

Research Article

On Radial Distribution of Julia Sets of Solutions to Certain Second Order Complex Linear Differential Equations

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We mainly investigate the radial distribution of the Julia set of entire solutions to a special second order complex linear differential equation, one of the entire coefficients of which has a finite deficient value.

1. Introduction and Main Results

In this paper, we assume the reader is familiar with standard notations and basic results of Nevanlinna's value distribution theory; see [1–5]. Some basic knowledge of complex dynamics of meromorphic functions is also needed; see [6, 7]. Let f be a meromorphic function in the whole complex plane. We use $\sigma(f)$ and $\mu(f)$ to denote the order and lower order of f , respectively; see [5] for the definitions.

We define that f^n , $n \in \mathbb{N}$, denotes the n th iterate of f . The Fatou set $F(f)$ of transcendental meromorphic function f is the subset of the plane \mathbb{C} where the iterates f^n of f form a normal family. The complement of $F(f)$ in \mathbb{C} is called the Julia set $J(f)$ of f . It is well known that $F(f)$ is open and completely invariant under f ; $J(f)$ is closed and nonempty.

We denote $\Omega(\alpha, \beta) = \{z \in \mathbb{C} \mid \arg z \in (\alpha, \beta)\}$, where $0 < \alpha < \beta < 2\pi$. Given $\theta \in [0, 2\pi)$, if $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap J(f)$ is unbounded, for any $\varepsilon > 0$, then the ray $\arg z = \theta$ is called the radial distribution of $J(f)$. Define

$$\Delta(f) = \{\theta \in [0, 2\pi) \mid J(f) \text{ has the radial distribution with respect to } \arg z = \theta\}. \quad (1)$$

Obviously, $\Delta(f)$ is closed and so measurable. We use $\text{mes}\Delta(f)$ to denote the linear measure of $\Delta(f)$. Many important results of radial distribution of transcendental meromorphic functions have been obtained, for example, [8–14]. Qiao [10] proved that $\text{mes}\Delta(f) = 2\pi$ if $\mu(f) < 1/2$

and $\text{mes}\Delta(f) \geq \pi/\mu(f)$ if $\mu(f) \geq 1/2$, where $f(z)$ is a transcendental entire function of finite lower order. Recently, Huang and Wang [15, 16] considered the radial distribution of the Julia sets of entire solutions to some special linear complex differential equations and obtained the lower bound of them.

In the present paper, we continue and extend the work of Huang and Wang. In fact, we are devoted to investigating the radial distribution of the Julia set of solutions to second order complex differential equations which is studied by Wu and Zhu [17]. One of coefficients of this equation has relation with deficient value. Actually, they proved the following.

Theorem A (see [17]). *Let $A(z)$ be an entire function with finite order having a finite deficient value and let $B(z)$ be a transcendental entire function with $\mu(B) < 1/2$. Then, every nontrivial solution f of equation*

$$f'' + A(z)f' + B(z)f = 0 \quad (2)$$

is of infinite order.

Our main results are about the lower bound of the radial distribution of the Julia set of solutions to (2).

Theorem 1. *Let f be a nontrivial solution of (2), where $A(z)$ is an entire function with finite order having a finite deficient value and $B(z)$ is a transcendental entire function with $\mu(B) < 1/2$; then $\text{mes}(\Delta(f)) \geq d > 0$, where d is defined in Remark 8, Section 2.*

Furthermore, we study the radial distribution of Julia set of the derivatives of the nontrivial entire solutions of (2). Indeed, we obtain the following results.

Theorem 2. *Let f be a nontrivial solution of (2), where $A(z)$ is an entire function with finite order having a finite deficient value and $B(z)$ is a transcendental entire function with $\mu(B) < 1/2$; then $\text{mes}(\Delta(f) \cap \Delta(f^{(k)})) \geq d$, where k is a positive integer.*

By Theorem 2, we immediately have the following.

Corollary 3. *Under the hypothesis of Theorem 2, one has $\text{mes}(\Delta(f^{(k)})) \geq d$, where k is a positive integer.*

Obviously, Theorem 1 becomes a corollary of Theorem 2, but we will use the result of Theorem 1 when proving Theorem 2.

2. Preliminary Lemmas

At first, we recall the Nevanlinna characteristic in an angle; see [1]. We set

$$\begin{aligned} \Omega(\alpha, \beta, r) &= \{z : z \in \Omega(\alpha, \beta), |z| < r\}; \\ \Omega(r, \alpha, \beta) &= \{z : z \in \Omega(\alpha, \beta), |z| \geq r\} \end{aligned} \tag{3}$$

and denote by $\overline{\Omega}(\alpha, \beta)$ the closure of $\Omega(\alpha, \beta)$. Let $g(z)$ be meromorphic on the angle $\overline{\Omega}(\alpha, \beta)$, where $\beta - \alpha \in (0, 2\pi]$. Following [1], we define

$$\begin{aligned} A_{\alpha, \beta}(r, g) &= \frac{w}{\pi} \int_1^r \left(\frac{1}{t^w} - \frac{t^w}{r^{2w}} \right) \\ &\quad \times \left\{ \log^+ |g(te^{i\alpha})| + \log^+ |g(te^{i\beta})| \right\} \frac{dt}{t}; \\ B_{\alpha, \beta}(r, g) &= \frac{2w}{\pi r^w} \int_\alpha^\beta \log^+ |g(re^{i\theta})| \sin w(\theta - \alpha) d\theta; \\ C_{\alpha, \beta}(r, g) &= 2 \sum_{1 < |b_n| < r} \left(\frac{1}{|b_n|^w} - \frac{|b_n|^w}{r^{2w}} \right) \sin w(\beta_n - \alpha), \end{aligned} \tag{4}$$

where $w = \pi/(\beta - \alpha)$ and $b_n = |b_n|e^{i\beta_n}$ are poles of $g(z)$ in $\overline{\Omega}(\alpha, \beta)$ appearing according to their multiplicities. The Nevanlinna angular characteristic is defined as

$$S_{\alpha, \beta}(r, g) = A_{\alpha, \beta}(r, g) + B_{\alpha, \beta}(r, g) + C_{\alpha, \beta}(r, g). \tag{5}$$

In particular, we denote the order of $S_{\alpha, \beta}(r, g)$ by

$$\sigma_{\alpha, \beta}(g) = \limsup_{r \rightarrow \infty} \frac{\log S_{\alpha, \beta}(r, g)}{\log r}. \tag{6}$$

If $\overline{\mathbb{C}} \setminus W$ contains at least three points, where $\overline{\mathbb{C}}$ is the extended complex plane, then W is called a hyperbolic domain. For $a \in \mathbb{C} \setminus W$, define

$$C_W(a) = \inf \{ \lambda_W(z) |z - a| : \forall z \in W \}, \tag{7}$$

where $\lambda_W(z)$ is the hyperbolic density on W . It is well known that if every component of W is simply connected, then $C_W(a) \geq 1/2$.

Lemma 4 (see [14, Lemma 2.2]). *Let $f(z)$ be an analytic in $\Omega(r_0, \theta_1, \theta_2)$; let U be a hyperbolic domain and $f : \Omega(r_0, \theta_1, \theta_2) \rightarrow U$. If there exists a point $a \in \partial U \setminus \{\infty\}$ such that $C_U(a) > 0$, then there exists a constant $d > 0$ such that, for sufficiently small $\varepsilon > 0$, one has*

$$|f(z)| = O(|z|^d), \quad z \in \Omega(r_0, \theta_1 + \varepsilon, \theta_2 - \varepsilon), |z| \rightarrow \infty. \tag{8}$$

The next lemma shows some estimates for the logarithmic derivative of functions being analytic in an angle. Before this, we recall the definition of an R -set; for reference, see [3]. Set $B(z_n, r_n) = \{z : |z - z_n| < r_n\}$. If $\sum_{n=1}^\infty r_n < \infty$ and $z_n \rightarrow \infty$, then $\cup_{n=1}^\infty B(z_n, r_n)$ is called an R -set. Clearly, the set $\{|z| : z \in \cup_{n=1}^\infty B(z_n, r_n)\}$ is of finite linear measure.

Lemma 5 (see [16, Lemma 2.2]). *Let $z = re^{i\psi}$, $r_0 + 1 < r$, and $\alpha \leq \psi \leq \beta$, where $0 < \beta - \alpha \leq 2\pi$. Suppose that $n (\geq 2)$ is an integer and that $g(z)$ is analytic in $\Omega(r_0, \alpha, \beta)$ with $\sigma_{\alpha, \beta}(g) < \infty$. Choose $\alpha < \alpha_1 < \beta_1 < \beta$. Then, for every $\varepsilon_j \in (0, (\beta_j - \alpha_j)/2)$ ($j = 1, 2, \dots, n - 1$) outside a set of linear measure zero with*

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2, 3, \dots, n - 1, \tag{9}$$

there exist $K > 0$ and $M > 0$ only depending on $g, \varepsilon_1, \dots, \varepsilon_{n-1}$, and $\Omega(\alpha_{n-1}, \beta_{n-1})$ and not depending on z , such that

$$\begin{aligned} \left| \frac{g'(z)}{g(z)} \right| &\leq Kr^M (\sin k(\psi - \alpha))^{-2}, \\ \left| \frac{g^{(n)}(z)}{g(z)} \right| &\leq Kr^M \left(\sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_{\varepsilon_j}(\psi - \alpha_j) \right)^{-2} \end{aligned} \tag{10}$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ outside an R -set D , where $k = \pi/(\beta - \alpha)$ and $k_{\varepsilon_j} = \pi/(\beta_j - \alpha_j)$ ($j = 1, 2, \dots, n - 1$).

Lemma 6 (see [18]). *Let $g(z)$ be an entire function with $0 \leq \mu(g) < 1$. Then, for every $\alpha \in (\mu(g), 1)$, there exists a set $E \subset [0, \infty)$ such that $\log \text{dens } E \geq 1 - (\mu(g)/\alpha)$, where $E = \{r \in [0, \infty) : m(r) > M(r) \cos \pi\alpha\}$. Also $m(r) = \inf_{|z|=r} \log |g(z)|$ and $M(r) = \sup_{|z|=r} \log |g(z)|$.*

In the above lemma, the upper logarithmic densities of $E \subset [1, \infty)$ are defined by

$$\overline{\log \text{dens}} E = \limsup_{r \rightarrow \infty} \frac{m_1(E \cap [0, r])}{\log r}, \tag{11}$$

where $m_1(E) = \int_E (dt/t)$ is logarithmic measure of E .

Lemma 7 (see [17]). *Let $A(z)$ be a meromorphic function with $\sigma(A) < +\infty$ and let $B(z)$ be an entire function with*

$0 < \mu(B) < 1/2$. If $A(z)$ has finite deficient value a with deficiency $\delta = \delta(a, A) > 0$, then, for any given constant $\varepsilon > 0$, there exists a sequence R_n with $R_n \rightarrow \infty$, such that the following two inequalities hold:

$$\begin{aligned} \text{mes}(F_n) =: \text{mes} \left\{ \theta \in [0, 2\pi) : \log |A(R_n e^{i\theta}) - a| \right. \\ \left. \leq -\frac{\delta}{4} T(R_n, A) \right\} \geq d > 0; \end{aligned} \tag{12}$$

$$|B(R_n e^{i\varphi})| \geq \exp \{R_n^{\mu(B)-\varepsilon}\}, \quad \varphi \in [0, 2\pi), \tag{13}$$

for all sufficiently large n , where d is a constant depending only on $\sigma(A)$, $\mu(B)$, and δ .

Remark 8. For the sake of simplicity, we denote $\sigma(A)$, $\mu(B)$ by σ , μ , respectively. From the proof of Lemma 2.4 in [17], we know that

$$d := \frac{\delta/4}{(8^{H_0}/2\pi) \{5 + (2 \log 10eL_0) / (\log(4/3))\}}, \tag{14}$$

where $H_0 = 2\sigma/(1 - 2\mu)$ and $L_0 = 2 + (2/\log 2)((2\mu + 1)/(2\mu - 1)^2)$.

Remark 9 (see [17, Remark 2]). If $B(z)$ is an entire function with $\mu(B) = 0$, according to Lemma 6, we only need to give an appropriate modification; then Lemma 7 still holds.

3. Proof of Theorems

Proof of Theorem 1. By Theorem A, we have already known that every nontrivial solution f of (2) is an entire function of infinite order. We will obtain the assertion by reduction to contradiction. At first, we suppose that $\text{mes}(\Delta(f)) < d$, so $\zeta := d - \text{mes}(\Delta(f)) > 0$. Since $\Delta(f)$ is closed, obviously $S = [0, 2\pi) \setminus \Delta(f)$ is open, so it consists of at most countably many open intervals. We can finitely choose many open intervals $I_i = (\alpha_i, \beta_i)$, $i = 1, 2, \dots, m$, satisfying $[\alpha_i, \beta_i] \subset S$ and $\text{mes}(S \setminus \cup_{i=1}^m I_i) < \zeta/4$. For the angular domain $\Omega(\alpha_i, \beta_i)$, it is easy to see that $(\alpha_i, \beta_i) \cap \Delta(f) = \emptyset$ and $\Omega(r, \alpha_i, \beta_i) \cap J(f) = \emptyset$ for sufficiently large r . This implies that, for each $i = 1, 2, \dots, m$, there exist the corresponding r_i and unbounded Fatou component U_i of $F(f)$ such that $\Omega(r_i, \alpha_i, \beta_i) \subset U_i$; see [19]. We take an unbounded and connected section Γ_i of ∂U_i ; then the mapping $f : \Omega(r_i, \alpha_i, \beta_i) \rightarrow \mathbb{C} \setminus \Gamma_i$ is analytic. Since we have chosen Γ_i such that $\mathbb{C} \setminus \Gamma_i$ is simply connected, for any $a \in \Gamma_i \setminus \{\infty\}$, we have $C_{\mathbb{C} \setminus \Gamma_i}(a) \geq 1/2$. By applying Lemma 4 to f in every $\Omega(r_i, \alpha_i, \beta_i)$, there exists a positive constant d_1 such that, for $z \in \cup_{i=1}^m \Omega(r_i, \alpha_i + \varepsilon, \beta_i - \varepsilon)$,

$$|f(z)| = O(|z|^{d_1}), \quad \text{as } |z| \rightarrow \infty, \tag{15}$$

where $0 < \varepsilon < \min\{\zeta/(16m), (\beta_i - \alpha_i)/8\}$, $i = 1, 2, \dots, m$. Thus, recalling the definition of $S_{\alpha, \beta}(r, f)$, we immediately have that

$$S_{\alpha_i + \varepsilon, \beta_i - \varepsilon}(r, f) = O(1), \quad (i = 1, 2, \dots, m). \tag{16}$$

So $\sigma_{\alpha_i + \varepsilon, \beta_i - \varepsilon}(r, f)$ is finite. Therefore, by Lemma 5, there exist two constants $M > 0$ and $K > 0$ such that

$$\left| \frac{f^{(s)}(z)}{f(z)} \right| \leq Kr^M, \quad (s = 1, 2), \tag{17}$$

for all $z \in \cup_{i=1}^m \Omega(r_i, \alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$, outside a R -set H . By Lemma 7, for sufficiently large n , we have

$$\text{mes}(F_n) > d - \frac{\zeta}{4}. \tag{18}$$

Therefore, we have

$$\begin{aligned} \text{mes}(F_n \cap S) &= \text{mes}(F_n \setminus (\Delta(f) \cap F_n)) \\ &\geq \text{mes}(F_n) - \text{mes}(\Delta(f)) > \frac{3\zeta}{4} > 0. \end{aligned} \tag{19}$$

Then, for each n , we have

$$\begin{aligned} \text{mes} \left(\left(\bigcup_{i=1}^m I_i \right) \cap F_n \right) \\ &= \text{mes}(S \cap F_n) - \text{mes} \left(\left(S \setminus \bigcup_{i=1}^m I_i \right) \cap F_n \right) \\ &> \frac{3\zeta}{4} - \frac{\zeta}{4} \\ &= \frac{\zeta}{2}. \end{aligned} \tag{20}$$

Thus, there exists an open interval $I_{i_0} = (\alpha, \beta) \subset \cup_{i=1}^m I_i \subset S$ such that, for infinitely many n ,

$$\text{mes}(F_n \cap (\alpha, \beta)) > \frac{\zeta}{2m} > 0. \tag{21}$$

Without loss of generality, we can assume that (21) holds for all n .

Let a be a finite deficient value of $A(z)$ with deficiency $\delta = \delta(a, A)$. From (2), we have the following inequality:

$$|B(z)| \leq \left| \frac{f''(z)}{f(z)} \right| + \left| \frac{f'(z)}{f(z)} \right| (|A(z) - a| + |a|). \tag{22}$$

In the following, we consider two cases.

Case 1: $0 < \mu(B) < 1/2$. By the definition (12) of F_n and (13), there exists a sequence R_n , which is outside a R -set H , with $R_n < R_{n+1}$ and $R_n \rightarrow \infty$ such that, for every n and $\varphi \in F_n \cap (\alpha, \beta)$, we have

$$\log |A(R_n e^{i\varphi}) - a| \leq -\frac{\delta}{4} T(R_n, A); \tag{23}$$

$$|B(R_n e^{i\varphi})| > \exp \{R_n^{(1/2)\mu(B)}\}. \tag{24}$$

From (17), (22), and (23), for every $n \geq n_0$, we get

$$|B(R_n e^{i\varphi})| \leq KR_n^M \left(1 + \exp \left\{ -\frac{\delta}{4} T(R_n, f) \right\} + |a| \right). \tag{25}$$

Thus, by (24), we obtain

$$\exp \left\{ R_n^{(1/2)\mu(B)} \right\} < KR_n^M \left(1 + \exp \left\{ -\frac{\delta}{4} T(R_n, f) \right\} + |a| \right). \tag{26}$$

Obviously, when n is sufficiently large, this is a contradiction.

Case 2: $\mu(B) = 0$. By using Lemma 6, there exists a set $E \subset [0, +\infty)$ with $\overline{\log \text{dens}} E = 1$ such that, for all z satisfying $|z| = r \in E$, we have

$$\log |B(z)| > \frac{\sqrt{2}}{2} \log M(r, B), \tag{27}$$

where $M(r, B) = \max_{|z|=r} |B(z)|$. It follows from Remark 9 that there exists a sequence R_n , which is contained in E and outside a R -set H , such that (17), (23), and (27) hold for $z = R_n e^{i\varphi}$, $\varphi \in F_n \cap (\alpha, \beta)$. From (17), (22), and (23), we obtain (25). Hence, from (25) and (27), we get

$$M(R_n, B)^{\sqrt{2}/2} \leq KR_n^M \left(1 + \exp \left\{ -\frac{\delta}{4} T(R_n, f) \right\} + |a| \right). \tag{28}$$

Since $B(z)$ is a transcendental entire function, we have

$$\liminf_{r \rightarrow \infty} \frac{\log M(r, B)}{\log r} = +\infty. \tag{29}$$

Therefore, we can easily obtain a contradiction from (28), when n is sufficiently large. Thus, we complete the proof of Theorem 1. \square

Proof of Theorem 2. We know that every nontrivial solution f of (2) is an entire function with infinite order. We also obtain the assertion by reduction to contradiction. Assume that

$$\text{mes} \left(\Delta(f) \cap \Delta(f^{(k)}) \right) < d \tag{30}$$

and so

$$\xi := d - \text{mes} \left(\Delta(f) \cap \Delta(f^{(k)}) \right) > 0. \tag{31}$$

We will show that there must exist an open interval

$$I = (\alpha, \beta) \subset \Delta(f^{(k)})^c, \quad 0 < \beta - \alpha < d \tag{32}$$

such that

$$\lim_{n \rightarrow \infty} \text{mes} \left(\Delta(f) \cap F_n \cap I \right) > 0, \tag{33}$$

where $\Delta(f^{(k)})^c := [0, 2\pi) \setminus \Delta(f^{(k)})$ and F_n is as defined in (12). In order to achieve this goal, we will firstly prove the following:

$$\lim_{n \rightarrow \infty} \text{mes} \left(F_n \setminus \Delta(f) \right) = 0. \tag{34}$$

Otherwise, suppose that there is a subseries $\{r_{n_k}\}$ such that

$$\lim_{k \rightarrow \infty} \text{mes} \left(F_{n_k} \setminus \Delta(f) \right) > 0; \tag{35}$$

then there exist $\theta_0 \in \Delta(f)^c$ and $\eta > 0$ satisfying

$$\lim_{k \rightarrow \infty} \text{mes} \left((\theta_0 - \eta, \theta_0 + \eta) \cap (F_{n_k} \setminus \Delta(f)) \right) > 0. \tag{36}$$

Since $\arg z = \theta_0$ is not a radial distribution of $J(f)$, there exists $r_0 > 0$ such that

$$\Omega(r_0, \theta_0 - \eta, \theta_0 + \eta) \cap J(f) = \emptyset. \tag{37}$$

This implies that there exists an unbounded component U of Fatou set $F(f)$, such that $\Omega(r_0, \theta_0 - \eta, \theta_0 + \eta) \subset U$. Take an unbounded and connected set $\Gamma \subset \partial U$; the mapping $f : \Omega(r_0, \theta_0 - \eta, \theta_0 + \eta) \rightarrow \mathbb{C} \setminus \Gamma$ is analytic. Since $\mathbb{C} \setminus \Gamma$ is simply connected, then, for any $a \in \Gamma \setminus \{\infty\}$, we have $C_{\mathbb{C} \setminus \Gamma}(a) \geq 1/2$. Now, by applying Lemma 4 to f in $\Omega(r_0, \theta_0 - \eta, \theta_0 + \eta)$, for any $\zeta > 0$, $\zeta < \eta$, we have

$$|f(z)| = O(|z|^{d_1}), \quad z \in \Omega(r_0, \theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta),$$

$$|z| \rightarrow \infty, \tag{38}$$

where d_1 is a positive constant. Recalling the definition of $S_{\alpha, \beta}(r, f)$, we immediately get that

$$S_{\theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta}(r, f) = O(1). \tag{39}$$

Thus, $\sigma_{\theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta}(r, f)$ is finite. Therefore, by Lemma 5, there exist constants $M > 0$ and $K > 0$ such that

$$\left| \frac{f^{(s)}(z)}{f(z)} \right| \leq Kr^M, \quad (s = 1, 2), \tag{40}$$

for all $z \in \Omega(r_0, \theta_0 - \eta + 2\zeta, \theta_0 + \eta - 2\zeta)$, outside a R -set H .

Since ζ can be chosen sufficiently small, from (36), we have

$$\lim_{k \rightarrow \infty} \text{mes} \left((\theta_0 - \eta + 2\zeta, \theta_0 + \eta - 2\zeta) \cap F_{n_k} \right) > 0. \tag{41}$$

Thus, we can find an infinite series $\{r_{n_k} e^{i\theta_{n_k}}\}$ such that, for all sufficiently large k , (23), (24), and (27) hold when $\theta_{n_k} \in (\theta_0 - \eta + 2\zeta, \theta_0 + \eta - 2\zeta) \cap F_{n_k}$. From (22), (23), (24), (27), and (40) and by the same argument as in Cases 1 and 2 in the proof of Theorem 1, we can obtain contradictions. This implies that (34) is valid.

By Theorem 1, we know that

$$\text{mes} \Delta(f) \geq d. \tag{42}$$

From Lemma 7, we have, for all sufficiently large n and any positive ε ,

$$\text{mes}(F_n) > d - \varepsilon. \tag{43}$$

Combining (34), (42), and (43), it follows that, for all sufficiently large n ,

$$\text{mes} \left(\Delta(f) \cap F_n \right) \geq d - \frac{\xi}{4}, \tag{44}$$

where ξ is defined in (31). Since $\Delta(f^{(k)})$ is closed, clearly $\Delta(f^{(k)})^c$ is open, so it consists of at most countably open intervals. We can choose finitely many open intervals I_j ($j = 1, 2, \dots, m$) satisfying

$$I_j \subset \Delta(f^{(k)})^c, \quad \text{mes} \left(\Delta(f^{(k)})^c \setminus \bigcup_{i=1}^m I_i \right) < \frac{\xi}{4}. \quad (45)$$

Since, for sufficiently large n ,

$$\begin{aligned} & \text{mes} \left(\Delta(f) \cap F_n \cap \left(\bigcup_{i=1}^m I_i \right) \right) + \text{mes} \left(\Delta(f) \cap F_n \cap \Delta(f^{(k)}) \right) \\ &= \text{mes} \left(\Delta(f) \cap F_n \cap \left(\Delta(f^{(k)}) \cup \left(\bigcup_{i=1}^m I_i \right) \right) \right) \geq d - \frac{\xi}{2}, \end{aligned} \quad (46)$$

we have

$$\begin{aligned} & \text{mes} \left(\Delta(f) \cap F_n \cap \left(\bigcup_{i=1}^m I_i \right) \right) \\ & \geq d - \frac{\xi}{2} - \text{mes} \left(\Delta(f) \cap F_n \cap \Delta(f^{(k)}) \right) \\ & \geq d - \frac{\xi}{2} - \text{mes} \left(\Delta(f) \cap \Delta(f^{(k)}) \right) = \frac{\xi}{2}. \end{aligned} \quad (47)$$

Thus, there exists an open interval $I_{i_0} = (\alpha, \beta) \subset \cup_{i=1}^m I_i \subset \Delta(f^{(k)})^c$ such that, for infinitely many sufficiently large n ,

$$\text{mes} \left(\Delta(f) \cap F_n \cap I_{i_0} \right) \geq \frac{\xi}{2m} > 0. \quad (48)$$

Then, we prove that (33) holds.

From (33), we know that there are $\widetilde{\theta}_0$ and $\widetilde{\eta} > 0$ such that

$$(\widetilde{\theta}_0 - \widetilde{\eta}, \widetilde{\theta}_0 + \widetilde{\eta}) \subset I, \quad (49)$$

$$\lim_{n \rightarrow \infty} \text{mes} \left(\Delta(f) \cap F_n \cap (\widetilde{\theta}_0 - \widetilde{\eta}, \widetilde{\theta}_0 + \widetilde{\eta}) \right) > 0.$$

Then, there exists \widetilde{r}_0 such that $\Omega(\widetilde{r}_0, \widetilde{\theta}_0 - \widetilde{\eta}, \widetilde{\theta}_0 + \widetilde{\eta}) \cap J(f^{(k)}) = \emptyset$. By similar argument between (37) and (38), for any $\widetilde{\zeta} > 0$, $\widetilde{\zeta} < \widetilde{\eta}$, we have

$$\begin{aligned} |f^{(k)}(z)| &= O(|z|^{d_2}), \quad z \in \Omega(\widetilde{r}_0, \widetilde{\theta}_0 - \widetilde{\eta} + 2\widetilde{\zeta}, \widetilde{\theta}_0 + \widetilde{\eta} - 2\widetilde{\zeta}), \\ |z| &\longrightarrow \infty, \end{aligned} \quad (50)$$

where d_2 is a positive constant.

Fix $r_N e^{i\theta_N} \in \{r_n e^{i\theta_n}\}$, and take a $r_n e^{i\theta_n} \in \{r_n e^{i\theta_n}\}$, $n > N$. Take a simple Jordan arc γ in $\Omega(\widetilde{r}_0, \widetilde{\theta}_0 - \widetilde{\eta}, \widetilde{\theta}_0 + \widetilde{\eta})$ which connects $r_N e^{i\theta_N}$ to $r_n e^{i\theta_n}$ along $|z| = r_N$ and connects $r_N e^{i\theta_N}$ to $r_n e^{i\theta_n}$ along $\arg z = \theta_n$. For any $z \in \gamma$, γ_z denotes a part of γ , which connects $r_N e^{i\theta_N}$ to z . Let $L(\gamma)$ be the length of γ . Clearly,

$$L(\gamma) = O(r_n), \quad n \longrightarrow \infty. \quad (51)$$

By (50), it follows that

$$\begin{aligned} |f^{(k-1)}(z)| &\leq \int_{\gamma_z} |f^{(k)}(z)| |dz| + c_k \\ &\leq O(|z|^{d_2} L(\gamma)) + c_k \\ &\leq O(r_n^{d_2+1}), \quad n \longrightarrow \infty. \end{aligned} \quad (52)$$

Similarly, we have

$$\begin{aligned} |f^{(k-2)}(z)| &\leq \int_{\gamma_z} |f^{(k-1)}(z)| |dz| + c_{k-1} \\ &\leq O(r_n^{d_2+2}), \quad n \longrightarrow \infty \\ &\vdots \end{aligned} \quad (53)$$

$$\begin{aligned} |f(z)| &\leq \int_{\gamma_z} |f'(z)| |dz| + c_1 \\ &\leq O(r_n^{d_2+k}), \quad n \longrightarrow \infty, \end{aligned}$$

where c_1, c_2, \dots, c_k are constants, which are independent of n . Therefore,

$$S_{\widetilde{\theta}_0 - \widetilde{\eta} + \widetilde{\zeta}, \widetilde{\theta}_0 + \widetilde{\eta} - \widetilde{\zeta}}(r, f) = O(1), \quad (54)$$

By Lemma 5, we know that (40) also holds for all $z \in \Omega(\widetilde{r}_0, \widetilde{\theta}_0 - \widetilde{\eta} + 2\widetilde{\zeta}, \widetilde{\theta}_0 + \widetilde{\eta} - 2\widetilde{\zeta})$, outside a R -set H . By applying similar argument as in Cases 1 and 2 in the proof of Theorem 1, we can deduce contradictions. Therefore, it follows that

$$\text{mes} \left(\Delta(f) \cap \Delta(f^{(k)}) \right) \geq d. \quad (55)$$

The proof is complete. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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