $|\partial_x V(t,x)-\partial_x v(t,x)|$, where the derivatives are understood in the sense of the distributions. We thus have to check that the probabilistic interpretations, in terms of BSDEs, of $V(t,x)$ and of $v(t,x)$, are differentiable in the sense of the distributions, and to exhibit formulae which are suitable to estimate $|\partial_x V(t,x)-\partial_x v(t,x)|$. Unfortunately, so far we are able to deal with one-dimensional problems only. which means that $\mathcal{O}$ is reduced to a bounded interval (d,d'). Two main reasons explain the limitation to one-dimensional problems: first, we need to prove an explicit representation of the derivative $\partial_x X_t^{t,x}$, where $X^{t,x}$ is as in (3); this representation appears to be simple and of exponential type; exhibiting such an explicit formula seems difficult for general multi-dimensional flows[1] (Malliavin derivatives were also explicited by Lépingle, Nualart and Sanz [10] in the one-dimensional case only); second, in order to get stochastic representations for $\partial_x v(t,x)$ when $h\neq 0$, that is, in the case of nonhomogeneous Neumann boundary conditions, we use an integration by parts technique which seems limited to the one-dimensional case (see Lemma 3.7).

We aim to provide a stochastic representation for $\partial_x v(t,x)$ in terms of the derivative of the solution $(\underline{\mathcal{Y}}^{t,x},\underline{\mathcal{Z}}^{t,x},\underline{\mathcal{R}}^{t,x})$ of the reflected BSDE with the reflected forward diffusion $X^{t,x}$

$$\begin{cases} \underline{\mathcal{Y}}_s^{t,x} = g\big(X_T^{t,x}\big) + \int_s^T f\big(r,X_r^{t,x},\underline{\mathcal{Y}}_r^{t,x},\underline{\mathcal{Z}}_r^{t,x}\big)\,\mathrm{d}r + \int_s^T h\big(r,X_r^{t,x}\big)\,\mathrm{d}K_r^{t,x}\\ \qquad\quad + \underline{\mathcal{R}}_T^{t,x} - \underline{\mathcal{R}}_s^{t,x} - \int_s^T \underline{\mathcal{Z}}_r^{t,x}\,\mathrm{d}W_r,\\ \underline{\mathcal{Y}}_s^{t,x} \ge L\big(s,X_s^{t,x}\big) \quad \text{for all } 0\le t\le s\le T,\\ \big(\underline{\mathcal{R}}_s^{t,x}, 0\le t\le s\le T\big) \text{ is a continuous increasing process such that}\\ \quad \int_t^T \big(\underline{\mathcal{Y}}_r^{t,x} - L\big(s,X_r^{t,x}\big)\big)\,\mathrm{d}\underline{\mathcal{R}}_r^{t,x} = 0. \end{cases}$$

As we suppose that the coefficients b and σ are only Lipschitz (and not necessarily differentiable), we need to extend various approaches developed to solve problems without or with reflexion: Bouleau and Hirsch [6] have explicited the derivatives w.r.t. the initial data of the solutions of nonreflected forward SDEs with Lipschitz coefficients; Lépingle et al. [10] have explicited the Malliavin derivatives of the solutions of one-dimensional reflected forwards SDEs. Pardoux and Zhang [19] have established stochastic representations, in terms of BSDEs driven by forward reflected SDEs, for viscosity solutions of semilinear partial differential equations with Neumann boundary conditions. In [12] and [13] Ma and Zhang have represented, without differentiating the coefficients g and f, derivatives of solutions of BSDEs and reflected BSDEs driven by nonreflected forward SDEs with differentiable coefficients. N'Zi, Ouknine and Sulem [16] have extended Ma and Zhang's results for nonreflected BSDEs to the case where the coefficients of the nonreflected forward SDEs are supposed Lipschitz only.

The paper is organized as follows. In Section 2 we explicit the derivative of the flow of the reflected flow $(X^{t,x})$ defined in (3). In Section 3 we get two stochastic representations for derivatives of solutions of semilinear parabolic partial differential equations (which corresponds to the case where $L(t,x)\equiv-\infty$ and $\mathcal{R}\equiv 0$): the first representation involves the gradient of f, the second one does not involve it. We distinguish the homogeneous Neumann boundary

[1]The differentiability, in the sense of the distributions, seems easy to get by localization procedures when the boundary of the domain is smooth.

condition case, that is, the case where $h(t,x) \equiv 0$, and the inhomogeneous case. In Section 4 we get stochastic representations for derivatives of parabolic variational inequalities. We conclude by using our representations to estimate $|\partial_x V(t,x) - \partial_x v(T,x)|$.

Notation. *In all the paper we denote by C, C_1, C_2 positive constants which may vary from line to line but only depend on d, d', T, the L^∞-norms and the Lipschitz constants of the functions b, σ, g, f and h, and the strong ellipticity constant α_* which appears in the inequality (6) below.*

2. Derivative of the flow of the reflected diffusion X

2.1. *Main result and examples*

From now on we consider a one-dimensional stochastic differential equation in the interval $[d,d']$, with reflection at points d and d':

$$\begin{cases} X_s^{t,x} = x + \int_t^s b\big(X_r^{t,x}\big)\,\mathrm{d}r + \int_t^s \sigma\big(X_r^{t,x}\big)\,\mathrm{d}W_r + K_s^{t,x}, & \\ K_s^{t,x} = \int_t^s \eta\big(X_r^t\big)\,\mathrm{d}|K|_r^{t,x} & \text{with } |K|_s^{t,x} = \int_t^s \mathbb{I}_{\{X_r^{t,x}\in\{d,d'\}\}}\,\mathrm{d}|K|_r^{t,x}, \end{cases} \tag{4}$$

where $\eta(d) = 1$ and $\eta(d') = -1$. Our objective in this section is to explicit the derivative w.r.t. x of the stochastic flow $X^{t,x}$.

We start with introducing some notation coming from [6]. We equip the space $\widetilde{\Omega} := (d,d') \times \Omega$ with its natural σ-field and the measure $\mathrm{d}\widetilde{\mathbb{P}} := \mathrm{d}x \otimes \mathrm{d}\mathbb{P}$. Let $\widetilde{D}_1$ be the space of functions $\gamma(x,\omega)$ satisfying: there exists a measurable function $\widetilde{\gamma}: \widetilde{\Omega} \to \mathbb{R}$ such that $\gamma = \widetilde{\gamma}$, $\widetilde{\mathbb{P}}$-a.s, and, for all (x,ω), the map $y \to \widetilde{\gamma}(x+y,\omega)$ is locally absolutely continuous.

For $\gamma \in \widetilde{D}_1$, set

$$\partial_x \gamma(x,\omega) := \liminf_{y\to 0} \frac{\widetilde{\gamma}(x+y,\omega) - \widetilde{\gamma}(x,\omega)}{y}.$$

Bouleau and Hirsch [5] have shown that this definition is proper in the sense that, $\widetilde{\mathbb{P}}$-a.s., $\partial_x\gamma(x,\omega)$ is well defined and does not depend on the choice of $\widetilde{\gamma}$. Finally, set

$$\widetilde{D} := \big\{\gamma \in \mathbb{L}^2(\widetilde{\mathbb{P}}) \cap \widetilde{D}_1;\ \partial_x\gamma \in \mathbb{L}^2(\widetilde{\mathbb{P}})\big\}$$

and

$$\|\gamma\|_{\widetilde{D}} = \left(\int_{(d,d')\times\Omega} \gamma^2\,\mathrm{d}\widetilde{\mathbb{P}} + \int_{(d,d')\times\Omega} (\partial_x\gamma)^2\,\mathrm{d}\widetilde{\mathbb{P}}\right)^{1/2}.$$

As in Lépingle et al. [10] we introduce the random set

$$\mathcal{E}_s^{t,x} := \Big\{\omega \in \Omega:\ d < \inf_{r\in[t,s]} X_r^{t,x}(\omega) \le \sup_{r\in[t,s]} X_r^{t,x}(\omega) < d'\Big\}. \tag{5}$$

Our main result in this section is the following statement.

Theorem 2.1. *Suppose that b and σ are bounded Lipschitz functions, and that*

$$\exists \alpha_* > 0, \forall x \in \big[d,d'\big] \quad \sigma(x) > \alpha_*. \tag{6}$$

Denote by b' and σ' versions of the a.e. derivatives of b and σ. Then the flow $X^{t,x}$ belongs to $\widetilde{D}$ and [2]

$$\partial_x X_s^{t,x} = J_s^{t,x}\,\mathbb{I}_{\mathcal{E}_s^{t,x}}, \quad \widetilde{\mathbb{P}}\text{-a.s.}, \tag{7}$$

[2] We recall that $\mathrm{d}\widetilde{\mathbb{P}} := \mathrm{d}x \otimes \mathrm{d}\mathbb{P}$.

where

$$J_s^{t,x} = \exp\left\{ \int_t^s \sigma'\big(X_r^{t,x}\big)\,\mathrm{d}W_r + \int_s^t \left(b'\big(X_r^{t,x}\big) - \frac{1}{2}\sigma'^2\big(X_r^{t,x}\big)\right) \mathrm{d}r \right\}. \tag{8}$$

Before proceeding to the proof of this theorem we illustrate it with two examples.

Example 2.2. *Brownian motion reflected at* 0. *Let* $x > 0$. *The resolution of the Skorokhod problem* (*see*, *e.g.*, *Karatzas and Shreve* [8]), *shows that the adapted increasing process*

$$k_s^x(\omega) := \sup\left\{0, -x + \sup_{0\le r\le s} W_r(\omega)\right\} \tag{9}$$

is such that the process $X_s^x := x - W_s + k_s^x$ *is positive and satisfies*

$$\int_0^T \mathbb{I}_{(0,\infty)}\big(X_s^x\big)\,\mathrm{d}k_s^x = 0.$$

We obviously have

$$\partial_x X_s^x = 1 + \frac{\partial}{\partial x} k_s^x = \mathbb{I}_{\inf_{0\le r\le s} X_r^x > 0}.$$

Example 2.3. *Brownian motion reflected at points* d *and* d'. *Let* $x \in (d, d')$. *Kruk et al.* [9] *have solved explicitly the Skorokhod problem corresponding to a two-sided reflection. To simplify the notation we suppose here that* $d = 0$. *We therefore consider the process* $X_s^x := x - W_s + \tilde{k}_s^x$, *where we define the increasing process* $\tilde{k}_s^s$ *by*

$$-\tilde{k}_s^x := \Big[0 \wedge \inf_{0\le r\le s}(x - W_r)\Big] \vee \sup_{0\le r\le s}\Big[\big(x - W_r - d'\big) \wedge \inf_{r\le\theta\le s}(x - W_\theta)\Big].$$

Notice that, *on the event* $\mathcal{E}_s^{0,x}$ *the process* $(\tilde{k}_r^x, r \le s)$ *is null and thus* $\frac{\partial}{\partial x} X_s^x = 1$, *whereas on* $\Omega - \mathcal{E}_s^{0,x}$ *one has* $-\tilde{k}_s^x = x + G_s$ *for some random variable* G_s *independent of* x, *and thus* $\frac{\partial}{\partial x} X_s^x = 0$.

We start the proof of Theorem 2.1 with checking that the right-hand side of equality (7), that we will denote by $\Phi^{t,x}(s)$, is properly defined.

Proposition 2.4. *The process* $(\Phi^{t,x}(s), t \le s \le T)$ *is well defined in the sense that it does not depend on the Borel versions of the a.e. derivatives of* b *and* σ.

Proof. Observe that $\Phi^{t,x}(s) = 0$ on the event $\Omega - \mathcal{E}_s^{t,x}$. To prove the desired result on the event $\mathcal{E}_s^{t,x}$, consider two Borel versions b_1' and b_2' (respectively, σ_1' and σ_2') of the a.e. derivative of b (respectively, σ). For $i = 1, 2$ set

$$\widetilde{\Phi}^i(s) := \exp\left\{ \int_t^s \sigma_i'\big(X_r^{t,x}\big)\,\mathrm{d}W_r + \int_t^s \left(b_i'\big(X_r^{t,x}\big) - \frac{1}{2}\big(\sigma_i'\big)^2\big(X_r^{t,x}\big)\right) \mathrm{d}r \right\},$$

and $\Phi^i(s) = \widetilde{\Phi}^i(s)\mathbb{I}_{\mathcal{E}_s^{t,x}}$, so that

$$\begin{aligned}
&\mathbb{E}\sup_{t\le s\le T}\big|\Phi^1(s) - \Phi^2(s)\big|^2 \\
&\quad\le \mathbb{E}\sup_{t\le s\le T}\big|\widetilde{\Phi}^1(s) - \widetilde{\Phi}^2(s)\big|^2 \\
&\quad\le C\left\{\mathbb{E}\int_t^T \big|b_1'\big(X_r^{t,x}\big) - b_2'\big(X_r^{t,x}\big) - \big(\big(\sigma_1'\big)^2\big(X_r^{t,x}\big) - \big(\sigma_2'\big)^2\big(X_r^{t,x}\big)\big)\big|^4\,\mathrm{d}r\right\}^{1/2} \\
&\qquad + C\left\{\mathbb{E}\int_t^T \big|\sigma_1'\big(X_r^{t,x}\big) - \sigma_2'\big(X_r^{t,x}\big)\big|^4\,\mathrm{d}r\right\}^{1/2}.
\end{aligned}$$

The hypotheses made in Theorem 2.1 allow us to apply the Proposition 4.1 in Lépingle et al. [10]: for all $s > t$ and x, the probability distribution of $X_r^{t,x}$ has a density $p^{t,x}(r,\cdot)$ w.r.t. Lebesgue's measure. To conclude, it then remains to use $b_1' \equiv b_2'$ and $\sigma_1' \equiv \sigma_2'$ a.e. □

2.2. *On approximations by penalization*

The proof of Theorem 2.1 essentially consists in approximating X by the solution of a penalized stochastic differential equation. We use the following proposition which precises the convergence rate of $\mathbb{E}\sup_{t\le s\le T}|X_s^{t,x} - X_s^{t,x,n}|^p$ for $p > 2$ and is easily derived from the inequality (3.23) in Menaldi [14]:

Proposition 2.5 ([14]). *For $n \ge 1$ define the function β_n by*

$$\beta_n(y) := \begin{cases} -n(y-d') & \text{if } y \ge d', \\ 0 & \text{if } d \le y \le d', \\ n(d-y) & \text{if } y \le d. \end{cases}$$

Then the solution $X^{t,x,n}$ to

$$X_s^{t,x,n} = x + \int_t^s b\left(X_r^{t,x,n}\right)\mathrm{d}r + \int_t^s \sigma\left(X_r^{t,x,n}\right)\mathrm{d}W_r + \int_t^s \beta_n\left(X_r^{t,x,n}\right)\mathrm{d}r \tag{10}$$

satisfies, for all $p \ge 1$,

$$\forall t \le T \quad \lim_{n\to\infty} \sup_{x\in(d,d')} \mathbb{E} \sup_{t\le s\le T} \left|X_s^{t,x} - X_s^{t,x,n}\right|^p = 0. \tag{11}$$

In order to explicit the limit of $\partial_x X_s^{x,n}$ we use the following convergence criterion used in Bouleau and Hirsch [6], p. 49.

Proposition 2.6. *Let $(H_s^{x,n}, s \in [0,T], n \ge 1)$ be a sequence of random fields which are time continuous from $[0,T]$ to $\widetilde{D}$. Suppose that*

$$\sup_{n\ge1}\sup_{s\in[0,T]}\left[\int_d^{d'} \mathbb{E}\left|H_s^{x,n}\right|^2 \mathrm{d}x + \int_d^{d'} \mathbb{E}\left|\partial_x H_s^{x,n}\right|^2 \mathrm{d}x\right] < +\infty. \tag{12}$$

Suppose that there exists a stochastic flow H_s^x continuous in (s,x) such that

$$\int_d^{d'} \mathbb{E} \sup_{s\in[0,T]} \left|H_s^{x,n} - H_s^x\right|^2 \mathrm{d}x \xrightarrow[n\to+\infty]{} 0. \tag{13}$$

Then, for all $s \in [0,T]$, $\partial_x H_s^x$ is well defined $\widetilde{\mathbb{P}}$-a.s., H_s^x is in $\widetilde{D}$ and

$$\int_d^{d'} \mathbb{E}\left|H_s^x\right|^2 \mathrm{d}x + \int_d^{d'} \mathbb{E}\left|\partial_x H_s^x\right|^2 \mathrm{d}x < +\infty. \tag{14}$$

In addition, $H_s^{x,n}$ converges weakly to H_s^x in the following sense: for all stochastic flow U_s^x such that $\partial_x U_s^x$ is well defined $\widetilde{\mathbb{P}}$-a.s. and

$$\int_d^{d'} \mathbb{E}\left|U_s^x\right|^2 \mathrm{d}x + \int_d^{d'} \mathbb{E}\left(\partial_x U_s^x\right)^2 \mathrm{d}x < +\infty,$$

then

$$\int_d^{d'} \mathbb{E}\left[U_s^x\left(H_s^{x,n} - H_s^x\right) + \partial_x U_s^x\left(\partial_x H_s^{x,n} - \partial_x H_s^x\right)\right]\mathrm{d}x \xrightarrow[n\to+\infty]{} 0.$$

The next lemma states that the process $X^{t,x,n}$ satisfies (12).

Lemma 2.7. *For all $p \geq 1$ we have*

$$\sup_{n\geq 1}\sup_{x\in(d,d')}\Big[\mathbb{E}\sup_{s\in[t,T]}\big|X_s^{t,x,n}\big|^p + \mathbb{E}\sup_{s\in[t,T]}\big|\partial_x X_s^{t,x,n}\big|^p\Big] < +\infty.$$

Proof. In view of (11) we only need to estimate $\mathbb{E}\sup_{s\in[t,T]}|\partial_x X_s^{t,x,n}|^p$. Set $b_n := b + \beta_n$. From the Theorem 1 and the discussion in [6], p. 56, we deduce that, $\mathbb{P}$-a.s., the derivative $\partial_s X_s^{t,x,n}$ in the sense of the distributions is well defined and satisfies

$$\partial_x X_s^{t,x,n} = \exp\left\{\int_t^s \sigma'\big(X_r^{t,x,n}\big)\,\mathrm{d}W_r + \int_t^s \left(b_n'\big(X_r^{t,x,n}\big) - \frac{1}{2}\sigma'\big(X_r^{t,x,n}\big)^2\right)\mathrm{d}r\right\}.$$

It then suffices to use the one-side bound from above $b_n'(y) \leq \|b'\|_\infty$ for all integer n and all $y \in \mathbb{R}$ to get

$$\sup_{n\geq 1}\sup_{x\in(d,d')}\mathbb{E}\sup_{s\in[t,T]}\big|\partial_x X_s^{t,x,n}\big|^p < +\infty. \tag{15}$$

□

Our next step consists in identifying the process $\partial_x X_s^{t,x}$.

2.3. *Proof of Theorem 2.1: The one-sided reflection case*

We are now in a position to explicit the derivative of $X_s^{t,x}$. We start with the case of the reflection at the sole point d.

Proposition 2.8. *Let $x \in (d, d')$ and $\widehat{X}^{t,x}$ be the solution to*

$$\widehat{X}_s^{t,x} = x + \int_t^s b\big(\widehat{X}_r^{t,x}\big)\,\mathrm{d}r + \int_t^s \sigma\big(\widehat{X}_r^{t,x}\big)\,\mathrm{d}W_r + \Lambda_s^d\big(\widehat{X}^{t,x}\big),$$

where $\Lambda^d(\widehat{X}^{t,x})$ is the local time at point d of the semi-martingale $\widehat{X}^{t,x}$. The flow $\widehat{X}^{t,x}$ belongs to $\widetilde{D}$ and, setting

$$\widehat{\mathcal{E}}_s^{t,x} := \left\{\omega \in \Omega,\ \inf_{t\leq r\leq s}\widehat{X}_r^{t,x}(\omega) > d\right\},$$

we have: for all $t \leq s \leq T$, $\widetilde{\mathbb{P}}$-a.s.,

$$\partial_x \widehat{X}_s^{t,x} = \exp\left\{\int_t^s \sigma'\big(\widehat{X}_r^{t,x}\big)\,\mathrm{d}W_r + \int_t^s \left(b'\big(\widehat{X}_r^{t,x}(\omega)\big) - \frac{1}{2}\sigma'^2\big(\widehat{X}_r^{t,x}\big)\right)\mathrm{d}r\right\}\mathbb{I}_{\widehat{\mathcal{E}}_s^{t,x}}. \tag{16}$$

Proof. For all $n \geq 1$ consider the solutions $(\widehat{X}^{t,x,n})$ to

$$\widehat{X}_s^{t,x,n} = x + \int_t^s b\big(\widehat{X}_r^{t,x,n}\big)\,\mathrm{d}r + \int_t^s \sigma\big(\widehat{X}_r^{t,x,n}\big)\,\mathrm{d}W_r + \int_t^s n\big(d - \widehat{X}_r^{t,x,n}\big)^+\,\mathrm{d}r.$$

In view of Theorem 1 in [6], the stochastic flow $\widehat{X}^{t,x,n}$ is differentiable in the sense of the distributions, and its derivative, denoted by $\partial_x \widehat{X}_s^{t,x,n}$, satisfies $\widetilde{\mathbb{P}}$-a.s.,

$$\begin{aligned}\partial_x \widehat{X}_s^{t,x,n} = {}& \exp\left\{\int_t^s \sigma'\big(\widehat{X}_r^{t,x,n}\big)\,\mathrm{d}W_r + \int_t^s \left(b'\big(\widehat{X}_r^{t,x,n}\big) - \frac{1}{2}\sigma'^2\big(\widehat{X}_r^{t,x,n}\big)\right)\mathrm{d}r\right\}\\ &\times \exp\left\{-n\int_t^s \mathbb{I}_{\widehat{X}_r^{t,x,n}<d}\,\mathrm{d}r\right\}.\end{aligned}$$

We can easily get a result similar to Lemma 2.7, that is,

$$\sup_{n\geq 1}\sup_{x\in(d,d')}\Big[\mathbb{E}\sup_{s\in[t,T]}\big|\widehat{X}_s^{t,x,n}\big|^2+\mathbb{E}\sup_{s\in[t,T]}\big|\partial_x\widehat{X}_s^{t,x,n}\big|^2\Big]<+\infty, \tag{17}$$

which establishes (12) with $H_s^{x,n}\equiv\widehat{X}_s^{t,x,n}$. To obtain (13) we observe that we may substitute $\widehat{X}$ to X into (11): indeed, in [14] the diffusion process is reflected at the boundary of a bounded domain whereas, here, the domain is the infinite interval $(d,+\infty)$; however, it is easy to see that Menaldi's proof of inequality (3.23) also applies in this latter case.[3] Therefore, in view of Proposition 2.6, for all $t\leq s\leq T$, $\widetilde{\mathbb{P}}$-a.s., $\partial_x\widehat{X}_s^{t,x,n}$ converges weakly into some process that we denote by $\partial_x\widehat{X}_s^{t,x}$ and $\widehat{X}_s^{t,x}\in\widetilde{D}$. Suppose now that we have proven, for all x in (d,d'):

$$A_s^{t,x,n}:=\exp\bigg\{-n\int_t^s\mathbb{I}_{\widehat{X}_r^{t,x,n}(\omega)<d}\bigg\}\mathbb{I}_{\widehat{\mathcal{E}}_s^{t,x}}\xrightarrow[n\to+\infty]{}\mathbb{I}_{\widehat{\mathcal{E}}_s^{t,x}},\quad\mathbb{P}\text{-a.s.}, \tag{18}$$

and

$$\mathbb{E}B_s^{t,x,n}\xrightarrow[n\to+\infty]{}0, \tag{19}$$

where

$$B_s^{t,x,n}:=\exp\bigg\{-n\int_t^s\mathbb{I}_{\widehat{X}_r^{t,x,n}(\omega)<d}\bigg\}\mathbb{I}_{\Omega-\widehat{\mathcal{E}}_s^{t,x}}.$$

Let us check that we then could deduce (16). Indeed, denoting by $\widehat{G}_s^{t,x}$ the r.h.s. of (16), it suffices to prove that, for all stochastic field U_s^x as in Proposition 2.6,

$$\begin{aligned}&\mathbb{E}\int_d^{d'}U_s^x\big(\widehat{X}_s^{t,x,n}-\widehat{X}_s^{t,x}\big)\,\mathrm{d}x+\mathbb{E}\int_d^{d'}\partial_xU_s^x\,\big(\partial_x\widehat{X}_s^{t,x,n}-\widehat{G}_s^{t,x}\big)\mathbb{I}_{\widehat{\mathcal{E}}_s^{t,x}}\,\mathrm{d}x\\&\quad+\mathbb{E}\int_d^{d'}\partial_xU_s^x\,\partial_x\widehat{X}_s^{t,x,n}\,\mathbb{I}_{\Omega-\widehat{\mathcal{E}}_s^{t,x}}\,\mathrm{d}x\end{aligned}$$

tends to 0 as n tends to infinity. Now, it is easy to check that each one of the three terms in the right-hand side tends to 0: for example, one has

$$\bigg|\mathbb{E}\int_d^{d'}\partial_xU_s^x\,\partial_x\widehat{X}_s^{t,x,n}\,\mathbb{I}_{\Omega-\widehat{\mathcal{E}}_s^{t,x}}\,\mathrm{d}x\bigg|^2\leq C\int_d^{d'}\mathbb{E}\big(\partial_xU_s^x\big)^2\,\mathrm{d}x\int_d^{d'}\mathbb{E}\big(B_s^{t,x,n}\big)^2\,\mathrm{d}x,$$

and the right-hand side tends to 0 in view of (19).

Therefore it now remains to prove (18) and (19).

We start with (18). It suffices to prove that, on the event $\{\inf_{t\leq r\leq s}\widehat{X}_r^{t,x}>d\}$, for all n large enough, $\widetilde{\mathbb{P}}$-a.s., $\inf_{t\leq r\leq s}\widehat{X}_r^{t,x,n}>d$. A sufficient condition is

$$\sup_{t\leq r\leq s}\big|\widehat{X}_r^{t,x,n}-\widehat{X}_r^{t,x}\big|\leq\frac{1}{2}\Big(\inf_{t\leq r\leq s}\widehat{X}_r^{t,x}-d\Big).$$

In view of Menaldi [14], Remark 3.1, p. 742, for all $2<2q<p$ there exists $C>0$ such that, for all n,

$$\mathbb{E}\sup_{t\leq s\leq T}\big|\widehat{X}_s^{t,x,n}-\widehat{X}_s^{t,x}\big|^p\leq\frac{C}{n^q}.$$

Thus Borel–Cantelli's lemma implies that $\sup_{t\leq s\leq T}|\widehat{X}_s^{t,x,n}-\widehat{X}_s^{t,x}|$ tends to 0 almost surely. We thus have proven (18).

---

[3] The properties (3.1) and (3.2) of the penalization function in [14] are clearly satisfied by β_n.

Let us now prove that $\mathbb{E}B_s^{t,x,n}$ converges to 0. The comparison theorem for stochastic differential equations shows that, for all $m<n$ and $t<r<T$, $\widehat{X}_r^{t,x,m} \le \widehat{X}_r^{t,x,n}$; therefore, for all n and $t<r<T$, $\widehat{X}_r^{t,x,n} \le \widehat{X}_r^{t,x}$. Thus

$$\mathbb{E}B_s^{t,x,n} \le \mathbb{E}\left[\exp\left\{-n\int_t^s \mathbb{I}_{\widehat{X}_r^{t,x,n}<d}\,\mathrm{d}r\right\}\mathbb{I}_{\inf_{t\le r\le s}\widehat{X}_r^{t,x,n}\le d}\right].$$

Let φ be the increasing one-to-one map $\varphi(z) := \int_0^z \frac{1}{\sigma(y)}\,\mathrm{d}y$. Set $\bar{X}_s^{t,x,n} := \varphi(\widehat{X}_s^{t,x,n})$. We have:

$$\begin{aligned}\bar{X}_s^{t,x,n} &= \varphi(x) + \int_t^s \left[\frac{b(\varphi^{-1}(\bar{X}_r^{t,x,n})) + n(d-\varphi^{-1}(\bar{X}_r^{t,x,n}))^+}{\sigma(\varphi^{-1}(\bar{X}_r^{t,x,n}))} - \frac{1}{2}\sigma'\big(\varphi^{-1}\big(\bar{X}_r^{t,x,n}\big)\big)\right]\mathrm{d}r\\ &\quad + W_s - W_t.\end{aligned}$$

Using the Girsanov transformation removing the drift coefficient of $(\bar{X}_s^{t,x,n})$ and denoting by $\mathbb{E}_{t,\varphi(x)}$ the conditional expectation knowing that $W_t=\varphi(x)$ we get

$$\mathbb{E}B_s^{t,x,n} \le \mathbb{E}_{t,\varphi(x)}\left[M_s^n \exp\left\{-n\int_t^s \mathbb{I}_{W_r<\varphi(d)}\,\mathrm{d}r\right\}\mathbb{I}_{\inf_{t\le r\le s}W_r\le\varphi(d)}\right],$$

where

$$\begin{aligned}M_s^n &= \exp\left\{\int_t^s\left[\frac{b(\varphi^{-1}(W_r)) + n(d-\varphi^{-1}(W_r))^+}{\sigma(\varphi^{-1}(W_r))} - \frac{1}{2}\sigma'\big(\varphi^{-1}(W_r)\big)\right]\mathrm{d}W_r\right\}\\ &\quad\times\exp\left\{-\frac{1}{2}\int_t^s\left[\frac{b(\varphi^{-1}(W_r)) + n(d-\varphi^{-1}(W_r))^+}{\sigma(\varphi^{-1}(W_r))} - \frac{1}{2}\sigma'\big(\varphi^{-1}(W_r)\big)\right]^2\mathrm{d}r\right\}.\end{aligned}$$

Set $\Phi_\sigma(z) := \int_z^{\varphi(d)} \frac{(d-\varphi^{-1}(y))^+}{\sigma(\varphi^{-1}(y))}\,\mathrm{d}y$. Then

$$\begin{aligned}\Phi_\sigma(W_s) &= \Phi_\sigma(W_t) - \int_t^s \frac{(d-\varphi^{-1}(W_r))^+}{\sigma(\varphi^{-1}(W_r))}\,\mathrm{d}W_r + \frac{1}{2}\int_t^s \mathbb{I}_{\varphi^{-1}(W_r)<d}\,\mathrm{d}r\\ &\quad + \frac{1}{2}\int_t^s \frac{\sigma'(\varphi^{-1}(W_r))}{\sigma(\varphi^{-1}(W_r))}\big(d-\varphi^{-1}(W_r)\big)^+\,\mathrm{d}r.\end{aligned}$$

In addition, observe that, for all $x\in(d,d')$, $\Phi_\sigma(\varphi(x))=0$, and that $\Phi_\sigma(z)\ge 0$ for all z in $\mathbb{R}$. Therefore, $\mathbb{P}_{t,\varphi(x)}$-a.s.,

$$\begin{aligned}n\int_t^s \frac{(d-\varphi^{-1}(W_r))^+}{\sigma(\varphi^{-1}(W_r))}\,\mathrm{d}W_r &\le \frac{n}{2}\int_t^s \mathbb{I}_{W_r<\varphi(d)}\,\mathrm{d}r + \frac{n}{2}\int_t^s \frac{\sigma'(\varphi^{-1}(W_r))}{\sigma(\varphi^{-1}(W_r))}\big(d-\varphi^{-1}(W_r)\big)^+\,\mathrm{d}r\\ &\le \frac{n}{2}\int_t^s \mathbb{I}_{W_r<\varphi(d)}\,\mathrm{d}r + K\frac{n}{2\alpha_*}\int_t^s \big(d-\varphi^{-1}(W_r)\big)^+\,\mathrm{d}r,\end{aligned}\tag{20}$$

where K is the Lipschitz constant of σ, and α_* is as in (6). We deduce that, for some positive constants C_1 and C_2 and bounded continuous functions ρ_1 and ρ_2, all of them independent of n, $\mathbb{P}_{t,\varphi(x)}$-a.s.,

$$\begin{aligned}M_s^n &\le \exp\left\{\frac{n}{2}\int_t^s \mathbb{I}_{W_r<\varphi(d)}\,\mathrm{d}r + C_1 n\int_t^s \big(d-\varphi^{-1}(W_r)\big)^+\,\mathrm{d}r\right.\\ &\quad\left. - C_2 n^2\int_t^s \big(\big(d-\varphi^{-1}(W_r)\big)^+\big)^2\,\mathrm{d}r + \int_t^s \rho_1(W_r)\,\mathrm{d}W_r + \int_t^s \rho_2(W_r)\,\mathrm{d}r\right\}.\end{aligned}\tag{21}$$

As there exists $C_0>0$ such that $C_1 nY - C_2 n^2 Y^2 < C_0$ for all integer n and all $Y\ge 0$, Cauchy–Schwartz inequality implies

$$\mathbb{E}B_s^{t,x,n} \le C\sqrt{\Upsilon^n},\tag{22}$$

where

$$\Upsilon^n := \mathbb{E}_{t,\varphi(x)}\left(\exp\left\{-n\int_t^s \mathbb{I}_{W_r<\varphi(d)}\,\mathrm{d}r\right\}\mathbb{I}_{\inf_{t\le r\le s} W_r\le\varphi(d)}\right).$$

Now set $x_0 := \varphi(d)-\varphi(x)$ and let $\tau_{x_0} := \inf\{r\ge 0,\, W_r = x_0\}$ be the first passage time of the Brownian motion W at point x_0. The strong Markov property and the definition of x_0 imply that

$$\Upsilon^n \le \int_0^{s-t} \mathbb{E}_{x_0}\exp\left\{-n\int_0^{s-t-\theta}\mathbb{I}_{W_r\le x_0}\,\mathrm{d}r\right\}\mathrm{d}\mathbb{P}^W_{\tau_{x_0}}(\theta),$$

where (see, e.g., Borodin and Salminen [4], p. 198)

$$\mathrm{d}\mathbb{P}^W_{\tau_{x_0}}(\theta) = \frac{|x_0|}{\sqrt{2\pi\theta^3}}\exp\left(-\frac{x_0^2}{2\theta}\right)\mathrm{d}\theta.$$

Using formula (1.5.3) in Borodin and Salminen [4], p. 160, we deduce

$$\Upsilon^n \le \int_0^{s-t} I_0\left(\frac{n(s-t-\theta)}{2}\right)\exp\left(-\frac{n}{2}(s-t-\theta)\right)\frac{|x_0|}{\sqrt{2\pi\theta^3}}\exp\left(-\frac{x_0^2}{2\theta}\right)\mathrm{d}\theta, \tag{23}$$

where I_0 is a Bessel function whose definition can be found in, e.g., Abramowitz and Stegun [1], p. 375. We split the integral in the right-hand side of (23) into the two following terms:

$$\Upsilon_1^n := \int_0^{s-t-1/\sqrt{n}} I_0\left(\frac{n(s-t-\theta)}{2}\right)\exp\left(-\frac{n}{2}(s-t-\theta)\right)\frac{|x_0|}{\sqrt{2\pi\theta^3}}\exp\left(-\frac{x_0^2}{2\theta}\right)\mathrm{d}\theta,$$

$$\Upsilon_2^n := \int_{s-t-1/\sqrt{n}}^{s-t} I_0\left(\frac{n(s-t-\theta)}{2}\right)\exp\left(-\frac{n}{2}(s-t-\theta)\right)\frac{|x_0|}{\sqrt{2\pi\theta^3}}\exp\left(-\frac{x_0^2}{2\theta}\right)\mathrm{d}\theta.$$

For all θ in $(0, s-t-\frac{1}{\sqrt{n}})$ one has $\frac{n(s-t-\theta)}{2}\ge\frac{\sqrt{n}}{2}$; in addition (see, e.g., Borodin and Salminen [4], p. 638),

$$I_0(y)\approx\frac{\mathrm{e}^y}{\sqrt{2\pi y}}\quad\text{as } y\to+\infty.$$

Therefore, there exists $C>0$, uniformly bounded in $x_0\in(\varphi(d)-\varphi(d'),0)$ such that, for all n large enough,

$$\Upsilon_1^n \le \frac{C}{n^{1/4}}.$$

Now, we use that $I_0(y)\mathrm{e}^{-y}\le 1$ for all $y\ge 0$ (see, e.g., Abramowitz and Stegun [1], p. 375) and deduce that

$$\Upsilon_2^n \le \int_{s-t-1/\sqrt{n}}^{s-t}\frac{|x_0|}{\sqrt{2\pi\theta^3}}\exp\left(-\frac{x_0^2}{2\theta}\right)\mathrm{d}\theta \le \frac{C}{\sqrt{n}},$$

from which

$$\Upsilon^n \le \frac{C}{n^{1/4}}, \tag{24}$$

where C is uniformly bounded in $x\in(d,d')$. In view of (22) we thus have obtained

$$\mathbb{E}B_s^{t,x,n}\le\frac{C}{n^{1/8}},$$

which ends the proof. □

2.4. *Proof of Theorem* 2.1: *The two-sided reflection case*

We now consider the penalized system (10).

With the arguments used at the beginning of the proof of Proposition 2.8 one can deduce that, $\mathbb{P}$-a.s., the map $x \mapsto X_s^{t,x}$ belongs to the Sobolev space

$$H^1(d, d') = \{ f \in L(d, d');\ f' \in L(d, d') \}.$$

We now aim to prove the representation formula (7). We first consider the event $\mathcal{E}_s^{t,x}$. On this event $X_s^{t,x,n}$ satisfies

$$X_s^{t,x,n} = x + \int_t^s b(X_r^{t,x,n})\,\mathrm{d}r + \int_t^s \sigma(X_r^{t,x,n})\,\mathrm{d}W_r.$$

Pathwise uniqueness holds for both the above stochastic differential equation and Eq. (4). Therefore $(X_r^{t,x,n}, r \in [t,s])$ and $(X_r^{t,x}, r \in [t,s])$ coincide on $\mathcal{E}_s^{t,x}$. We deduce the equality (7) on $\mathcal{E}_s^{t,x}$.

We next consider the event $\Omega - \mathcal{E}_s^{t,x}$. We are inspired by Lépingle et al. [10] to reduce our study to the one-sided reflection case. For all rational numbers c_1 and s_1 such that $d < c_1 < d'$ and $t < s_1 < s$ set

$$\mathcal{A}_{s_1}^{d,c_1} := \Big\{ \omega \in \Omega \colon d = \inf_{r \in [t,s_1]} X_r^{t,x},\ \sup_{r \in [t,s_1]} X_r^{t,x} = c_1 \Big\},$$

$$\mathcal{A}_{s_1}^{c_1,d'} := \Big\{ \omega \in \Omega \colon \inf_{r \in [t,s_1]} X_r^{t,x} = c_1,\ \sup_{r \in [t,s_1]} X_r^{t,x} = d' \Big\}.$$

Set also

$$\mathcal{A}^d := \{ \omega \in \Omega \colon \forall r \in [t,s),\ d < X_r^{t,x} < d',\ X_s^{t,x} = d \},$$

$$\mathcal{A}^{d'} := \{ \omega \in \Omega \colon \forall r \in [t,s),\ d < X_r^{t,x} < d',\ X_s^{t,x} = d' \}.$$

We have

$$\Omega - \mathcal{E}_s^{t,x} = \mathcal{A}^d \cup \mathcal{A}^{d'} \bigcup_{\substack{d<c_1<d' \\ t<s_1<s}} \big(\mathcal{A}_{s_1}^{d,c_1} \cup \mathcal{A}_{s_1}^{c_1,d'} \big).$$

Let $X^{t,x,n}$ be defined as in (10). As observed in the proof of Lemma 2.7, setting $b_n := b + \beta_n$ we have, $\widetilde{\mathbb{P}}$-a.s.,

$$\partial_x X_s^{t,x,n} = \exp\left\{ \int_t^s \sigma'(X_r^{t,x,n})\,\mathrm{d}W_r + \int_t^s \left(b_n'(X_r^{t,x,n}) - \frac{1}{2}\sigma'(X_r^{t,x,n})^2 \right) \mathrm{d}r \right\}.$$

Let $\widehat{X}^{t,x}$ be the one-sided reflected diffusion process defined in Proposition 2.8, and, as in the proof of Proposition 2.8, let $\widehat{X}^{t,x,n}$ be the corresponding penalized process. On the event $\mathcal{A}_{s_1}^{d,c_1}$ we have $X^{t,x} = \widehat{X}^{t,x}$ and, as already noticed, we also have $\widehat{X}^{t,x,n} \le \widehat{X}^{t,x}$; therefore, on $\mathcal{A}_{s_1}^{d,c_1}$ the paths of $\widehat{X}^{t,x,n}$ do not hit the point d', which implies that $X^{t,x,n} = \widehat{X}^{t,x,n}$ on this event, from which, by a classical local property of Brownian stochastic integrals,

$$\partial_x X_{s_1}^{t,x,n} \mathbb{I}_{\mathcal{A}_{s_1}^{d,c_1}} = \exp\left\{ \int_t^{s_1} \sigma'(\widehat{X}_r^{t,x,n})\,\mathrm{d}W_r + \int_t^{s_1} \left(b_n'(\widehat{X}_r^{t,x,n}) - \frac{1}{2}\sigma'(\widehat{X}_r^{t,x,n})^2 \right) \mathrm{d}r \right\} \mathbb{I}_{\mathcal{A}_{s_1}^{d,c_1}}.$$

Moreover, the arguments used to prove (19) imply that

$$\mathbb{E}\left[\exp\left\{ \int_t^{s_1} \sigma'(\widehat{X}_r^{t,x,n})\,\mathrm{d}W_r + \int_t^{s_1} \left(b_n'(\widehat{X}_r^{t,x,n}) - \frac{1}{2}\sigma'(\widehat{X}_r^{t,x,n})^2 \right) \mathrm{d}r \right\} \mathbb{I}_{\mathcal{A}_{s_1}^{d,c_1}} \right] \xrightarrow[n\to+\infty]{} 0.$$

We deduce that

$$\partial_x X_s^{t,x} \mathbb{I}_{\mathcal{A}_{s_1}^{d,c_1}} = 0, \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

We readily conclude that

$$\partial_x X_s^{t,x} \mathbb{I}_{\Omega - \mathcal{E}_s^{t,x}} = 0, \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

3. Stochastic representations of derivatives of solutions of semilinear parabolic PDEs

Consider the semilinear parabolic PDE in an interval with a Neumann boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{A}u(t,x) + f\big(t,x,u(t,x),\sigma(x)\,\partial_x u(t,x)\big) = 0, & (t,x) \in [0,T) \times \big(d,d'\big), \\ u(T,x) = g(x), & x \in \big[d,d'\big], \\ \dfrac{\partial u}{\partial x}(t,x) + h(t,x) = 0, & (t,x) \in [0,T) \times \big\{d,d'\big\}, \end{cases} \tag{25}$$

where h is such that $h(T,\cdot) = -g'(\cdot)$ and

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}.$$

We aim to prove that $u(t,x)$ is in $H^1(d,d')$ for all $0 \le t < T$ and to exhibit probabilistic representation formulae for its derivative in the sense of the distributions, respectively when g is a bounded differentiable function and when g is only supposed Lipschitz. We start with the case of an homogeneous Neumann boundary condition, that is, the case where $h \equiv 0$.

3.1. *Homogeneous Neumann boundary condition: A representation involving g' and ∇f*

Consider the BSDE driven by the diffusion process $X^{t,x}$ reflected at points d and d':

$$Y_s^{t,x} = g\big(X_T^{t,x}\big) + \int_t^T f\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\,\mathrm{d}r - \int_t^T Z_r^{t,x}\,\mathrm{d}W_r. \tag{26}$$

Pardoux and Zhang [19] have shown that, under the hypotheses made in this section, the BSDE (26) has a unique progressively measurable solution such that

$$\mathbb{E}\sup_{t\le s\le T}\big|Y_s^{t,x}\big|^2 + \mathbb{E}\int_0^T \big|Z_r^{t,x}\big|^2\,\mathrm{d}r < \infty,$$

and the deterministic function $u(t,x) := Y_t^{t,x}$ is a viscosity solution to (25). The uniqueness issue has been studied by Barles [2], Theorem 2.1.

The aim of this subsection is to prove the following theorem which expresses the fact that, formally, the derivative of a parabolic PDE with a Neumann boundary condition solves a new parabolic PDE, driven with a Dirichlet boundary condition.

Theorem 3.1. *Suppose that $h \equiv 0$, and that b and σ are bounded Lipschitz functions. Suppose that σ satisfies (6). Suppose that the function f is in $\mathcal{C}^{0,1,1,1}([0,T] \times [d,d'] \times \mathbb{R} \times \mathbb{R})$ bounded with bounded derivatives. Suppose that g is a continuously differentiable function satisfying $g'(d) = g'(d') = 0$. Let $\tau^{t,x}$ be the first time that the process*

$X^{t,x}$ hits the boundary $\{d, d'\}$. Then the process $Y^{t,x}$ is in $\widetilde{D}$ and the function $u(t,x) := Y_t^{t,x}$ is in $H^1(d,d')$ for all $0 \le t \le T$. Moreover, for almost all x in (d,d'),

$$\partial_x u(t,x) = \mathbb{E}\biggl\{ g'\bigl(X_T^{t,x}\bigr)\,\partial_x X_T^{t,x}\,\mathbb{I}_{\{\tau^{t,x}>T\}} + \int_t^{T\wedge\tau^{t,x}} \bigl[\partial_x f\bigl(r,\Theta_r^{t,x}\bigr)\,\partial_x X_r^{t,x} + \partial_y f\bigl(r,\Theta_r^{t,x}\bigr)\Psi_r^{t,x} + \partial_z f\bigl(r,\Theta_r^{t,x}\bigr)\Gamma_r^{t,x}\bigr]\,\mathrm{d}r\biggr\}, \tag{27}$$

where $\Theta^{t,x} := (X^{t,x}, Y^{t,x}, Z^{t,x})$ solves (4) and (26), and $(\Psi_s^{t,x}, \Gamma_s^{t,x})$ is the unique adapted process satisfying

$$\mathbb{E}\sup_{t\le s\le T}\bigl|\Psi_s^{t,x}\bigr|^2 + \mathbb{E}\int_0^T \bigl|\Gamma_r^{t,x}\bigr|^2\,\mathrm{d}r < \infty,$$

and solution of the following BSDE:

$$\Psi_s^{t,x} = g'\bigl(X_T^{t,x}\bigr)\,\partial_x X_T^{t,x} + \int_s^T \bigl[\partial_x f\bigl(r,\Theta_r^{t,x}\bigr)\,\partial_x X_r^{t,x} + \partial_y f\bigl(r,\Theta_r^{t,x}\bigr)\Psi_r^{t,x} + \partial_z f\bigl(r,\Theta_r^{t,x}\bigr)\Gamma_r^{t,x}\bigr]\,\mathrm{d}r - \int_s^T \Gamma_r^{t,x}\,\mathrm{d}W_r. \tag{28}$$

Remark 3.2. *Existence and uniqueness of the solution of the linear BSDE (28) is a classical result: see Pardoux and Peng [18]. Proceeding as in the proof of Proposition 2.4, we can easily prove that the process $(\Psi_s^{t,x}, \Gamma_s^{t,x}, t\le s\le T)$ is well defined in the sense that, up to indistinguishability, it does not depend on the Borel versions of the a.e. derivatives of b and σ. In addition, notice that, in view of (7), the process $\partial_x X^{t,x}$ is null after $\tau^{t,x}$; therefore, for all functions f and g, the solution $(\Psi^{t,x}, \Gamma^{t,x})$ of (28) is also null for all $s \ge \tau^{t,x}$ if $\tau^{t,x} \le T$. We thus may rewrite (28) under the following form which will be useful in the sequel:*

$$\Psi_s^{t,x} = g'\bigl(X_T^{t,x}\bigr)\,\partial_x X_T^{t,x}\,\mathbb{I}_{\{\tau^{t,x}>T\}} + \int_{s\wedge\tau^{t,x}}^{T\wedge\tau^{t,x}} \bigl[\partial_x f\bigl(r,\Theta_r^{t,x}\bigr)\,\partial_x X_r^{t,x} + \partial_y f\bigl(r,\Theta_r^{t,x}\bigr)\Psi_r^{t,x} + \partial_z f\bigl(r,\Theta_r^{t,x}\bigr)\Gamma_r^{t,x}\bigr]\,\mathrm{d}r - \int_{s\wedge\tau^{t,x}}^{T\wedge\tau^{t,x}} \Gamma_r^{t,x}\,\mathrm{d}W_r. \tag{29}$$

Proof of Theorem 3.1. We only sketch the proof which closely follows the method developed by N'Zi et al. [16] to prove the equalities (32), (33) below when $X^{t,x}$ is valued in the whole space and, as in our context, b and σ are solely supposed Lipschitz.[4]

To prove the a.e. differentiability w.r.t. x of $Y^{t,x}$, we aim to use Proposition 2.6. To this end, consider $X^{t,x,n}$ defined as in (10) and the BSDE

$$Y_s^{t,x,n} = g\bigl(X_T^{t,x,n}\bigr) + \int_s^T f\bigl(r, X_r^{t,x,n}, Y_r^{t,x,n}, Z_r^{t,x,n}\bigr)\,\mathrm{d}r - \int_s^T Z_r^{t,x,n}\,\mathrm{d}W_r. \tag{30}$$

For all $t\le s\le T$ and x in (d,d'), we set $\Theta^{t,x,n} = (X^{t,x,n}, Y^{t,x,n}, Z^{t,x,n})$, and

$$\begin{aligned}
&\mathbb{E}\bigl|Y_s^{t,x,n} - Y_s^{t,x}\bigr|^2 + \mathbb{E}\int_s^T \bigl|Z_r^{t,x,n} - Z_r^{t,x}\bigr|^2\,\mathrm{d}r \\
&\quad\le \mathbb{E}\bigl|g\bigl(X_T^{t,x,n}\bigr) - g\bigl(X_T^{t,x}\bigr)\bigr|^2 + 2\mathbb{E}\int_s^T \bigl|\bigl(Y_r^{t,x,n} - Y_r^{t,x}\bigr)\bigl(f\bigl(r,\Theta_r^{t,x,n}\bigr) - f\bigl(r,\Theta_r^{t,x}\bigr)\bigr)\bigr|\,\mathrm{d}r
\end{aligned}$$

[4] We draw the reader's attention to the fact that, in [16], the parameter n concerns smooth approximations of b and σ, whereas here it concerns the approximation of the reflection by penalization.

$$\le C\mathbb{E}\left|X_T^{t,x,n} - X_T^{t,x}\right|^2 + C\left(1 + \frac{1}{\varepsilon}\right)\mathbb{E}\int_s^T \left|Y_r^{t,x,n} - Y_r^{t,x}\right|^2 \mathrm{d}r$$
$$+ C\mathbb{E}\int_s^T \left|X_r^{t,x,n} - X_r^{t,x}\right|^2 \mathrm{d}r + C\varepsilon\mathbb{E}\int_s^T \left|Z_r^{t,x,n} - Z_r^{t,x,n}\right|^2 \mathrm{d}r.$$

We choose $\varepsilon = 1/(2C)$ and apply Grönwall's lemma. It comes

$$\mathbb{E}\sup_{t\le s\le T}\left|Y_s^{t,x,n} - Y_s^{t,x}\right|^2 + \mathbb{E}\int_t^T \left|Z_r^{t,x,n} - Z_r^{t,x}\right|^2 \mathrm{d}r \le C\mathbb{E}\sup_{t\le s\le T}\left|X_s^{t,x,n} - X_s^{t,x}\right|^2.$$

In view of (11), we thus get by Lebesgue's Dominated Convergence theorem

$$\lim_{n\to+\infty}\int_d^{d'} \mathbb{E}\left(\sup_{t\le s\le T}\left|Y_s^{t,x,n} - Y_s^{t,x}\right|^2 + \int_t^T \left|Z_r^{t,x,n} - Z_r^{t,x}\right|^2 \mathrm{d}r\right)\mathrm{d}x = 0. \tag{31}$$

Now, from N'Zi et al. [16], Theorem 3.2, $\widetilde{\mathbb{P}}$-a.s.,

$$\partial_x X_s^{t,x,n} = \exp\left\{\int_t^s \sigma'\left(X_r^{t,x,n}\right)\mathrm{d}W_r + \int_t^s \left(b_n'\left(X_r^{t,x,n}\right) - \frac{1}{2}\left(\sigma'\right)^2\left(X_r^{t,x,n}\right)\right)\mathrm{d}r\right\}, \tag{32}$$

and the BSDE

$$\begin{aligned}\Psi_s^{t,x,n} = {} & g'\left(X_T^{t,x,n}\right)\partial_x X_T^{t,x,n} + \int_s^T \left[\partial_x f\left(r, \Theta_r^{t,x,n}\right)\Phi_r^{t,x,n} + \partial_y f\left(r, \Theta_r^{t,x,n}\right)\Psi_r^{t,x,n}\right.\\ & \left. + \partial_z f\left(r, \Theta_r^{t,x,n}\right)\Gamma_r^{t,x,n}\right]\mathrm{d}r - \int_s^T \Gamma_r^{t,x,n}\,\mathrm{d}W_r\end{aligned} \tag{33}$$

has a unique solution; in addition, $\Psi^{t,x,n} = \partial_x Y^{t,x,n}$. In view of Lemma 2.7, standard calculations lead to

$$\sup_{n\ge1}\int_d^{d'} \mathbb{E}\left(\sup_{t\le s\le T}\left|\Psi_s^{t,x,n}\right|^2 + \int_t^T \left|\Gamma_r^{t,x,n}\right|^2 \mathrm{d}r\right)\mathrm{d}x < +\infty.$$

Thus all the hypotheses of Proposition 2.6 are satisfied by the sequence of random fields $(Y^{t,x,n})$. To explicit the derivative of $Y_s^{t,x}$ we observe that

$$\begin{aligned}&\int_d^{d'} \mathbb{E}\left(\sup_{t\le s\le T}\left|\Psi_s^{t,x,n} - \Psi_s^{t,x}\right|^2 + \int_t^T \left|\Gamma_r^{t,x,n} - \Gamma_r^{t,x}\right|^2 \mathrm{d}r\right)\mathrm{d}x\\ &\quad\le C\int_d^{d'} \mathbb{E}\left(\left|\xi_T^{t,x,n}\right|^2 + \int_t^T \left|\delta_r^{t,x,n}\right|^2 \mathrm{d}r\right)\mathrm{d}x,\end{aligned} \tag{34}$$

where

$$\begin{aligned}\xi_T^{t,x,n} &:= g'\left(X_T^{t,x,n}\right)\partial_x X_T^{t,x,n} - g'\left(X_T^{t,x}\right)\partial_x X_T^{t,x},\\ \delta_r^{t,x,n} &:= \left(\partial_x f\left(r, \Theta_r^{t,x,n}\right) - \partial_x f\left(r, \Theta_r^{t,x}\right)\right)\partial_x X_r^{t,x} + \left(\partial_y f\left(r, \Theta_r^{t,x,n}\right) - \partial_y f\left(r, \Theta_r^{t,x}\right)\right)\Psi_r^{t,x}\\ &\quad + \left(\partial_z f\left(r, \Theta_r^{t,x,n}\right) - \partial_z f\left(r, \Theta_r^{t,x}\right)\right)\Gamma_r^{t,x}.\end{aligned}$$

In view of (31), (11) and Lemma 3.3 below we easily observe that the right-hand side of (34) tends to 0 when n tends to infinity, which ends the proof. □

Lemma 3.3. *The processes $X^{t,x}$ and $X^{t,x,n}$ being defined as in Proposition 2.5, we have: for all $t \le s \le T$,*

$$\sup_{x\in(d,d')}\mathbb{E}\left|\partial_x X_s^{t,x,n} - \partial_x X_s^{t,x}\right|^2 \xrightarrow[n\to+\infty]{} 0.$$

Proof. For $\mathcal{E}_s^{t,x}$ as in (5), we have

$$\mathbb{E}\big|\partial_x X_s^{t,x,n} - \partial_x X_s^{t,x}\big|^2 \le \mathbb{E}\big[\big|\partial_x X_s^{t,x,n} - \partial_x X_s^{t,x}\big|^2 \mathbb{I}_{\mathcal{E}_s^{t,x}}\big] + \mathbb{E}\big[\big|\partial_x X_s^{t,x,n} - \partial_x X_s^{t,x}\big|^2 \mathbb{I}_{\Omega - \mathcal{E}_s^{t,x}}\big]. \tag{35}$$

The first term of the right-hand side is null since the processes $X^{t,x,n}$ and $X^{t,x}$ are pathwise identical on the event $\mathcal{E}_s^{t,x}$. Now, in view of Theorem 2.1 one has

$$\mathbb{E}\big[\big|\partial_x X_s^{t,x,n} - \partial_x X_s^{t,x}\big|^2 \mathbb{I}_{\Omega - \mathcal{E}_s^{t,x}}\big] = \mathbb{E}\big[\big|\partial_x X_s^{t,x,n}\big|^2 \mathbb{I}_{\Omega - \mathcal{E}_s^{t,x}}\big] := \mathbb{A}_s^{t,x,n}.$$

Define the stopping times τ_d and $\tau_{d'}$ as

$$\begin{aligned}
\tau_d &:= \inf\big\{r \ge t, X_r^{t,x} = d\big\} \wedge T, \\
\tau_{d'} &:= \inf\big\{r \ge t, X_r^{t,x} = d'\big\} \wedge T.
\end{aligned}$$

As noticed in Section 2.3, on the event $\{\tau_d < \tau_{d'}\} \cap (\Omega - \mathcal{E}_s^{t,x})$, the sequence of processes $(X_r^{t,x,n}, t \le r \le \tau_{d'})$ increases to $(X_r^{t,x}, t \le r \le \tau_{d'})$, and therefore this event is included in $\{\inf_{t \le r \le s} X_r^{t,x,n} \le d\}$. Similarly, the event $\{\tau_{d'} < \tau_d\} \cap (\Omega - \mathcal{E}_s^{t,x})$ in included in $\{\sup_{t \le r \le s} X_r^{t,x,n} \ge d'\}$. It comes:

$$\begin{aligned}
\mathbb{A}_s^{t,x,n} \le{}& C\mathbb{E}\left[\exp\left\{-2n\int_t^s (\mathbb{I}_{X_r^{t,x,n} < d} + \mathbb{I}_{X_r^{t,x,n} > d'})\,\mathrm{d}r\right\}\mathbb{I}_{\{\tau_d < \tau_{d'}\} \cap (\Omega - \mathcal{E}_s^{t,x})}\right] \\
&+ C\mathbb{E}\left[\exp\left\{-2n\int_t^s (\mathbb{I}_{X_r^{t,x,n} < d} + \mathbb{I}_{X_r^{t,x,n} > d'})\,\mathrm{d}r\right\}\mathbb{I}_{\{\tau_{d'} < \tau_d\} \cap (\Omega - \mathcal{E}_s^{t,x})}\right] \\
\le{}& C\mathbb{E}\left[\exp\left\{-2n\int_t^s (\mathbb{I}_{X_r^{t,x,n} < d} + \mathbb{I}_{X_r^{t,x,n} > d'})\,\mathrm{d}r\right\}\mathbb{I}_{\inf_{t \le r \le s} X_r^{t,x,n} \le d}\right] \\
&+ C\mathbb{E}\left[\exp\left\{-2n\int_t^s (\mathbb{I}_{X_r^{t,x,n} < d} + \mathbb{I}_{X_r^{t,x,n} > d'})\,\mathrm{d}r\right\}\mathbb{I}_{\sup_{t \le r \le s} X_r^{t,x,n} \ge d'}\right].
\end{aligned}$$

We now only sketch the calculations since we proceed as in the proof of Proposition 2.8: using again the Lamperti transform φ and a Girsanov transformation

$$\begin{aligned}
\mathbb{A}_s^{t,x,n} \le{}& C\mathbb{E}_{t,\varphi(x)}\left[\mathbb{M}_s^n \exp\left\{-2n\int_t^s (\mathbb{I}_{W_r < \varphi(d)} + \mathbb{I}_{W_r > \varphi(d')})\,\mathrm{d}r\right\}\mathbb{I}_{\inf_{t \le r \le s} W_r \le \varphi(d)}\right] \\
&+ C\mathbb{E}_{t,\varphi(x)}\left[\mathbb{M}_s^n \exp\left\{-2n\int_t^s (\mathbb{I}_{W_r < \varphi(d)} + \mathbb{I}_{W_r > \varphi(d')})\,\mathrm{d}r\right\}\mathbb{I}_{\sup_{t \le r \le s} W_r \ge \varphi(d')}\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{M}_s^n = \exp\Bigg\{&\int_t^s \left[\frac{b(\varphi^{-1}(W_r)) + n(d - \varphi^{-1}(W_r))^+ - n(\varphi^{-1}(W_r) - d')^+}{\sigma(\varphi^{-1}(W_r))} - \frac{1}{2}\sigma'\big(\varphi^{-1}(W_r)\big)\right]\mathrm{d}W_r \\
&- \frac{1}{2}\int_t^s \left[\frac{b(\varphi^{-1}(W_r)) + n(d - \varphi^{-1}(W_r))^+ - n(\varphi^{-1}(W_r) - d')^+}{\sigma(\varphi^{-1}(W_r))} - \frac{1}{2}\sigma'\big(\varphi^{-1}(W_r)\big)\right]^2 \mathrm{d}r\Bigg\}.
\end{aligned}$$

The exponential martingale is bounded from above as in (21) by using (20) and the following analogous inequality:

$$-n\int_t^s \frac{(\varphi^{-1}(W_r) - d')^+}{\sigma(\varphi(W_r))}\,\mathrm{d}W_r \le \frac{n}{2}\int_t^s \mathbb{I}_{W_r > \varphi(d')}\,\mathrm{d}r + K\frac{n}{2\alpha_*}\int_t^s \big(\varphi^{-1}(W_r) - d'\big)^+\,\mathrm{d}r.$$

It then remains to use (24). We omit the details. $\square$

3.2. *Homogeneous Neumann boundary condition: A representation without g' and ∇f*

Inspired by the results in Ma and Zhang [13], we now aim to prove a formula of Elworthy's type for $\partial_x u(t,x)$ which does not suppose that the function f is everywhere differentiable.

Theorem 3.4. *Suppose that $h \equiv 0$. Suppose that b and σ are bounded Lipschitz functions. Suppose that σ satisfies (6). Suppose that the function f is in $\mathcal{C}([0,T] \times [d,d'] \times \mathbb{R} \times \mathbb{R})$, bounded and uniformly Lipschitz w.r.t. the space variables. Suppose that g is a continuously differentiable function satisfying $g'(d) = g'(d') = 0$. Then the function $u(t,x) := Y_t^{t,x}$ is in $H^1(d,d')$ for all $0 \le t \le T$. Moreover, for almost all x in (d,d'),*

$$\partial_x u(t,x) = \mathbb{E}\left(g\left(X_T^{t,x}\right) N_T^{t,x} + \int_t^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) N_r^{t,x} \, \mathrm{d}r \right), \tag{36}$$

where, for all $0 \le t < s \le T$,

$$N_s^{t,x} := \frac{1}{s-t} \int_t^s \sigma^{-1}\left(X_r^{t,x}\right) \partial_x X_r^{t,x} \, \mathrm{d}W_r. \tag{37}$$

Proof. Set $u^n(t,x) := Y_t^{t,x,n}$ where $Y^{t,x,n}$ is as in (30). In view of N'zi et al. [16], Theorem 4.1, we have

$$\partial_x u^n(t,x) = \mathbb{E}\left(g\left(X_T^{t,x,n}\right) N_T^{t,x,n} + \int_t^T f\left(r, X_r^{t,x,n}, Y_r^{t,x,n}, Z_r^{t,x,n}\right) N_r^{t,x,n} \, \mathrm{d}r \right), \tag{38}$$

where, for all $0 \le t < s \le T$,

$$N_s^{t,x,n} := \frac{1}{s-t} \int_t^s \sigma^{-1}\left(X_r^{t,x,n}\right) \partial_x X_r^{t,x,n} \, \mathrm{d}W_r. \tag{39}$$

We first need to show that the deterministic version of Proposition 2.6 is satisfied by $u^n(t,x)$, that is,

$$\sup_{n \ge 1} \left[\int_d^{d'} \left|u^n(t,x)\right|^2 \mathrm{d}x + \int_d^{d'} \left|\partial_x u^n(t,x)\right|^2 \mathrm{d}x \right] < +\infty \tag{40}$$

and

$$\int_d^{d'} \left|u^n(t,x) - u(t,x)\right|^2 \mathrm{d}x \xrightarrow[n \to +\infty]{} 0. \tag{41}$$

In view of (6), (7) and Lemma 2.7, one observes that, for all $t < r < T$ and $p \ge 1$,

$$\sup_{x \in (d,d)'} \mathbb{E}\left(\left|N_r^{t,x,n}\right|^{2p} + \left|N_r^{t,x}\right|^{2p}\right) \le \frac{C}{(r-t)^p}, \tag{42}$$

from which we deduce (40). Now, we observe that, to obtain (31), we only used that f is a Lipschitz function; therefore (41) holds true, and $u(t,x)$ is in $H^1(d,d')$. It thus remains to identify $\partial_x u(t,x)$ by letting n go to infinity in (38).

From Lemma 3.3 and (11), we easily get that, for all $0 \le t < s \le T$,

$$\sup_{x \in (d,d')} \mathbb{E}\left|N_s^{t,x,n} - N_s^{t,x}\right|^2 \xrightarrow[n \to +\infty]{} 0. \tag{43}$$

Therefore $\int_d^{d'} |\mathbb{E}(g(X_T^n) N_T^{t,x,n} - g(X_T) N_T^{t,x})|^2 \, \mathrm{d}x$ tends to 0 and

$$\int_d^{d'} \left| \mathbb{E} \int_t^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)\left(N_r^{t,x,n} - N_r^{t,x}\right) \mathrm{d}r \right|^2 \mathrm{d}x \le C \int_d^{d'} \left| \int_t^T \sqrt{\mathbb{E}\left|N_r^{t,x,n} - N_r^t\right|^2} \, \mathrm{d}r \right|^2 \mathrm{d}x$$

tends also to 0 by Lebesgue's Dominated Convergence theorem. In view of (31), we are in a position to conclude that the right-hand side of (38) converges to the right-hand side of (36). □

3.3. *Extension to nonhomogeneous Neumann boundary conditions*

Consider the BSDE

$$\underline{Y}_s^{t,x} = g\big(X_T^{t,x}\big) + \int_s^T h\big(r, X_r^{t,x}\big)\,\mathrm{d}K_r^{t,x} + \int_s^T f\big(r, X_r^{t,x}, \underline{Y}_r^{t,x}, \underline{Z}_r^{t,x}\big)\,\mathrm{d}r - \int_s^T \underline{Z}_r^{t,x}\,\mathrm{d}W_r. \tag{44}$$

Under the hypotheses made in this subsection, Pardoux and Zhang [19] have shown that there exists a unique adapted solution $(\underline{Y}^{t,x}, \underline{Z}^{t,x})$ to (44) such that

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\big|\underline{Y}_t^{t,x}\big|^2 + \int_0^T \big|\underline{Y}_r^{t,x}\big|^2\,\mathrm{d}K_r^{t,x} + \int_0^T \big|\underline{Z}_r^{t,x}\big|^2\,\mathrm{d}r\bigg) < +\infty,$$

and, in addition, the function $u(t,x) := \underline{Y}_t^{t,x}$ is a viscosity solution to the parabolic PDE with nonhomogeneous Neumann boundary condition

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \mathcal{A}u(t,x) + f\big(t,x,u(t,x),\partial_x u(t,x)\sigma(x)\big) = 0, & (t,x)\in[0,T)\times\big(d,d'\big),\\ u(T,x) = g(x), & x\in\big[d,d'\big],\\ \frac{\partial u}{\partial x}(t,x) + h(t,x) = 0, & (t,x)\in[0,T)\times\big\{d,d'\big\}. \end{cases} \tag{45}$$

For uniqueness results for this PDE, we again refer to Barles [2], Theorem 2.1. We easily extend the representation formula in Theorem 3.1.

Theorem 3.5. *Let the assumptions of Theorem 3.1 hold true. In addition, suppose that the function h is continuous on $[0,T]\times[d,d']$. Suppose also that $g'(x) = -h(T,x)$ for $x = d$ or $x = d'$. Then the function $u(t,x) := \underline{Y}_t^{t,x}$ is in $H^1(d,d')$ for all $0\le t\le T$, and, for almost all x in (d,d'),*

$$\begin{aligned} \partial_x u(t,x) = \mathbb{E}\bigg\{ & g'\big(X_T^{t,x}\big)\,\partial_x X_T^{t,x}\,\mathbb{I}_{\{\tau^{t,x}>T\}} - h\big(\tau^{t,x}, X_{\tau^{t,x}}^{t,x}\big) J_{\tau^{t,x}}^{t,x}\,\mathbb{I}_{\{\tau^{t,x}\le T\}} \\ & + \int_t^{T\wedge\tau^{t,x}} \big[\partial_x f\big(r,\underline{\Theta}_r^{t,x}\big)\,\partial_x X_r^{t,x} + \partial_y f\big(r,\underline{\Theta}_r^{t,x}\big)\underline{\Psi}_r^{t,x} + \partial_z f\big(r,\underline{\Theta}_r^{t,x}\big)\underline{\Gamma}_r^{t,x}\big]\,\mathrm{d}r\bigg\}, \end{aligned} \tag{46}$$

where $\underline{\Theta}_s^{t,x} := (X_s^{t,x}, \underline{Y}_s^{t,x}, \underline{Z}_s^{t,x})$ solves (4) and (44), $J_r^{t,x}$ is as in Theorem 2.1, and $(\underline{\Psi}_s^{t,x}, \underline{\Gamma}_s^{t,x})$ is the unique adapted process satisfying

$$\mathbb{E}\sup_{t\le s\le T}\big|\underline{\Psi}_s^{t,x}\big|^2 + \mathbb{E}\int_0^T \big|\underline{\Gamma}_r^{t,x}\big|^2\,\mathrm{d}r < \infty,$$

and, for all $0\le s\le T$,

$$\begin{aligned} \underline{\Psi}_s^{t,x} = {} & g'\big(X_T^{t,x}\big)\,\partial_x X_T^{t,x}\,\mathbb{I}_{\{\tau^{t,x}>T\}} - h\big(\tau^{t,x}, X_{\tau^{t,x}}^{t,x}\big) J_{\tau^{t,x}}^{t,x}\,\mathbb{I}_{\{\tau^{t,x}\le T\}} \\ & + \int_{s\wedge\tau^{t,x}}^{T\wedge\tau^{t,x}} \big[\partial_x f\big(r,\underline{\Theta}_r^{t,x}\big)\,\partial_x X_r^{t,x} + \partial_y f\big(r,\underline{\Theta}_r^{t,x}\big)\underline{\Psi}_r^{t,x} + \partial_z f\big(r,\underline{\Theta}_r^{t,x}\big)\underline{\Gamma}_r^{t,x}\big]\,\mathrm{d}r \\ & - \int_{s\wedge\tau^{t,x}}^{T\wedge\tau^{t,x}} \underline{\Gamma}_r^{t,x}\,\mathrm{d}W_r. \end{aligned} \tag{47}$$

Proof. Without loss of generality, we can suppose that $h(t,d)$ and $h(t,d')$ are continuously differentiable on $[0,T]$. If not, we approximate them by a sequence of continuously differentiable functions (that converges uniformly on

$[0,T]$) and apply Proposition 1.2 in Pardoux and Zhang [19] in order to satisfy the requirements of Proposition 2.6. Interpolate the functions $h(t,d)$ and $h(t,d')$ by a function h of class $\mathcal{C}^{1,2}([0,T]\times[d,d'])$ and Lipschitz w.r.t. x with a Lipschitz constant which is uniform in time, and set

$$H(r,x) := \int_d^x h(r,\xi)\,\mathrm{d}\xi,$$

and

$$\mathcal{L}_r H(r,x) := \frac{\partial H}{\partial r}(r,x) + b(x)h(r,x) + \frac{1}{2}\sigma^2(x)\frac{\partial h}{\partial x}(r,x).$$

For all $t \le s \le T$ one has

$$\int_s^T h\big(r, X_r^{t,x}\big)\,\mathrm{d}K_r^{t,x} = H\big(T, X_T^{t,x}\big) - H\big(s, X_s^{t,x}\big) - \int_s^T h\big(r, X_r^{t,x}\big)\sigma\big(X_r^{t,x}\big)\,\mathrm{d}W_r - \int_s^T \mathcal{L}_r H\big(r, X_r^{t,x}\big)\,\mathrm{d}r,$$

from which

$$\begin{aligned}\underline{Y}_s^{t,x} &= g\big(X_T^{t,x}\big) + H\big(T, X_T^{t,x}\big) - H\big(s, X_s^{t,x}\big) + \int_s^T f\big(r, \underline{\Theta}_r^{t,x}\big)\,\mathrm{d}r - \int_s^T \mathcal{L}_r H\big(r, X_r^{t,x}\big)\,\mathrm{d}r \\ &\quad - \int_s^T Z_r^{t,x}\,\mathrm{d}W_r - \int_s^T h\big(r, X_r^{t,x}\big)\sigma\big(X_r^{t,x}\big)\,\mathrm{d}W_r.\end{aligned} \tag{48}$$

Notice that all the terms in the right-hand side of (48) are a.e. differentiable w.r.t. x. Moreover, the process

$$\begin{cases}\widehat{Y}_s^{t,x} := \underline{Y}_s^{t,x} + H\big(s, X_s^{t,x}\big), \\ \widehat{Z}_s^{t,x} := \underline{Z}_s^{t,x} + h\big(s, X_s^{t,x}\big)\sigma\big(X_s^{t,x}\big)\end{cases} \tag{49}$$

is the unique solution of a BSDE of the type (26) with the new coefficients

$$\begin{cases}\widehat{g}(x) := g(x) + H(T,x), \\ \widehat{f}(t,x,y,z) := f\big(t,x,y-H(t,x), z - h(t,x)\sigma(x)\big) - \mathcal{L}_r H(t,x).\end{cases} \tag{50}$$

Set $\widehat{\Theta}^{t,x} := (X^{t,x}, \widehat{Y}^{t,x}, \widehat{Z}^{t,x})$. We denote by $(\widehat{\Psi}^{t,x}, \widehat{\Gamma}^{t,x})$ the solution of the following BSDE analogous to (29) rewritten under the form (28):

$$\begin{aligned}\widehat{\Psi}_s^{t,x} &= \widehat{g}'\big(X_T^{t,x}\big)\,\partial_x X_T^{t,x} + \int_{s\wedge\tau^{t,x}}^{T\wedge\tau^{t,x}} \big[\partial_x \widehat{f}\big(r, \widehat{\Theta}_r^{t,x}\big)\,\partial_x X_r^{t,x} + \partial_y \widehat{f}\big(r, \widehat{\Theta}_r^{t,x}\big)\widehat{\Psi}_r^{t,x} \\ &\quad + \partial_z \widehat{f}\big(r, \widehat{\Theta}_r^{t,x}\big)\widehat{\Gamma}_r^{t,x}\big]\,\mathrm{d}r - \int_{s\wedge\tau^{t,x}}^{T\wedge\tau^{t,x}} \widehat{\Gamma}_r^{t,x}\,\mathrm{d}W_r.\end{aligned} \tag{51}$$

Now, as $K_s^{t,x} = 0$ for all $t \le s \le \tau^{t,x}$, for all $t \le s \le T\wedge\tau^{t,x}$ we have

$$\begin{aligned}&h\big(T\wedge\tau^{t,x}, X_{T\wedge\tau^{t,x}}^{t,x}\big)J_{T\wedge\tau^{t,x}}^{t,x} - h\big(s\wedge\tau^{t,x}, X_{s\wedge\tau^{t,x}}^{t,x}\big)J_{s\wedge\tau^{t,x}}^{t,x} \\ &\quad = \int_{s\wedge\tau^{t,x}}^{T\wedge\tau^{t,x}} \partial_x(\mathcal{L}_r H)\big(r, X_r^{t,x}\big)J_r^{t,x}\,\mathrm{d}r \\ &\qquad + \int_{s\wedge\tau^{t,x}}^{T\wedge\tau^{t,x}} \big(\partial_x h\big(r, X_r^{t,x}\big)\sigma\big(X_r^{t,x}\big) + h\big(r, X_r^{t,x}\big)\sigma'\big(X_r^{t,x}\big)\big)J_r^{t,x}\,\mathrm{d}W_r,\end{aligned} \tag{52}$$

where $J_r^{t,x}$ is as in Theorem 2.1 and is used here because the process $\partial_x X^{t,x}$ is discontinuous. Therefore, as in addition $\widehat{g}'(x) = g'(x) + h(T,x)$, the pair of processes

$$\begin{cases} \underline{\Psi}_s^{t,x} := \widehat{\Psi}_s^{t,x} - h\big(s \wedge \tau^{t,x}, X_{s\wedge\tau^{t,x}}^{t,x}\big) J_{s\wedge\tau^{t,x}}^{t,x}, \\ \underline{\Gamma}_s^{t,x} := \widehat{\Gamma}_s^{t,x} - \partial_x(h\sigma)\big(s, X_s^{t,x}\big) J_s^{t,x} \end{cases}$$

solves (47). It is the unique solution satisfying the conditions listed in the statement of the theorem; see Pardoux [17].

We are now in a position to get (46). Notice that Theorem 3.1 implies that $\widehat{\Psi}_t^{t,x} = \partial_x \widehat{Y}_t^{t,x}$, and consequently, $\underline{\Psi}_t^{t,x} = \partial_x \underline{Y}_t^{t,x}$. Moreover, as noticed in Remark 3.2, $(\partial_x X^{t,x}, \underline{\Psi}^{t,x}, \underline{\Gamma}^{t,x})$ is null after $\tau^{t,x}$ if $\tau^{t,x} \le T$. In view of the definition of $\underline{\Psi}^{t,x}$ in (47), we finally obtain

$$\begin{aligned} \partial_x \underline{Y}_t^{t,x} = \mathbb{E}\bigg\{ & g'\big(X_T^{t,x}\big)\,\partial_x X_T^{t,x}\,\mathbb{I}_{\{\tau^{t,x}>T\}} - h\big(\tau^{t,x}, X_{\tau^{t,x}}^{t,x}\big) J_{\tau^{t,x}}^{t,x}\,\mathbb{I}_{\{\tau^{t,x}\le T\}} \\ & + \int_t^{T\wedge\tau^{t,x}} \big[\partial_x f\big(r, \underline{\Theta}_r^{t,x}\big)\,\partial_x X_r^{t,x} + \partial_y f\big(r, \underline{\Theta}_r^{t,x}\big)\underline{\Psi}_r^{t,x} + \partial_z f\big(r, \underline{\Theta}_r^{t,x}\big)\underline{\Gamma}_r^{t,x}\big]\,\mathrm{d}r \bigg\}, \end{aligned}$$

which ends the proof. □

The next theorem explicits the derivative in the sense of the distributions of $u(t,x)$ without derivatives of f and g.

Theorem 3.6. *Let the assumptions of Theorem 3.5 hold true. For all $0 \le t < T$, for almost all x in (d,d'), it holds that*

$$\partial_x u(t,x) = \mathbb{E}\bigg[g\big(X_T^{t,x}\big)N_T^{t,x} - h\big(\tau^{t,x}, X_{\tau^{t,x}}^{t,x}\big) J_{\tau^{t,x}}^{t,x}\,\mathbb{I}_{\tau^{t,x}\le T} + \int_t^T f\big(r, X_r^{t,x}, \underline{Y}_r^{t,x}, \underline{Z}_r^{t,x}\big) N_r^{t,x}\,\mathrm{d}r\bigg].$$

Proof. Apply Theorem 3.4 after having substituted $\widehat{g}$ and $\widehat{f}$, defined as in (50) to g and f, respectively, and $\widehat{Y}^{t,x}$, defined as in (49), to $Y^{t,x}$. Then

$$\begin{aligned} \partial_x \widehat{Y}_t^{t,x} &= \partial_x u(t,x) + h(t,x) \\ &= \mathbb{E}\bigg[\widehat{g}\big(X_T^{t,x}\big)N_T^{t,x} + \int_t^T f\big(r, \underline{\Theta}_r^{t,x}\big)N_r^{t,x}\,\mathrm{d}r - \int_t^T \mathcal{L}_r H\big(r, X_r^{t,x}\big)N_r^{t,x}\,\mathrm{d}r\bigg]. \end{aligned}$$

It then remains to show:

$$\mathbb{E}\big[-h\big(\tau^{t,x}, X_{\tau^{t,x}}^{t,x}\big) J_{\tau^{t,x}}^{t,x}\,\mathbb{I}_{\tau^{t,x}\le T}\big] = \mathbb{E}\bigg[H\big(T, X_T^{t,x}\big)N_T^{t,x} - \int_t^T \mathcal{L}_r H\big(r, X_r^{t,x}\big)N_r^{t,x}\,\mathrm{d}r - h(t,x)\bigg].$$

Now, apply the Lemma 3.7 below; it comes

$$-\mathbb{E}\int_t^T \mathcal{L}_r H\big(r, X_r^{t,x}\big)N_r^{t,x}\,\mathrm{d}r - h(t,x) = -\mathbb{E}\int_t^{T\wedge\tau^{t,x}} \partial_x(\mathcal{L}_r H)\big(r, X_r^{t,x}\big)\,\partial X_r^{t,x}\,\mathrm{d}r - h(t,x).$$

Next, use the equality (52) at time $s = t$:

$$-\mathbb{E}\int_t^T \mathcal{L}_r H\big(r, X_r^{t,x}\big)N_r^{t,x}\,\mathrm{d}r - h(t,x) = -\mathbb{E}\big[h\big(T\wedge\tau^{t,x}, X_{T\wedge\tau^{t,x}}^{t,x}\big) J_{T\wedge\tau^{t,x}}^{t,x}\big].$$

Finally, observe that, $H(T,x) = -g(x) + g(d)$ and thus, using again the Lemma 3.7

$$\mathbb{E}\big(H\big(T, X_T^{t,x}\big)N_T^{t,x}\big) = -\mathbb{E}\big(g\big(X_T^{t,x}\big)N_T^{t,x}\big) = \mathbb{E}\big(h\big(T, X_T^{t,x}\big)J_T^{t,x}\,\mathbb{I}_{T<\tau^{t,x}}\big),$$

which ends the proof. □

Lemma 3.7. *For all differentiable function ϕ with bounded derivative and all $t \le r \le T$,*

$$\mathbb{E}\big[\phi'\big(X_r^{t,x}\big)\,\partial_x X_r^{t,x}\,\mathbb{I}_{r<\tau^{t,x}}\big] = \mathbb{E}\big[\phi\big(X_r^{t,x}\big)N_r^{t,x}\big].$$

Proof. Consider the following event:

$$\mathcal{E}_{\theta,r}^{t,x} := \Big\{\omega \in \Omega\colon\ d < \inf_{\theta\le s\le r} X_s^{t,x} \le \sup_{\theta\le s\le r} X_s^{t,x} < d'\Big\}.$$

Lépingle et al. [10] have shown that the Malliavin derivative of $X_r^{t,x}$ satisfies:

$$\forall t \le \theta < r \quad D_\theta X_r^{t,x} = \sigma\big(X_\theta^{t,x}\big)\frac{J_r^{t,x}}{J_\theta^{t,x}}\mathbb{I}_{\mathcal{E}_{\theta,r}^{t,x}}.$$

Therefore,

$$\frac{1}{\sigma(X_\theta^{t,x})}D_\theta\phi\big(X_r^{t,x}\big)J_\theta^{t,x} = \phi'\big(X_r^{t,x}\big)J_r^{t,x}\mathbb{I}_{\mathcal{E}_{\theta,r}^{t,x}}.$$

We now slightly modify the proof of Elworthy's formula (see, e.g., Nualart [15]) by integrating the previous equality w.r.t. θ between times t and $r \wedge \tau^{t,x}$. Notice that

$$\mathbb{I}_{r\ge\tau^{t,x}}\int_t^{\tau^{t,x}}\mathbb{I}_{\mathcal{E}_{\theta,r}^{t,x}}\,\mathrm{d}\theta = 0$$

and

$$\mathbb{I}_{r<\tau^{t,x}}\int_t^{\tau^{t,x}}\mathbb{I}_{\mathcal{E}_{\theta,r}^{t,x}}\,\mathrm{d}\theta = (r-t)\mathbb{I}_{\tau^{t,x}>r}.$$

It comes:

$$\frac{1}{r-t}\int_t^{r\wedge\tau^{t,x}}\frac{1}{\sigma(X_\theta^{t,x})}D_\theta\phi\big(X_r^{t,x}\big)J_\theta^{t,x}\,\mathrm{d}\theta = \phi'\big(X_r^{t,x}\big)J_r^{t,x}\mathbb{I}_{r<\tau^{t,x}}.$$

It now remains to use the duality relation between the Malliavin derivative and the Skorokhod integral to get

$$\frac{1}{r-t}\mathbb{E}\bigg[\phi\big(X_r^{t,x}\big)\int_t^r \mathbb{I}_{\theta\le r\wedge\tau^{t,x}}\frac{1}{\sigma(X_\theta^{t,x})}J_\theta^{t,x}\,\mathrm{d}W_\theta\bigg] = \mathbb{E}\big[\phi'\big(X_r^{t,x}\big)J_r^{t,x}\mathbb{I}_{\tau^{t,x}>r}\big].$$

We again use that $\partial_x X_\theta^{t,x} = 0$ when $\theta \ge \tau^{t,x}$ and $\partial_x X_\theta^{t,x} = J_\theta^{t,x}$ when $\theta < \tau^{t,x}$, and finally obtain

$$\frac{1}{r-t}\mathbb{E}\bigg[\phi\big(X_r^{t,x}\big)\int_t^r \frac{1}{\sigma(X_\theta^{t,x})}\,\partial_x X_\theta^{t,x}\,\mathrm{d}W_\theta\bigg] = \mathbb{E}\big[\phi'\big(X_r^{t,x}\big)\,\partial_x X_r^{t,x}\,\mathbb{I}_{\tau^{t,x}>r}\big].$$

□

4. Stochastic representations of derivatives of solutions of variational parabolic inequalities

In this section we aim to establish stochastic representations for the derivative $\partial_x v(t,x)$ in the sense of the distribution of the solution of variational inequality (2). We successively examine the case of an homogeneous Neumann boundary condition ($h \equiv 0$), and the case of a nonhomogeneous Neumann boundary condition.

4.1. *The case of homogeneous Neumann boundary conditions*

Consider the reflected BSDE

$$\begin{cases} X_s^{t,x} = x + \int_t^s b\big(X_r^{t,x}\big)\,\mathrm{d}r + \int_t^s \sigma\big(X_r^{t,x}\big)\,\mathrm{d}W_r + K_s^{t,x}, \\ \mathcal{Y}_s^{t,x} = g\big(X_T^{t,x}\big) + \int_s^T f\big(r, X_r^{t,x}, \mathcal{Y}_r^{t,x}, \mathcal{Z}_r^{t,x}\big)\,\mathrm{d}r - \int_s^T \mathcal{Z}_r^{t,x}\,\mathrm{d}W_r + \mathcal{R}_T^{t,x} - \mathcal{R}_s^{t,x}, \\ \mathcal{Y}_s^{t,x} \ge L\big(s, X_s^{t,x}\big) \quad \text{for all } 0 \le t \le s \le T, \\ \big(\mathcal{R}_s^{t,x}, 0 \le t \le s \le T\big) \text{ is a continuous increasing process such that} \\ \quad \int_t^T \big(\mathcal{Y}_s^{t,x} - L\big(s, X_s^{t,x}\big)\big)\,\mathrm{d}\mathcal{R}_s^{t,x} = 0. \end{cases} \tag{53}$$

In all this section, in addition to the assumptions made in Theorem 3.4 we suppose that the function L is in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R};\mathbb{R})$, bounded with bounded derivatives. Adapting a technique due to Cvitanić and Ma [11], Berthelot et al. [3] have shown existence and uniqueness of an adapted solution $(\mathcal{Y}^{t,x}, \mathcal{Z}^{t,x}, \mathcal{R}^{t,x})$, and that the function $v:[0,T]\times[d,d']\to\mathbb{R}$ defined by

$$v(t,x) = \mathcal{Y}_t^{t,x}$$

is the unique continuous viscosity solution of (2). We will need the following estimates.

Proposition 4.1. *There exist* $0<\beta<1$ *and* $C>0$ *such that, for all* x *in* (d,d'),

$$\text{for all } t \le r \le T \qquad \sup_{x\in(d,d')} \mathbb{E}\big(|K|_r^{t,x}\big)^2 \le C(r-t), \tag{54}$$

$$\mathbb{E}\bigg(\sup_{t\le s\le T}\big|\mathcal{Y}_s^{t,x}\big|^2 + \int_t^T \big|\mathcal{Z}_r^{t,x}\big|^2\,\mathrm{d}r + \big(|K|_T^{t,x}\big)^2 + \big|\mathcal{R}_T^{t,x}\big|^2\bigg) \le C \tag{55}$$

and

$$\mathbb{E}\bigg|\int_t^T N_r^{t,x}\,\mathrm{d}|K|_r^{t,x}\bigg| \le \frac{C}{[(x-d)\wedge(d'-x)]^{14/11}}(T-t)^\beta. \tag{56}$$

Proof. We start with proving (54). Consider $\psi(x) = \frac{1}{2(d-d')}((x-d')^2 + (x-d)^2)$, so that $\psi'(x) = \eta(x)$ for $x = d, d'$ and

$$|K|_r^{t,x} = \psi\big(X_r^{t,x}\big) - \psi(x) - \int_t^r \mathcal{A}\psi\big(X_s^{t,x}\big)\,\mathrm{d}s - \int_t^r \psi'\big(X_s^{t,x}\big)\sigma\big(X_s^{t,x}\big)\,\mathrm{d}W_s.$$

Then $\mathbb{E}(|K|_s^{t,x})^2 \le C\mathbb{E}|X_r^{t,x} - x|^2 + C(r-t)$. Moreover, as $(y-x)\eta(y) \le 0$ for $y = d, d'$,

$$\begin{aligned} \mathbb{E}\big(X_r^{t,x} - x\big)^2 &= \mathbb{E}\int_t^r \big\{2\big(X_s^{t,x} - x\big)b\big(X_s^{t,x}\big) + \sigma^2\big(X_s^{t,x}\big)\big\}\,\mathrm{d}s + 2\mathbb{E}\int_t^r \big(X_s^{t,x} - x\big)\eta\big(X_s^{t,x}\big)\,\mathrm{d}|K|_s^{t,x} \\ &\le C(r-t). \end{aligned}$$

We now prove (55). Proceeding as in the proof of Proposition 3.5 in El Karoui et al. [7] there exists $C>0$ such that, for all x in (d,d'),

$$\begin{aligned} &\mathbb{E}\bigg(\sup_{t\le s\le T}\big|\mathcal{Y}_s^{t,x}\big|^2 + \int_t^T \big|\mathcal{Z}_r^{t,x}\big|^2\,\mathrm{d}r + \big|\mathcal{R}_T^{t,x}\big|^2\bigg) \\ &\quad \le C\mathbb{E}g^2\big(X_T^{t,x}\big) + C\int_t^T f^2(r,0,0,0)\,\mathrm{d}r + C\mathbb{E}\sup_{t\le s\le T} L^2\big(s, X_s^{t,x}\big). \end{aligned}$$

To obtain the desired result, it then suffices to apply (54).

We now prove (56). Set

$$M_s^{t,x} := \int_t^s \sigma^{-1}\big(X_r^{t,x}\big)\,\partial_x X_r^{t,x}\,\mathrm{d}W_r,$$

$$A := \int_t^T N_r^{t,x}\,\mathrm{d}|K|_r^{t,x} = \int_t^T \frac{M_r^{t,x}}{(r-t)}\,\mathrm{d}|K|_r^{t,x}.$$

As $M_t^{t,x} = 0$ we have, for all $0 < \alpha < 1$,

$$|A| \le \sup_{t\le\theta_1\le\theta_2\le T} \frac{|M_{\theta_1}^{t,x} - M_{\theta_2}^{t,x}|}{|\theta_1-\theta_2|^{\alpha}} \int_t^T \frac{1}{(r-t)^{(1-\alpha)}}\,\mathrm{d}|K|_r^{t,x}.$$

Here, we have used an Hölder version of the stochastic integral $(M_\theta^{t,x})$ and used a trick from Ma and Zhang [13], p. 1406.

We choose $\alpha := \frac{2}{9}$ and set $\gamma := \frac{14}{3}$. As

$$\mathbb{E}\big|M_{\theta_1}^{t,x} - M_{\theta_2}^{t,x}\big|^{\gamma} \le C|\theta_1-\theta_2|^{\gamma/2}$$

and $\alpha < \frac{\gamma/2-1}{\gamma}$, we may apply the Theorem 2.1 in [20], Chapter 1. We get

$$\mathbb{E}|A| \le \big(M^*\big)^{1/\gamma}\left\{\mathbb{E}\left(\int_t^T \frac{1}{(r-t)^{(1-\alpha)}}\,\mathrm{d}|K|_r^{t,x}\right)^{\gamma/(\gamma-1)}\right\}^{(\gamma-1)/\gamma},$$

where

$$M^* := \mathbb{E}\sup_{t\le\theta_1<\theta_2\le T}\left(\frac{|M_{\theta_1}^{t,x} - M_{\theta_2}^{t,x}|}{(\theta_2-\theta_1)^{\alpha}}\right)^{\gamma} < \infty.$$

Now set

$$B := \mathbb{E}\left(\int_t^T \frac{1}{(r-t)^{(1-\alpha)}}\,\mathrm{d}|K|_r^{t,x}\right)^{\gamma/(\gamma-1)}.$$

From Slominski [21] we know that, for all x in (d,d') and integer $p \ge 1$ there exists $C > 0$ such that, for all $t < s < T$,

$$\mathbb{E}\big(|K|_s^{t,x}\big)^p \le C\frac{(s-t)^p}{(x-d)^p \wedge (d'-x)^p}. \tag{57}$$

Therefore, the Kolmogorov–Centsov criterion implies that, almost surely,

$$\lim_{s\to t}\left(\frac{|K|_s^{t,x}}{(s-t)^{(1-\alpha)}}\right)^{\gamma/(\gamma-1)} = 0.$$

By integration by parts, we thus get

$$\begin{aligned}
B &= \mathbb{E}\left\{\left[\frac{|K|_s^{t,x}}{(s-t)^{(1-\alpha)}}\right]_{s=t}^{s=T} + (1-\alpha)\int_t^T \frac{|K|_s^{t,x}}{(s-t)^{(2-\alpha)}}\,\mathrm{d}s\right\}^{\gamma/(\gamma-1)} \\
&\le C\mathbb{E}\left(\frac{|K|_T^{t,x}}{(T-t)^{(1-\alpha)}}\right)^{\gamma/(\gamma-1)} + C\mathbb{E}\int_t^T \frac{(|K|_s^{t,x})^{\gamma/(\gamma-1)}}{(s-t)^{\gamma(2-\alpha)/(\gamma-1)}}\,\mathrm{d}s \\
&=: B_1 + B_2.
\end{aligned}$$

From (57), it comes: $B_1 \le C(T-t)^{\alpha\gamma/(\gamma-1)}[(x-d)\wedge(d'-x)]^{-\gamma/(\gamma-1)}$. We finally observe that

$$
\begin{aligned}
B_2 &\le C\big[(x-d)\wedge\big(d'-x\big)\big]^{-\gamma/(\gamma-1)} \int_t^T \frac{(s-t)^{\gamma/(\gamma-1)}}{(s-t)^{\gamma(2-\alpha)/(\gamma-1)}}\,\mathrm{d}s \\
&\le C\big[(x-d)\wedge\big(d'-x\big)\big]^{-\gamma/(\gamma-1)}(T-t)^{1/99}. \qquad \square
\end{aligned}
$$

To get a probabilistic representation of the derivative $\partial_x v(t,x)$ in the sense of the distributions we need a precise information on the Stieljes measure $\mathrm{d}\mathcal{R}_r^{t,x}$. For the Stieljes measure $\mathrm{d}\mathcal{R}_r^{t,x}$ we have:

Lemma 4.2. *Let $\Theta^{t,x} := (X^{t,x}, \mathcal{Y}^{t,x}, \mathcal{Z}^{t,x})$. For all $t \le r \le T$,*

$$
\begin{aligned}
\mathrm{d}\mathcal{R}_r^{t,x} \le{}& \mathbb{I}_{\mathcal{Y}_r^{t,x}=L(r,X_r^{t,x})}\bigg[\frac{\partial L}{\partial r}\big(r,X_r^{t,x}\big)+\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)b\big(X_r^{t,x}\big) \\
&+\frac{1}{2}\frac{\partial^2 L}{\partial x^2}\big(r,X_r^{t,x}\big)\sigma^2\big(X_r^{t,x}\big)+f\big(r,\Theta_r^{t,x}\big)\bigg]^- \mathrm{d}r \\
&+\mathbb{I}_{\mathcal{Y}_r^{t,x}=L(r,X_r^{t,x})}\bigg[\bigg(\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)\bigg)\eta\big(X_r^{t,x}\big)\bigg]^- \mathrm{d}|K|_r^{t,x}.
\end{aligned}
\tag{58}
$$

Proof. We adapt a trick from El Karoui et al. [7]. Let $\Lambda_s^{t,x}$ be the local time at 0 of the semimartingale $(\mathcal{Y}_s^{t,x} - L(s, X_s^{t,x}))$. Itô–Tanaka's formula leads to

$$
\begin{aligned}
\mathrm{d}\big(\mathcal{Y}_r^{t,x}-L\big(r,X_r^{t,x}\big)\big)^+ ={}& -\mathbb{I}_{\mathcal{Y}_r^{t,x}>L(r,X_r^{t,x})}\bigg[\frac{\partial L}{\partial r}\big(r,X_r^{t,x}\big)+\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)b\big(X_r^{t,x}\big) \\
&+\frac{1}{2}\frac{\partial^2 L}{\partial x^2}\big(r,X_r^{t,x}\big)\sigma^2\big(X_r^{t,x}\big)+f\big(r,\Theta_r^{t,x}\big)\bigg]\mathrm{d}r \\
&-\mathbb{I}_{\mathcal{Y}_r^{t,x}>L(r,X_r^{t,x})}\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)\eta\big(X_r^{t,x}\big)\,\mathrm{d}|K|_r^{t,x} \\
&+\mathbb{I}_{\mathcal{Y}_r^{t,x}>L(r,X_r^{t,x})}\bigg(\mathcal{Z}_r^{t,x}-\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)\sigma\big(X_r^{t,x}\big)\bigg)\mathrm{d}W_r+\frac{1}{2}\,\mathrm{d}\Lambda_r^{t,x}.
\end{aligned}
$$

As

$$
\big(\mathcal{Y}_r^{t,x}-L\big(r,X_r^{t,x}\big)\big)^+ = \mathcal{Y}_r^{t,x}-L\big(r,X_r^{t,x}\big),
$$

Itô's formula applied to $L(r, X_r^{t,x})$ leads to

$$
\begin{aligned}
\mathrm{d}\mathcal{R}_r^{t,x}+\frac{1}{2}\,\mathrm{d}\Lambda_r^{t,x} ={}& -\mathbb{I}_{\mathcal{Y}_r^{t,x}=L(r,X_r^{t,x})}\bigg[\frac{\partial L}{\partial r}\big(r,X_r^{t,x}\big)+\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)b\big(X_r^{t,x}\big) \\
&+\frac{1}{2}\frac{\partial^2 L}{\partial x^2}\big(r,X_r^{t,x}\big)\sigma^2\big(X_r^{t,x}\big)+f\big(r,\Theta_r^{t,x}\big)\bigg]\mathrm{d}r \\
&-\mathbb{I}_{\mathcal{Y}_r^{t,x}=L(r,X_r^{t,x})}\bigg(\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)\bigg)\eta\big(X_r^{t,x}\big)\,\mathrm{d}|K|_r^{t,x} \\
&+\mathbb{I}_{\mathcal{Y}_r^{t,x}=L(r,X_r^{t,x})}\bigg(\mathcal{Z}_r^{t,x}-\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)\sigma\big(X_r^{t,x}\big)\bigg)\mathrm{d}W_r
\end{aligned}
$$

and

$$
\mathcal{Z}_r^{t,x}-\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)\sigma\big(X_r^{t,x}\big)=\mathbb{I}_{\mathcal{Y}_r^{t,x}>L(r,X_r^{t,x})}\bigg(\mathcal{Z}_r^{t,x}-\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)\sigma\big(X_r^{t,x}\big)\bigg),
$$

from which we deduce

$$\mathbb{I}_{\mathcal{Y}_r^{t,x}=L(r,X_r^{t,x})}\left(\mathcal{Z}_r^{t,x}-\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)\sigma\big(X_r^{t,x}\big)\right)=0$$

and

$$\begin{aligned}\mathrm{d}\mathcal{R}_r^{t,x}+\frac{1}{2}\,\mathrm{d}\Lambda_r^{t,x}=&-\mathbb{I}_{\mathcal{Y}_r^{t,x}=L(r,X_r^{t,x})}\bigg[\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)b\big(X_r^{t,x}\big)+\frac{1}{2}\frac{\partial^2 L}{\partial x^2}\big(r,X_r^{t,x}\big)\sigma^2\big(X_r^{t,x}\big)\\&+\frac{\partial L}{\partial r}\big(r,X_r^{t,x}\big)+f\big(r,\Theta_r^{t,x}\big)\bigg]\,\mathrm{d}r\\&-\mathbb{I}_{\mathcal{Y}_r^{t,x}=L(r,X_r^{t,x})}\frac{\partial L}{\partial x}\big(r,X_r^{t,x}\big)\eta\big(X_r^{t,x}\big)\,\mathrm{d}|K|_r^{t,x}.\end{aligned}$$

As local times are increasing we deduce (58). □

We now are in a position to prove the main result of this section.

Theorem 4.3. *Suppose that the function L is in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R};\mathbb{R})$, bounded with bounded derivatives. Under the assumptions of Theorem 3.4, for all $0\le t<T$ the function $v(t,x)$ is in $H^1(d,d')$ and, for almost all x in (d,d'),*

$$\partial_x v(t,x)=\mathbb{E}\bigg(g\big(X_T^{t,x}\big)N_T^{t,x}+\int_t^T f\big(r,X_r^{t,x},\mathcal{Y}_r^{t,x},\mathcal{Z}_r^{t,x}\big)N_r^{t,x}\,\mathrm{d}r+\int_t^T N_r^{t,x}\,\mathrm{d}\mathcal{R}_r^{t,x}\bigg),\tag{59}$$

where $N^{t,x}$ is as in (37).

Remark 4.4. *$\mathbb{E}\int_t^T N_r^{t,x}\,\mathrm{d}\mathcal{R}_r^{t,x}$ is well defined in view of (58) and (56).*

Proof of Theorem 4.3. We follow the same guidelines as the proof of Theorem 3.4. In this proof all the constants C are uniform w.r.t. x in (d,d').

Consider the system

$$\begin{cases}\mathcal{Y}_s^{t,x,n}=g\big(X_T^{t,x,n}\big)+\int_s^T f\big(r,X_r^{t,x,n},\mathcal{Y}_r^{t,x,n},\mathcal{Z}_r^{t,x,n}\big)\,\mathrm{d}r\\ \qquad\qquad -\int_s^T \mathcal{Z}_r^{t,x,n}\,\mathrm{d}W_r+\mathcal{R}_T^{t,x,n}-\mathcal{R}_s^{t,x,n},\quad \forall t\le s\le T,\ \mathcal{Y}_s^{t,x,n}\ge L\big(s,X_s^{t,x,n}\big),\\ \big\{\mathcal{R}_s^{t,x,n},t\le s\le T\big\}\text{ is an increasing continuous process such that}\\ \quad\int_t^T\big(\mathcal{Y}_s^{t,x,n}-L\big(s,X_s^{t,x,n}\big)\big)\,\mathrm{d}\mathcal{R}_s^{t,x,n}=0,\end{cases}$$

where $X^{t,x,n}$ is the solution to (10). Set $v^n(t,x):=\mathcal{Y}_t^{t,x,n}$. Ma and Zhang [13] have shown that, for almost all x in (d,d'),

$$\begin{aligned}\partial_x v^n(t,x)&=\mathcal{Z}_t^{t,x,n}\sigma^{-1}(x)\\&=\mathbb{E}\bigg(g\big(X_T^{t,x,n}\big)N_T^{t,x,n}+\int_t^T f\big(r,X_r^{t,x,n},\mathcal{Y}_r^{t,x,n},\mathcal{Z}_r^{t,x,n}\big)N_r^{t,x,n}\,\mathrm{d}r+\int_t^T N_r^{t,x,n}\,\mathrm{d}\mathcal{R}_r^{t,x,n}\bigg).\end{aligned}$$

Notice that, from El Karoui et al. [7]), we have: almost surely,

$$\begin{aligned}\mathrm{d}\mathcal{R}_r^{t,x,n}\le\mathbb{I}_{\mathcal{Y}_r^{t,x,n}=L(r,X_r^{t,x,n})}&\bigg[\frac{\partial L}{\partial r}\big(r,X_r^{t,x,n}\big)+\frac{\partial L}{\partial x}\big(r,X_r^{t,x,n}\big)b_n\big(X_r^{t,x,n}\big)\\&+\frac{1}{2}\frac{\partial^2 L}{\partial x^2}\big(r,X_r^{t,x,n}\big)\sigma^2\big(X_r^{t,x,n}\big)+f\big(r,\Theta_r^{t,x,n}\big)\bigg]^-\,\mathrm{d}r.\end{aligned}\tag{60}$$

Here, $\Theta^{t,x,n} := (X^{t,x,n}, \mathcal{Y}^{t,x,n}, \mathcal{Z}^{t,x,n})$. We aim to apply Proposition 2.6 (in its deterministic version) and to prove that the right-hand side of the preceding equality tends to the right-hand side of (59).

We first show that

$$\lim_{n\to\infty}\int_d^{d'} \left|v^n(t,x)-v(t,x)\right|^2 \mathrm{d}x = 0. \tag{61}$$

The following calculation is classical:

$$\begin{aligned}
&\mathbb{E}\left|\mathcal{Y}_s^{t,x,n}-\mathcal{Y}_s^{t,x}\right|^2+\mathbb{E}\int_s^T\left|\mathcal{Z}_r^{t,x,n}-\mathcal{Z}_r^{t,x}\right|^2\mathrm{d}r\\
&\quad\le \mathbb{E}\left|g\left(X_T^{t,x,n}\right)-g(X_T)\right|^2\\
&\qquad+2\mathbb{E}\int_s^T\left(\mathcal{Y}_r^{t,x,n}-\mathcal{Y}_r^{t,x}\right)\left(f\left(r,X_r^{t,x,n},\mathcal{Y}_r^{t,x,n},\mathcal{Z}_r^{t,x,n}\right)-f\left(r,X_r^{t,x},\mathcal{Y}_r^{t,x},\mathcal{Z}_r^{t,x}\right)\right)\mathrm{d}r\\
&\qquad+2\mathbb{E}\int_s^T\left(\mathcal{Y}_r^{t,x,n}-\mathcal{Y}_r^{t,x}\right)\left(\mathrm{d}\mathcal{R}_r^{t,x,n}-\mathrm{d}\mathcal{R}_r^{t,x}\right)\\
&\quad\le C\mathbb{E}\left|X_T^{t,x,n}-X_T\right|^2\\
&\qquad+C\mathbb{E}\int_s^T\left|\mathcal{Y}_r^{t,x,n}-\mathcal{Y}_r^{t,x}\right|\left(\left|X_r^{t,x,n}-X_r^{t,x}\right|+\left|\mathcal{Y}_r^{t,x,n}-\mathcal{Y}_r^{t,x}\right|+\left|\mathcal{Z}_r^{t,x,n}-\mathcal{Z}_r^{t,x}\right|\right)\mathrm{d}r\\
&\qquad+2\mathbb{E}\int_s^T\left(L\left(r,X_r^{t,x,n}\right)-L\left(r,X_r^{t,x}\right)\right)\left(\mathrm{d}\mathcal{R}_r^{t,x,n}-\mathrm{d}\mathcal{R}_r^{t,x}\right)\\
&\quad\le C\mathbb{E}\left|X_T^{t,x,n}-X_T\right|^2+C\left(1+\frac{1}{\varepsilon}\right)\mathbb{E}\int_s^T\left|\mathcal{Y}_r^{t,x,n}-\mathcal{Y}_r^{t,x}\right|^2\mathrm{d}r\\
&\qquad+C\mathbb{E}\int_s^T\left|X_r^{t,x,n}-X_r^{t,x}\right|^2\mathrm{d}r+C\varepsilon\mathbb{E}\int_s^T\left|\mathcal{Z}_r^{t,x,n}-\mathcal{Z}_r^{t,x}\right|^2\mathrm{d}r\\
&\qquad+2\mathbb{E}\int_s^T\left(L\left(r,X_r^{t,x,n}\right)-L\left(r,X_r^{t,x}\right)\right)\left(\mathrm{d}\mathcal{R}_r^{t,x,n}-\mathrm{d}\mathcal{R}_r^{t,x}\right).
\end{aligned}$$

Therefore, choosing ε small enough, say, $\varepsilon=\frac{1}{2C}$, we get

$$\begin{aligned}
&\mathbb{E}\left|\mathcal{Y}_s^{t,x,n}-\mathcal{Y}_s^{t,x}\right|^2+\mathbb{E}\int_s^T\left|\mathcal{Z}_r^{t,x,n}-\mathcal{Z}_r^{t,x}\right|^2\mathrm{d}r\\
&\quad\le C\mathbb{E}\left|X_T^{t,x,n}-X_T\right|^2+C\mathbb{E}\int_s^T\left|X_r^{t,x,n}-X_r^{t,x}\right|^2\mathrm{d}r\\
&\qquad+C\mathbb{E}\int_s^T\left|\mathcal{Y}_r^{t,x,n}-\mathcal{Y}_r^{t,x}\right|^2\mathrm{d}r\\
&\qquad+2\mathbb{E}\int_s^T\left(L\left(r,X_r^{t,x,n}\right)-L\left(r,X_r^{t,x}\right)\right)\left(\mathrm{d}\mathcal{R}_r^{t,x,n}-\mathrm{d}\mathcal{R}_r^{t,x}\right),
\end{aligned}$$

from which

$$\begin{aligned}
&\sup_{t\le s\le T}\mathbb{E}\left|\mathcal{Y}_s^{t,x,n}-\mathcal{Y}_s^{t,x}\right|^2+\mathbb{E}\int_t^T\left|\mathcal{Z}_r^{t,x,n}-\mathcal{Z}_r^{t,x}\right|^2\mathrm{d}r\\
&\quad\le C\mathbb{E}\sup_{t\le s\le T}\left|X_s^{t,x,n}-X_s^{t,x}\right|^2+C\mathbb{E}\sup_{t\le s\le T}\int_s^T\left|L\left(r,X_r^{t,x,n}\right)-L\left(r,X_r^{t,x}\right)\right|\left(\mathrm{d}\mathcal{R}_r^{t,x,n}+\mathrm{d}\mathcal{R}_r^{t,x}\right).
\end{aligned}$$

We now observe that the proof of the inequality (3.4) in Menaldi [14] leads to: for all $p \geq 1$ there exists $C > 0$ such that

$$\sup_{x\in(d,d')} \mathbb{E}\int_t^T \left|\beta_n\left(X_r^{t,x,n}\right)\right|^p \mathrm{d}r \leq C, \tag{62}$$

where the function β_n is defined as in Proposition 2.5. It then remains to use (11), (60) and (58) to conclude that

$$\lim_{n\to\infty} \sup_{x\in(d,d')} \left(\sup_{t\leq s\leq T} \mathbb{E}\left|\mathcal{Y}_s^{t,x,n} - \mathcal{Y}_s^{t,x}\right|^2 + \mathbb{E}\int_t^T \left|\mathcal{Z}_r^{t,x,n} - \mathcal{Z}_r^{t,x}\right|^2 \mathrm{d}r \right) = 0, \tag{63}$$

from which (61) follows.

We now aim to prove that

$$\lim_{n\to\infty} \int_d^{d'} \left|\partial_x v^n(t,x) - \zeta_t^{t,x}\right|^2 \mathrm{d}x = 0, \tag{64}$$

where

$$\zeta_t^{t,x} := \mathbb{E}\left(g\left(X_T^{t,x}\right) N_T^{t,x} + \int_t^T f\left(r, X_r^{t,x}, \mathcal{Y}_r^{t,x}, \mathcal{Z}_r^{t,x}\right) N_r^{t,x}\,\mathrm{d}r + \int_t^T N_r^{t,x}\,\mathrm{d}\mathcal{R}_r^{t,x} \right).$$

We have

$$\begin{aligned}
\left|\partial_x v^n(t,x) - \zeta_t^{t,x}\right| &\leq \left|\mathbb{E}\left(g\left(X_T^{t,x,n}\right)N_T^{t,x,n} - g\left(X_T^{t,x}\right)N_T^{t,x}\right)\right| \\
&\quad + \left|\mathbb{E}\int_t^T \left(f\left(r, X_r^{t,x,n}, \mathcal{Y}_r^{t,x,n}, \mathcal{Z}_r^{t,x,n}\right)N_r^{t,x,n} - f\left(r, X_r^{t,x}, \mathcal{Y}_r^{t,x}, \mathcal{Z}_r^{t,x}\right)N_r^{t,x}\right)\mathrm{d}r\right| \\
&\quad + \left|\mathbb{E}\int_t^T N_r^{t,x,n}\,\mathrm{d}\mathcal{R}_r^{t,x,n} - \mathbb{E}\int_t^T N_r^{t,x}\,\mathrm{d}\mathcal{R}_r^{t,x}\right| \\
&=: I_1^{t,x,n} + I_2^{t,x,n} + I_3^{t,x,n}.
\end{aligned}$$

Combining the inequality (63) and the arguments used at the end at the proof of Theorem 3.4 we obtain

$$\lim_{n\to\infty} \int_d^{d'} \left(\left(I_1^{t,x,n}\right)^2 + \left(I_2^{t,x,n}\right)^2\right)\mathrm{d}x = 0.$$

We now examine $I_3^{t,x,n}$:

$$\begin{aligned}
I_3 &\leq \mathbb{E}\int_t^T \left|N_r^{t,x,n} - N_r^{t,x}\right| \mathrm{d}\mathcal{R}_r^{t,x,n} + \left|\mathbb{E}\int_t^T N_r^{t,x}\left(\mathrm{d}\mathcal{R}_r^{t,x,n} - \mathrm{d}\mathcal{R}_r^{t,x}\right)\right| \\
&=: I_{31}^{t,x,n} + I_{32}^{t,x,n}.
\end{aligned}$$

In order to estimate $I_{31}^{t,x,n}$, we again use (39) and (11), and get

$$\begin{aligned}
\left|I_{31}^{t,x,n}\right| &\leq \mathbb{E}\int_t^T \left|N_r^{t,x,n} - N_r^{t,x}\right|\left(C + \left|b_n\left(X_r^{t,x,n}\right)\right|\right)\mathrm{d}r \\
&\leq C\int_t^T \sqrt{\mathbb{E}\left|N_r^{t,x,n} - N_r^{t,x}\right|^2}\left(1 + \sqrt{\mathbb{E}\left|b_n\left(X_r^{t,x,n}\right)\right|^2}\right)\mathrm{d}r \\
&\leq C\int_t^T \frac{1}{\sqrt{r-t}}\left(\left\{\mathbb{E}\sup_{t\leq r\leq T}\left|X_r^{t,x,n} - X_r^{t,x}\right|^4\right\}^{1/4} + \left\{\mathbb{E}\sup_{t\leq r\leq T}\left|X_r^{t,x,n} - X_r^{t,x}\right|^2\right\}^{1/2}\right) \\
&\quad \times \left(1 + \sqrt{\mathbb{E}\left|b_n\left(X_r^{t,x,n}\right)\right|^2}\right)\mathrm{d}r.
\end{aligned}$$

In view of (62) we deduce:

$$\lim_{n\to\infty}\int_d^{d'} \left|I_{31}^{t,x,n}\right|^2 \mathrm{d}x = 0.$$

We now turn to $I_{32}^{t,x,n}$. Notice that (57)–(58) imply $\mathbb{E}((N_{t+\varepsilon}^{t,x})^{2p}(\mathcal{R}_{t+\varepsilon}^{t,x,n} - \mathcal{R}_{t+\varepsilon}^{t,x})^{2p}) = \mathcal{O}(\varepsilon^p),\ p \geq 1$. Thus the Kolmogorov–Centsov criterion implies that $\lim_{\varepsilon\to 0}(\mathcal{R}_{t+\varepsilon}^{t,x,n} - \mathcal{R}_{t+\varepsilon}^{t,x})N_{t+\varepsilon}^{t,x} = 0$ a.s. Therefore we may integrate by parts to get

$$\left|\mathbb{E}\int_t^T N_r^{t,x}\,\mathrm{d}\big(\mathcal{R}_r^{t,x,n} - \mathcal{R}_r^{t,x}\big)\right| \leq \mathbb{E}\big|\big(\mathcal{R}_T^{t,x,n} - \mathcal{R}_T^{t,x}\big)N_T^{t,x}\big| + \int_t^T \mathbb{E}\left|N_r^{t,x}\frac{(\mathcal{R}_r^{t,x,n} - \mathcal{R}_r^{t,x})}{r-t}\right|\mathrm{d}r.$$

Now, a straightforward calculation leads to

$$\begin{aligned}\mathbb{E}\big(\mathcal{R}_s^{t,x,n} - \mathcal{R}_s^{t,x}\big)^2 \leq C \sup_{t\leq r\leq T}\mathbb{E}\big(\mathcal{Y}_r^{t,x,n} - \mathcal{Y}_r^{t,x}\big)^2 + C\int_t^s \mathbb{E}\big(X_r^{t,x,n} - X_r^{t,x}\big)^2\,\mathrm{d}r \\ + C\int_t^s \mathbb{E}\big(\mathcal{Z}_r^{t,x,n} - \mathcal{Z}_r^{t,x}\big)^2\,\mathrm{d}r,\end{aligned}$$

which, in view of (63) and (11), implies that

$$\int_d^{d'} \mathbb{E}\big|\big(\mathcal{R}_T^{t,x,n} - \mathcal{R}_T^{t,x}\big)N_T^{t,x}\big|^2\,\mathrm{d}x$$

tends to 0.

We finally consider

$$\lambda^{t,n} := \int_d^{d'}\left(\int_t^T \mathbb{E}\left|N_r^{t,x}\frac{(\mathcal{R}_r^{t,x,n} - \mathcal{R}_r^{t,x})}{r-t}\right|\mathrm{d}r\right)^2\mathrm{d}x.$$

Notice that (60) implies that $\sup_{x\in(d,d')}\mathbb{E}(\mathcal{R}_s^{t,x,n})^2 \leq C(s-t)^2$. Moreover,

$$\int_t^T \frac{1}{(r-t)}\mathbb{E}\big|N_r^{t,x}|K|_r^{t,x}\big|\,\mathrm{d}r \leq \int_t^T \frac{C}{(r-t)^{3/2}}\big\{\mathbb{E}\big(|K|_r^{t,x}\big)^2\big\}^{7/16}\big\{\mathbb{E}\big(|K|_r^{t,x}\big)^2\big\}^{1/16}\,\mathrm{d}r$$

with a constant C uniform in x. Using (54) and (57), we get

$$\big\{\mathbb{E}\big(|K|_r^{t,x}\big)^2\big\}^{7/16}\big\{\mathbb{E}\big(|K|_r^{t,x}\big)^2\big\}^{1/16} \leq C\frac{(r-t)^{9/16}}{[(x-d)\wedge(d'-x)]^{1/8}}$$

and by (58), $\mathbb{E}|N_r^{t,x}\mathcal{R}_r^{t,x}| \leq C((r-t)^{1/2} + (r-t)^{1/16}[(x-d)\wedge(d'-x)]^{-1/8}$. Therefore, the Lebesgue Dominated Convergence theorem allow us to deduce that $\lambda^{t,n}$ tends to 0. That ends the proof. □

4.2. *The case of nonhomogeneous Neumann boundary conditions*

Consider the system

$$\begin{cases}\underline{\mathcal{Y}}_s^{t,x} = g\big(X_T^{t,x}\big) + \int_s^T f\big(r, X_r^{t,x}, \underline{\mathcal{Y}}_r^{t,x}, \underline{\mathcal{Z}}_r^{t,x}\big)\,\mathrm{d}r + \int_s^T h\big(r, X_r^{t,x}\big)\,\mathrm{d}K_r^{t,x} \\ \qquad + \underline{\mathcal{R}}_T^{t,x} - \underline{\mathcal{R}}_s^{t,x} - \int_s^T \underline{\mathcal{Z}}_r^{t,x}\,\mathrm{d}W_r, \\ \underline{\mathcal{Y}}_s^{t,x} \geq L\big(s, X_s^{t,x}\big) \quad \text{for all } 0\leq t\leq s\leq T, \\ \big(\underline{\mathcal{R}}_s^{t,x}, 0\leq t\leq s\leq T\big) \text{ is a continuous increasing process such that} \\ \quad \int_t^T \big(\underline{\mathcal{Y}}_r^{t,x} - L\big(r, X_r^{t,x}\big)\big)\,\mathrm{d}\underline{\mathcal{R}}_r^{t,x} = 0.\end{cases} \tag{65}$$

Berthelot et al. [3] have shown that the function $v(t,x) := \underline{\mathcal{Y}}_t^{t,x}$ is the unique (in an appropriate space of functions) viscosity solution of the following parabolic system with a nonhomogeneous Neumann boundary condition:

$$\begin{cases} \min\big\{v(t,x) - L(t,x); -\frac{\partial v}{\partial t}(t,x) - \mathcal{A}v(t,x) \\ \quad - f\big(t,x,v(t,x),\partial_x v(t,x)\sigma(x)\big)\big\} = 0, & (t,x) \in [0,T) \times \big(d,d'\big), \\ v(T,x) = g(x), & x \in \big[d,d'\big], \\ \partial_x v(t,x) + h(t,x) = 0, & (t,x) \in [0,T) \times \big\{d,d'\big\}. \end{cases} \tag{66}$$

Proceeding as in Section 3.3, we readily deduce from Theorem 4.3 the following stochastic representation of $\partial_x v(t,x)$:

Theorem 4.5. *Suppose that the function L is in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R};\mathbb{R})$, bounded with bounded derivatives. Under the assumptions of Theorem 3.6, it holds that, for all t in $[0,T]$ and almost all x in (d,d'),*

$$\partial_x v(t,x) = \mathbb{E}\bigg[N_T^{t,x} g\big(X_T^{t,x}\big) - h\big(\tau^{t,x}, X_{\tau^{t,x}}^{t,x}\big) J_{\tau^{t,x}}^{t,x} \mathbb{I}_{\{\tau^{t,x}\le T\}} \\ + \int_t^T f\big(r, X_r^{t,x}, \underline{\mathcal{Y}}_r^{t,x}, \underline{\mathcal{Z}}_r^{t,x}\big) N_r^{t,x}\,\mathrm{d}r + \int_t^T N_r^{t,x}\,\mathrm{d}\underline{\mathcal{R}}_r^{t,x}\bigg].$$

5. Conclusion

Coming back to our original motivation described in the Introduction, we deduce from Theorem 4.5 a tractable estimate of the error induced by the artificial Neumann boundary condition $h(t,x)$. In this section, we suppose that $\partial_x V(t,d)$ and $\partial_x V(t,d')$ are well defined for all times $t \in [0,T]$. For example, if in addition of assumptions of Theorem 4.5, we suppose that b and σ are differentiable with bounded derivatives, Ma and Zhang [13], Theorem 5.1, have shown that $\partial_x V(\cdot,\cdot)$ is a bounded continuous function on $[0,T]\times\mathbb{R}$.

The following quantity represents the order of magnitude of the misspecification at the boundary $\{d,d'\}$:

$$\epsilon(h) := \sup_{t\le r\le T}\big(\big|V(r,d) - v(r,d)\big| + \big|V\big(r,d'\big) - v\big(r,d'\big)\big|\big) \\ + \sup_{t\le r\le T}\big(\big|\partial_x V(r,d) + h(r,d)\big| + \big|\partial_x V\big(r,d'\big) + h\big(r,d'\big)\big|\big).$$

We are in a position to prove the following estimate for the error induced by the artificial Neumann boundary condition $h(t,x)$.

Theorem 5.1. *Suppose that $\partial_x V(r,d)$ and $\partial_x V(r,d')$ are well defined for all times $r \in [t,T]$. Under the hypotheses of Theorem 4.5, there exists C independent of h such that, for all $\rho < \frac{1}{2}$,*

$$\int_d^{d'} \big|\partial_x V(t,x) - \partial_x v(t,x)\big|^2\,\mathrm{d}x \le C\epsilon(h)^\rho \wedge \epsilon(h).$$

Proof. The various constants C below are uniform w.r.t. $x \in [d,d']$ and $h(t,x)$.

As shown in [3], the viscosity solution of (1) is $V(t,x) = \check{\mathcal{Y}}_t^{t,x}$, where

$$\begin{cases} \check{\mathcal{Y}}_s^{t,x} = g\big(X_T^{t,x}\big) + \int_s^T f\big(r, X_r^{t,x}, \check{\mathcal{Y}}_r^{t,x}, \check{\mathcal{Z}}_r^{t,x}\big)\,\mathrm{d}r - \int_s^T \partial_x V\big(r, X_r^{t,x}\big)\,\mathrm{d}K_r^{t,x} \\ \qquad + \check{\mathcal{R}}_T^{t,x} - \check{\mathcal{R}}_s^{t,x} - \int_s^T \check{\mathcal{Z}}_r^{t,x}\,\mathrm{d}W_r, \\ \check{\mathcal{Y}}_s^{t,x} \ge L\big(s, X_s^{t,x}\big) \quad \text{for all } 0 \le t \le s \le T, \\ \big(\check{\mathcal{R}}_s^{t,x}, 0 \le t \le s \le T\big) \text{ is a continuous increasing process such that} \\ \quad \int_t^T \big(\check{\mathcal{Y}}_r^{t,x} - L\big(r, X_r^{t,x}\big)\big)\,\mathrm{d}\check{\mathcal{R}}_r^{t,x} = 0. \end{cases} \tag{67}$$

From Theorem 4.5 we get that

$$\partial_x V(t,x) - \partial_x v(t,x) = \mathbb{E}\bigg[\big(\partial_x V\big(\tau^{t,x}, X^{t,x}_{\tau^{t,x}}\big) + h\big(\tau^{t,x}, X^{t,x}_{\tau^{t,x}}\big)\big) J^{t,x}_{\tau^{t,x}} \mathbb{I}_{\{\tau^{t,x} \leq T\}} + \int_t^T N_r^{t,x}\big(f\big(r, X_r^{t,x}, \check{\mathcal{Y}}_r^{t,x}, \check{\mathcal{Z}}_r^{t,x}\big) - f\big(r, X_r^{t,x}, \underline{\mathcal{Y}}_r^{t,x}, \underline{\mathcal{Z}}_r^{t,x}\big)\big)\,\mathrm{d}r + \int_t^T N_r^{t,x}\big(\mathrm{d}\check{\mathcal{R}}_r^{t,x} - \mathrm{d}\underline{\mathcal{R}}_r^{t,x}\big)\bigg].$$

Since $\check{\mathcal{Y}}_r^{t,x}$ and $\underline{\mathcal{Y}}_r^{t,x}$ are larger than $L(r, X_r^{t,x})$, we have

$$\int_t^T \big(\check{\mathcal{Y}}_r^{t,x} - \underline{\mathcal{Y}}_r^{t,x}\big)\,\mathrm{d}\big(\check{\mathcal{R}}_r^{t,x} - \underline{\mathcal{R}}_r^{t,x}\big) \leq 0,$$

from which, by standard computations,

$$\begin{aligned}
&\sup_{t\leq s\leq T} \mathbb{E}\big|\check{\mathcal{Y}}_s^{t,x} - \underline{\mathcal{Y}}_s^{t,x}\big|^2 + \int_t^T \mathbb{E}\big|\check{\mathcal{Z}}_r^{t,x} - \underline{\mathcal{Z}}_r^{t,x}\big|^2\,\mathrm{d}r \\
&\quad\leq C\mathbb{E}\bigg[\int_t^T \big|V\big(r, X_r^{t,x}\big) - v\big(r, X_r^{t,x}\big)\big|\big|\partial_x V\big(r, X_r^{t,x}\big) + h\big(r, X_r^{t,x}\big)\big|\,\mathrm{d}|K|_r^{t,x}\bigg] \\
&\quad\leq C\mathbb{E}|K|_T^{t,x}\epsilon(h)^2.
\end{aligned}\tag{68}$$

Therefore,

$$\begin{aligned}
\sup_{t\leq s\leq T} \mathbb{E}\big(\check{\mathcal{R}}_s^{t,x} - \underline{\mathcal{R}}_s^{t,x}\big)^2 &\leq C \sup_{t\leq s\leq T} \mathbb{E}\big(\check{\mathcal{Y}}_s^{t,x} - \underline{\mathcal{Y}}_s^{t,x}\big)^2 + C\int_t^T \mathbb{E}\big|\check{\mathcal{Z}}_r^{t,x} - \underline{\mathcal{Z}}_r^{t,x}\big|^2\,\mathrm{d}r \\
&\quad + C\mathbb{E}\bigg(\int_t^T |\partial_x V + h|\big(r, X_r^{t,x}\big)\,\mathrm{d}|K|_r^{t,x}\bigg)^2 \\
&\leq C\mathbb{E}\big(|K|_T^{t,x}\big)^2\epsilon(h)^2.
\end{aligned}\tag{69}$$

Now, in view of (68),

$$\begin{aligned}
&\mathbb{E}\bigg|\int_t^T N_r^{t,x}\big(f\big(r, X_r^{t,x}, \check{\mathcal{Y}}_r^{t,x}, \check{\mathcal{Z}}_r^{t,x}\big) - f\big(r, X_r^{t,x}, \underline{\mathcal{Y}}_r^{t,x}, \underline{\mathcal{Z}}_r^{t,x}\big)\big)\,\mathrm{d}r\bigg| \\
&\quad\leq \int_t^T \frac{C}{\sqrt{r-t}}\big\{\big(\mathbb{E}\big(\check{\mathcal{Y}}_r^{t,x} - \underline{\mathcal{Y}}_r^{t,x}\big)^2\big)^{1/2} + \big(\mathbb{E}\big(\check{\mathcal{Z}}_r^{t,x} - \underline{\mathcal{Z}}_r^{t,x}\big)^2\big)^{1/2}\big\}\,\mathrm{d}r \\
&\quad\leq C\sqrt{\mathbb{E}|K|_T^{t,x}}\,\epsilon(h).
\end{aligned}$$

Finally, we proceed as at the end of the proof of Theorem 4.3:

$$\begin{aligned}
&\bigg|\mathbb{E}\bigg[\int_t^T N_r^{t,x}\big(\mathrm{d}\check{\mathcal{R}}_r^{t,x} - \mathrm{d}\underline{\mathcal{R}}_r^{t,x}\big)\bigg]\bigg| \\
&\quad\leq \mathbb{E}\big[\big|N_T^{t,x}\big|\big|\check{\mathcal{R}}_T^{t,x} - \underline{\mathcal{R}}_T^{t,x}\big|\big] + \mathbb{E}\bigg[\int_t^T \frac{|N_r^{t,x}|}{(r-t)}\big|\check{\mathcal{R}}_r^{t,x} - \underline{\mathcal{R}}_r^{t,x}\big|\,\mathrm{d}r\bigg] \\
&\quad\leq C\big\{\mathbb{E}\big|\check{\mathcal{R}}_T^{t,x} - \underline{\mathcal{R}}_T^{t,x}\big|^2\big\}^{1/2} + C\int_t^T \frac{1}{(r-t)^{3/2}}\big\{\mathbb{E}\big|\check{\mathcal{R}}_r^{t,x} - \underline{\mathcal{R}}_r^{t,x}\big|^2\big\}^{1/2}\,\mathrm{d}r.
\end{aligned}$$

In view of (69) and (54) we have

$$\left\{\mathbb{E}\left|\check{\mathcal{R}}_T^{t,x} - \underline{\mathcal{R}}_T^{t,x}\right|^2\right\}^{1/2} \le C\epsilon(h).$$

In addition, using again (57) we get, for all $\gamma < \frac{1}{4}$,

$$\begin{aligned}\int_t^T \frac{1}{(r-t)^{3/2}} \left\{\mathbb{E}\left|\check{\mathcal{R}}_r^{t,x} - \underline{\mathcal{R}}_r^{t,x}\right|^2\right\}^{1/2} \mathrm{d}r &\le C\epsilon(h)^{2\gamma} \int_t^T \frac{1}{(r-t)^{3/2}} \left\{\mathbb{E}\left|\check{\mathcal{R}}_r^{t,x} - \underline{\mathcal{R}}_r^{t,x}\right|^2\right\}^{1/2-\gamma} \mathrm{d}r \\ &\le C \frac{\epsilon(h)^{2\gamma}}{((x-d)\wedge(d'-x))^{1-2\gamma}} \int_t^T \frac{1}{(r-t)^{1/2+2\gamma}} \mathrm{d}r.\end{aligned}$$

That ends the proof. □

Remark 5.2. *A better estimate for semilinear PDEs can be derived from Section 3, namely,*

$$\int_d^{d'} \left|\partial_x V(t,x) - \partial_x v(t,x)\right|^2 \mathrm{d}x \le C\epsilon(h).$$

Two challenging questions, which are important issues for applications, need now to be treated for multi-dimensional PDEs or variational inequalities: first, the extension of our work to the multi-dimensional case; second, given a desired accuracy on the approximation of $\partial_x V(t,x)$ or of the hedging strategy of an American option, the relevant choice of a function $h(t,x)$ and of an artificial boundary.

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