AN EXTENSION OF TIETZE'S THEOREM

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1. Introduction. Let X be an arbitrary metric space, A a closed subset of X, and E^n the Euclidean n-space. Tietze's theorem asserts that any (continuous) $f: A \longrightarrow E^1$ can be extended to a (continuous) $F: X \longrightarrow E^1$. This theorem trivially implies that any $f: A \longrightarrow E^n$ and any $f: A \longrightarrow$ (Hilbert cube) can be extended; we merely decompose f into its coordinate mappings and observe that, in these cases, the continuity of each of the coordinate mappings is equivalent to that of the resultant map.

Where this equivalence is not true, for example mapping into the Hilbert space, the theorem has been neglected. We are going to prove that, in fact, Tietze's theorem is valid for continuous mappings of A into any locally convex linear space (4.1), (4.3). Two proofs of this result will be given; the second proof (4.3), although essentially the same as the first, is more direct; but it hides the geometrical motivation.

There are several immediate consequences of the above result. First we obtain a theorem on the simultaneous extension of continuous real-valued functions on a closed subset of a metric space (5.1). Secondly, we characterize completely those normed linear (not necessarily complete) spaces in which the Brouwer fixed-point theorem is true for their unit spheres (6.3). Finally, we can generalize the whole theory of locally connected spaces to arbitrary metric spaces. By way of illustration, we prove a theorem about absolute neighborhood retracts that is apparently new even in the separable metric case (7.5).

The idea of the proof of the main theorem is simple. Given A and X, we show how to replace $X - A$ by an infinite polytope; we extend f continuously first on the vertices of the polytope, and then over the entire polytope by linearity. For this we need several preliminary remarks on coverings and on polytopes.

2. On coverings and polytopes. If X is any space, a covering of X by an arbitrary collection $\{U\}$ of open sets is called a *locally finite covering* if, given any

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$x \in X$, there exists a nbd of x meeting only a finite number of the sets of $\{U\}$. If $\{U\}$, $\{V\}$ are any two coverings of X by open sets, $\{V\}$ is a *refinement* of $\{U\}$ if for each $V \in \{V\}$ there is a $U \in \{U\}$ containing it. A. H. Stone has proved [12] that every covering of an arbitrary metric space has a locally finite refinement.

2.1 LEMMA. *Let X be an arbitrary metric space, and A a closed subset of X; then there exists a covering $\{U\}$ of $X - A$ such that:*

2.11 *the covering $\{U\}$ is locally finite;*

2.12 *any* nbd *of $a \in (A -$ interior $A)$ contains infinitely many sets of $\{U\}$;*

2.13 *given any* nbd *W of $a \in A$, there exists a* nbd *W', $a \in W' \subset W$, such that $U \cap W' \neq 0$ implies $U \subset W$.*

Proof. Around each point $x \in (X - A)$, draw a nbd S_x such that diameter $S_x < (1/2)d(x, A)$, where d is the metric in X. This is a covering of $X - A$, since $X - A$ is open. By A. H. Stone's theorem, we can construct a locally finite refinement $\{U\}$. It is then evident that $\{U\}$ satisfies 2.11–2.13.

A covering of $X - A$ satisfying the conditions 2.11–2.13 will be called a *canonical covering* of $X - A$.

2.2 A *polytope* P is a point set composed of an arbitrary collection of closed Euclidean cells (higher dimensional analogs of a tetrahedron) satisfying (a) every face of a cell of the collection is itself a cell of the collection, and (b) the intersection of any two closed cells of P is a face of both of them. A *CW polytope* is a polytope with the *CW* topology of Whitehead [14]: a subset U of P is open if and only if the intersection $U \cap \bar{\sigma}$ of U with every closed cell $\bar{\sigma}$ is open in the Euclidean topology of $\bar{\sigma}$. It is easy to verify:

2.21 *a CW polytope is a Hausdorff space;*

2.22 *in a CW polytope, the star of any cell σ (the collection of all open cells having σ as a face) is an open set;*

2.23 *if Y is an arbitrary space, then $f: P \longrightarrow Y$ is continuous if and only if f is continuous on each cell.*

2.3 As a final preliminary, we need the "nerve" of a covering. Let X be a space, and $\{U\}$ a covering of X by open sets. Consider an abstract nontopologized real linear vector space R spanned by linearly independent vectors $\{p_U\}$ is a fixed one-to-one correspondence with the collection $\{U\}$; the elements of R will be called *points*. The $n+1$ points $p_{U_1}, \cdots, p_{U_n}$ determine an n-cell in the usual way if and only if the corresponding sets satisfy $U_1 \cap \cdots \cap U_n \neq 0$. The polytope determined in this way, with the CW topology, will be called the *nerve* of the covering $\{U\}$, and denoted by $N(U)$.

2.31 THEOREM. *If $\{U\}$ is a locally finite covering of a metric space X, and $N(U)$ the nerve of $\{U\}$, then there exists a continuous $K: X \longrightarrow N(U)$ such that $K^{-1}(\text{star } p_U) \subset U$ for every $U \in \{U\}$.*

Proof. (Cf. Dowker [4], where $N(U)$ is taken as a metric polytope.) Define for each $U \in \{U\}$,

$$\lambda_U(x) = \frac{d(x, X-U)}{\sum_U d(x, X-U)} \qquad (x \in X,\ d \text{ the metric in } X).$$

It is first necessary to investigate the nature of these functions. First we notice that $\Sigma_U d(x, X-U)$ is always a finite sum, since $d(x, X-U) \neq 0$ if and only if $x \in U$, and since the covering being locally finite means x lies in a finite number of U's. Further, since $\{U\}$ is a covering, we have $\Sigma_U d(x, X-U) \neq 0$ for every $x \in X$, and so $\lambda_U(x)$ is well-defined for each $x \in X$. Now each $\lambda_U(x)$ is continuous; in fact, for any $x \in X$ there is a nbd meeting only a finite number of the sets of $\{U\}$; in this nbd, $\lambda_U(x)$ is explicitly determined in terms of a finite number of continuous functions, so λ_U is continuous at each $x \in X$. Finally, it is evident that $\Sigma_U \lambda_U(x) = 1$ for each $x \in X$ and that only a finite number are not zero in some nbd of any point $x \in X$.

The mapping $K: X \longrightarrow N(U)$ is defined by setting

$$K(x) = \sum_U \lambda_U(x) p_U .$$

Now $\lambda_U(x) \neq 0$ if and only if $x \in U$; hence if $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these sets, then because $\Sigma_U \lambda_U(x) = 1$ for every x, $K(x)$ is the point in the interior of the cell spanned by $(p_{U_1}, \cdots, p_{U_n})$ with barycentric coordinates $\{\lambda_{U_i}(x)\}$. It follows readily that $K^{-1}(\text{star } p_U) \subset U$ for every U. Finally, K is continuous: for, given $x \in X$, let $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these sets; then $K(x)$ is in the interior of $\bar{\sigma} = (\overline{p_{U_1}, \cdots, p_{U_n}})$. Let V be any open set containing $K(x)$; then $V \cap \bar{\sigma}$ is open in the Euclidean topology of $\bar{\sigma}$, and so the continuity of each λ_U shows the existence of an open $W \supset x$ with $K(W) \subset V \cap \bar{\sigma} \subset V$. This proves the assertion. (See also 7.4 in this connection.)

3. The replacement by polytopes. After the above preliminaries, we are ready to perform the "replacement" mentioned in the introduction.

3.1 THEOREM. *Let X be a metric space and A a closed subset of X; then there exists a space Y (not necessarily metrizable) and a continuous $\mu: X \longrightarrow Y$ with the properties*:

3.11 *$\mu | A$ is a homeomorphism and $\mu(A)$ is closed in Y*;

3.12 *$Y - \mu(A)$ is an infinite polytope, and $\mu(X - A) \subset [Y - \mu(A)]$*;

3.13 *each* nbd *of $a \in [\mu(A)$-interior $\mu(A)]$ contains infinitely many cells of $Y - \mu(A)$.*

Proof. Let $\{U\}$ be a canonical covering of $X - A$, and $N(U)$ the nerve of this covering. The set Y consists of the set A and a set of points in a one-to-one correspondence with the points of $N(U)$; to avoid extreme symbolism we denote this set Y by $A \cup N(U)$. The topology in $A \cup N(U)$ is determined as follows:

a. $N(U)$ is taken with the CW topology.

b. A subbasis for nbds of $a \in A$ in $A \cup N(U)$ is determined by selecting a nbd W of a *in* X and taking in $A \cup N(U)$ the set of points $W \cap A$ together with the star of every vertex of $N(U)$ corresponding to a set of the covering $\{U\}$ contained in W. This nbd is denoted by $\widetilde{W}$.

It is not hard to verify that $A \cup N(U)$ with this topology is a Hausdorff space, and that both A and $N(U)$, as subspaces, preserve their original topologies. We now define

$$\mu(x) = \begin{cases} x & (x \in A), \\ K(x) & [x \in (X - A)]. \end{cases}$$

Because of 2.31 and the preceding remarks, the continuity of $\mu(x)$ will be proved as soon as we show it continuous at points of $A \cap (\overline{X - A})$. Let $a \in A \cap \overline{X - A}$, and let $\widetilde{W}$ be a subbasic nbd of $\mu(a)$ in $A \cup N(U)$; this is determined by a nbd W of a in X. Now (2.13) we can determine a nbd W', $a \in W' \subset W$, such that $U \cap W' \neq 0$ implies $U \subset W$, since $\{U\}$ is canonical, and clearly $\{U \mid U \subset W' \cap (X - A)\}$ is not vacuous. We now prove $\mu(W') \subset \widetilde{W}$. In fact, if $x \in W' \cap (X - A)$ let $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these; then $K(x)$ is in the interior of the cell spanned by $p_{U_1}, \cdots, p_{U_n}$, and therefore $K(x)$ is in the star of, say, p_{U_1}. But since $U_1 \cap W' \neq 0$, we have $U_1 \subset W$, and so $K(x) \in \widetilde{W}$. This shows

$$K[W' \cap (X - A)] = \mu[W' \cap (X - A)] \subset \widetilde{W}.$$

Finally, since $W' \subset W$ we have $\mu(W' \cap A) \subset W' \cap A \subset \widetilde{W}$, and so $\mu(W') \subset \widetilde{W}$. This proves that μ is continuous. The properties 3.11–3.13 now follow at once.

4. Extension of Tietze's theorem. Let X, Y be arbitrary spaces, and $A \subset X$. Let $f: A \longrightarrow Y$ be continuous. A continuous $F: X \longrightarrow Y$ is called an *extension* of f if $F(a) = f(a)$ for every $a \in A$. We now prove:

4.1 THEOREM. *Let X be an arbitrary metric space, A a closed subset of X, L a locally convex linear space* [10, p. 72], *and $f: A \longrightarrow L$ a continuous map. Then there exists an extension $F: X \longrightarrow L$ of f; furthermore, $F(X) \subset$ [convex hull of $f(A)$].*

Proof. Let us form the space $A \cup N(U)$ of Theorem 3.1. It is sufficient to prove that every continuous $f: A \longrightarrow L$ extends to a continuous $F: A \cup N(U) \longrightarrow L$. In fact, to handle the general case we first define, on $A \subset A \cup N(U)$, the map $\bar{f}(a) = f[\mu^{-1}(a)]$; extending $\bar{f}$ to $\bar{F}$ we can write $F(x) = \bar{F}[\mu(x)]$; it is evident that F is the desired extension of f.

Let then $N(U)_0$ denote the collection of all vertices of $N(U)$; we first define an extension of f to an $f_0: A \cup N(U)_0 \longrightarrow L$ as follows: in each set of $\{U\}$ select a point x_U; then choose an $a_U \in A$ such that $d(x_U, a_U) < 2d(x_U, A)$; if p_U is the vertex of $N(U)$ corresponding to U, set

$$f_0(p_U) = f(a_U)$$
$$f_0(a) = f(a) \qquad (a \in A).$$

We now prove f_0 continuous. It is clearly so on $N(U)$, since the vertices of $N(U)$ are an isolated set (the star of any one vertex excludes all the others). Thus continuity of f_0 need only be checked at A.

Select any nbd V of $f_0(a) = f(a)$; since f is continuous on A, there is a $\delta > 0$ such that $d(a, a') < \delta$ implies $f(a') \in V$. Let W be any nbd of a in X of radius $< \delta/3$. If $U \in \{U\}$ and $U \subset W$, then clearly $d(x_U, a) < \delta/3$, and so $d(a_U, a) \leq d(a_U, x_U) + d(x_U, a) < 2d(x_U, A) + \delta/3 < 2\delta/3 + \delta/3 = \delta$. Thus all vertices of $N(U)_0$ in the nbd $\widetilde{W}$ satisfy $f_0(p_U) = f(a_U) \subset V$. Hence for all $\tilde{x} \in \widetilde{W} \cap [A \cup N(U)_0]$ we have $f_0(\tilde{x}) \in V$ and continuity is proved.

We now extend linearly over each cell of $N(U)$ the mapping already given on the vertices, and thus obtain an F mapping $A \cup N(U)$ into L. This map we now prove continuous; on the basis of 2.23 we need prove F continuous only at points of A.

Let V be a convex nbd of $f(a) = F(a)$. Since f_0 is continuous at a, there is a nbd $\widetilde{W}$ with $f_0\{\widetilde{W} \cap [A \cup N(U)_0]\} \subset V$. Construct now a nbd $W' \subset W$ of a in X such that $U \cap W' \neq 0$ implies $U \subset W$. It follows that all vertices corresponding to sets in the nbd W' have images lying in the convex set V. If p_U is any vertex in the closure of the star of a vertex $p_{U'}$ with $U' \subset W'$, we observe that $U \cap W' \neq 0$ and so $p_U \subset \widetilde{W}$. Thus the vertices of any cell belonging to the closure of the star of any vertex $p_{U'}$ are sent into the convex set $V \subset L$ and therefore the linear extensions over these cells have images lying in V; this shows $F(W') \subset V$. Since L is locally convex, this result implies that F is continuous. It is evident, finally, from the construction, that $F(X) \subset$ [convex hull of $f(A)$], and that F is an extension of f. The theorem is proved.

If Y is a space with the property that, given any metric space X and any closed $A \subset X$, every continuous $f: A \longrightarrow Y$ extends to a continuous $F: X \longrightarrow Y$, we call Y an *absolute retract.* Thus Theorem 4.1 asserts that any locally convex linear space is an absolute retract. The conclusions of the theorem give a slight extension.

4.2 COROLLARY. *Let C be a convex set in a locally convex linear space L. Then C is an absolute retract.*

Proof. This is immediate from the construction of Theorem 4.1, since the extension has an image lying in the convex hull of $f(A)$, and so in C.

Note that C is not required to be closed in L.

4.3 It is possible to give an elementary direct proof of Theorem 4.1 not explicitly involving the space $A \cup N(U)$, by merely explicitly exhibiting the resulting extension that was constructed in 4.2. It has the advantage of exhibiting a certain kind of "linearity" in the constructed extension, which is sometimes more amenable to applications. In fact, using the notations of Theorem 2.31 and Theorem 4.1, we find it is simple to verify directly that

$$F(x) = \sum_U \lambda_U(x) f(a_U) \qquad [x \in (X - A)],$$

$$= f(x) \qquad (x \in A)$$

is the extension of f which we have constructed. The proof of the continuity is essentially a repetition of the last part of 4.1, and is as follows: By the considerations of 2.31, the continuity of F need be proved only at points of A. Select any convex nbd V of $F(a) = f(a)$; we are to find a nbd $W'' \supset a$ with $F(W'') \subset V$.

Since f is continuous on A, there exists a $\delta > 0$ such that $d(a, a') < \delta$ implies $f(a') \in V$. Now let W be a nbd of a in X of radius $< \delta/3$; since $\{U\}$ is canonical, we can find a nbd W', $a \in W' \subset W$, such that whenever $U \cap W' \neq 0$, then $U \in W$. It follows that for any $x_U \in W'$ we have $U \subset W$ and so $d(x_U, a) < \delta/3$; this shows that $d(a_U, a) \leq d(a_U, x_U) + d(x_U, a) < \delta$ and therefore we conclude:

(*) *Whenever* $x_U \in W'$, *then* $F(x_U) = f(a_U) \in V$.

Construct, finally, a nbd W'' such that $a \in W'' \subset W'$ and such that whenever $U \cap W'' \neq 0$, then $U \subset W'$. We are going to show that $F(W'') \subset V$.

In fact, if $x \in W'' \cap (X - A)$, let $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these sets; since $\Sigma_U \lambda_U(x) = 1$ for every $x \in (X - A)$ and $\lambda_U(x) \neq 0$ only if $U = U_i$, $i = 1, \cdots, n$, it follows that $F(x)$ belongs to the (perhaps degenerate) cell in L spanned by $f(a_{U_1}), \cdots, f(a_{U_n})$; and since $U_i \cap W'' \neq 0$ for $i = 1, \cdots, n$, we see from (*) that $f(a_{U_i}) \subset V$, $i = 1, \cdots, n$. This means that the vertices of the cell spanned by $f(a_{U_1}), \cdots, f(a_{U_n})$ are all in the convex set V, so the linear extension lies in V also, and therefore $F(x) \in V$. Since x is arbitrary, we see that

$$F[W'' \cap (X - A)] \subset V.$$

But also, since we have diameter $W'' < \delta$, it follows that $F(W'' \cap A) = f(W'' \cap A) \subset V$, and so $F(W'') \subset V$, as stated. Since L is locally convex, this proves F continuous at points of A, and, as remarked, continuous on X. (See also Kuratowski [9]).

We note that to prove Theorem 4.1 our method requires essentially three things: (1) the existence of a canonical covering of $X - A$, (2) the possibility of mapping $X - A$ into the nerve of a canonical covering, and (3) the possibility retracting the set $\{x_U\}$ into A; for (3) allows an extension over the vertices of $N(U)$, and then with a linear extension over the cells the theorem follows at once from (1) and (2). The metric enters in obtaining (1) and (3), while the paracompactness comes into play only in establishing (2) (Dowker [4]; Stone [12]). It should be remarked that, after Theorem 4.1 was communicated to R. Arens, he was able to demonstrate that the method used here applies in the case where X is paracompact (but not metric), provided L is a Banach space. Arens' result coincides with one by Dowker (oral communication).

5. Application to the simultaneous extension of continuous functions. The explicit form of the extension given in 4.3 immediately permits us to answer a question of Borsuk [2]. Let Z be a metric space; denote by $C(Z)$ the Banach space of all bounded real-valued continuous functions on Z. We prove, as a first application:

5.1 THEOREM. *Let A be a closed subset of a metric space X; then there exists a linear operation ϕ which makes correspond to each $f \in C(A)$ an extension $\phi(f) \in C(X)$.*

Proof. With the notations of Theorem 4.1, having selected the points a_U once for all, define for every $f \in C(A)$,

$$\phi(f) = \sum_U \lambda_U(x) f(a_U).$$

Then $\phi(f)$ is clearly an extension of f for every f (see 4.3). We have evidently

$$\phi(f + g) = \phi(f) + \phi(g),$$
$$\|\phi(f)\| = \|f\|,$$

and so ϕ is additive and continuous, hence a linear operation.

The restriction of Borsuk [2] that A be separable is thus not necessary. This result extends, naturally, to Banach space valued functions.

6. Application to normed linear spaces. To give another application, we characterize those normed linear spaces for which Brouwer's fixed-point theorem holds in their unit spheres.

6.1 LEMMA. *Let L be a normed linear space, and $C \subset L$ the set*

$$\{x \mid \|x\| = 1\}.$$

Let $\bar{\sigma}^n$ be any n-cell, and $\beta\bar{\sigma}^n$ its boundary. If C is not compact, then any $f: \beta\sigma^n \longrightarrow C$ can be extended to an $F: \bar{\sigma}^n \longrightarrow C$.

Proof. By a known theorem [1, p. 502] it is enough to show that $f(\beta\bar{\sigma}^n)$ can be contracted to a point over C. Now, since $\beta\sigma^n$ is compact and C is not, it follows that $f(\beta\bar{\sigma}^n)$ cannot cover all of C, so that there exists at least one point $x_0 \in [C - f(\beta\sigma^n)]$. Select its antipode $-x_0$ and define

$$\phi(x, t) = \frac{t(-x_0) + (1-t)f(x)}{\|t(-x_0) + (1-t)f(x)\|} \qquad (0 \leq t \leq 1,\ x \in \beta\sigma^n).$$

Then ϕ is continuous in x and t, since the denominator cannot vanish for any x because $-x_0$ and $f(x)$ are never antipodal. Since $\phi(x, 0) = f(x)$, $\phi(\beta\bar{\sigma}^n, 1) = -x_0$, and $\|\phi(x, t)\| = 1$ always, ϕ exhibits the desired contraction.

6.2 THEOREM. *Let L be a normed linear space, and $C = \{x \mid \|x\| = 1\}$. If C is not compact, then C is an absolute retract.*

Proof. With the notations of Theorem 4.1, let us take the space $A \cup N(U)$ and the mapping $f: A \longrightarrow C$. By the construction of Theorem 4.1, we extend f to $F: A \cup N(U) \longrightarrow L$ and notice that $F[A \cup N(U)] \subset \widetilde{C} = \{x \mid \|x\| \leq 1\}$. Let $C' = \{x \mid \|x\| \leq 1/2\}$; then $\widetilde{C} - C'$ is an open set and $F^{-1}(\widetilde{C} - C')$ is an open set containing A. Let us consider the totality of all closed cells contained in $F^{-1}(\widetilde{C} - C')$; this is a closed subpolytope Q of $N(U)$, and because $\{U\}$ is canonical it is easily verified that no point of A can be a limit point of $N(U) - Q$; furthermore, $A \cup Q$ is a closed subset of $A \cup N(U)$.

Let $r(l) = l/\|l\|$; then taking $rF \mid (A \cup Q)$ we observe that this is an extension of $f: A \longrightarrow C$ over the closed set $A \cup Q$, with values in C. We shall now extend $rF \mid (A \cup Q)$ over $N(U) - Q$ with values in C; this is the desired extension of f.

Define

$$\begin{aligned} \phi_0(p) &= rF(p) && (p \text{ a vertex of } N(U) - Q), \\ &= rF(x) && (x \in A \cup Q). \end{aligned}$$

Then ϕ_0 is an extension of $rF|(A \cup Q)$ over the vertices of $N(U) - Q$ with values in C; the continuity is evident since we have $rF(p) = F(p)$ for all vertices, and since F is continuous.

We proceed by induction. Let ϕ_n be an extension of ϕ_{n-1} over all $A \cup Q \cup$ [n-cells of $N(U) - Q$], with values in C. We construct ϕ_{n+1} as follows: for any $(n+1)$-cell of $N(U) - Q$, we have $\phi_n(\beta\bar{\sigma}^{n+1}) \subset C$; applying Lemma 6.1, we obtain an extension $\phi_{n+1}: \bar{\sigma}^{n+1} \longrightarrow C$; extending over every $n+1$-cell, with values in C, we obtain ϕ_{n+1}. Now, ϕ_{n+1} is continuous, in virtue of 2.23 and because no point of A is a limit point of $N(U) - Q$. Defining

$$\phi(x) = \lim_n \phi_n(x)$$

for each $x \in A \cup N(U)$, we observe that ϕ is continuous; further, ϕ is an extension with values in C of $rF|(A \cup Q)$, and hence of $f: A \longrightarrow C$. This proves the assertion.

6.3 THEOREM. *Let L be a normed linear space, and $S = \{x \mid \|x\| \leq 1\}$. A necessary and sufficient condition that every continuous $f: S \longrightarrow S$ have a fixed point is that S be compact.*

Proof. If S is compact, the result comes from Tychonoff's Theorem [13]. If S is not compact, it follows readily that $C: \{x \mid \|x\| = 1\}$ is not compact either. Let $F: S \longrightarrow C$ be an extension of the identity map $I: C \longrightarrow C$ (6.2 Theorem). Setting $\phi(x) = -F(x)$, we see that ϕ has no fixed point.

In particular (Banach, [2, p. 84]) this proves that the Brouwer fixed-point theorem for the unit sphere of any infinite dimensional Banach space is not true. This is a partial answer to a question of Kakutani [6] who showed that in the Hilbert space a fixed-point free map of the unit sphere in itself can in fact be selected to to be a homeomorphism.

6.4 COROLLARY. *Let L be a normed linear space with noncompact*

$$C: \{x \mid \|x\| = 1\}.$$

Then C is contractible on itself to a point.

Proof. Form the metric space $C \times I$, I the unit interval, and map $C \times 0$ by the identity, $C \times 1$ by a constant map. Since C is an absolute retract, the map on $C \times 0 \cup C \times 1 \subset C \times I$ extends to a $\phi: C \times I \longrightarrow C$, and this ϕ gives the required deformation.

7. Application to a generalization of the theory of locally connected spaces. For our final application, we show that the entire theory of locally connected spaces can be extended to arbitrary metric spaces. In this development, as in that for the separable metric spaces (Fox [5]), the role of the Hilbert cube in the classical theory is taken over by a whole class of "universal" spaces. Kuratowski [8] has shown that any metric space Z can be embedded in the Banach space $C(Z)$ of all bounded continuous real-valued functions on Z. Subsequently, Wojdyslawski [15] has pointed out that, in the Kuratowski embedding of $Z \longrightarrow C(Z)$, Z is a closed subset of its convex hull $H(Z)$. The "universal" spaces in our development are the convex sets in Banach spaces. We shall illustrate the technique by proving a theorem (7.5) about "factorization" of mappings into absolute nbd retracts.

If A is a subset of X, A is called a *retract* of X if there exists a continuous $r: X \longrightarrow A$ such that $r(a) = a$ for each $a \in A$; if X is a Hausdorff space, it follows that a retract of X is closed in X. Now we prove the following result.

7.1 THEOREM. *The following two properties of a metric space Y are equivalent:*

7.11 *In every metric space $Z \supset Y$ in which Y is closed, there is a* nbd *$V \supset Y$ of which Y is a retract.*

7.12 *If X is any metric space, A a closed subset of X, and $f: A \longrightarrow Y$, there exists a* nbd *$W \supset A$ and an extension $F: W \longrightarrow Y$ of f.*

Proof. We need only prove that 7.11 implies 7.12, the converse implication being trivial. Let Y be embedded in $H(Y)$ as a closed subset. By Corollary 4.2, we get an extension of $f: A \longrightarrow Y$ to $F: X \longrightarrow H(Y)$. Let V be a nbd of Y in $H(Y)$ which retracts onto Y, and r the retracting function. Then $F^{-1}(V) = W$ is open in X and contains A, and $rF: W \longrightarrow Y$ is an extension of f.

A metric space Y with the properties 7.11, 7.12 is called an *absolute nbd retract*, abbreviated ANR. They are thus characterized as nbd retracts of the set $H(Y)$ in $C(Y)$.

7.2 LEMMA. *Let Y be an ANR. Then given any covering $\{U\}$ of Y, there exists a refinement $\{W\}$ with the property: If X is any metric space and f_0, $f_1: X \longrightarrow Y$ are such that $f_0(x)$, $f_1(x)$ lie in a common set of $\{W\}$ for each $x \in X$, then f_0 is homotopic to f_1, and the homotopy $\phi(x,t)$, $0 \leq t \leq 1$, can be selected so that $\phi(x,I) \subset$ some U for each $x \in X$, where I denotes the* *unit interval.*

Proof. We consider Y embedded in $H(Y) \subset C(Y)$. Since Y is closed in $H(Y)$, and Y is an ANR, there is retraction r of a nbd $V \supset Y$ in $H(Y)$ onto Y. To simplify the terminology, we let a spherical nbd of $y \in H(Y)$ be the intersection of a spherical nbd of y in $C(Y)$ with $H(y)$. For each $y \in Y$, select a spherical nbd $S(y)$ in $H(y)$ such that $S(y) \subset V$ and $S(y) \cap Y \subset$ some U. Finally, for each y, select a spherical nbd $T(y) \subset S(y)$ in $H(Y)$ such that $r[T(y)] \subset S(y)$. The desired covering is $\{T(y) \cap Y\}$; it clearly refines $\{U\}$. If $f_0, f_1 : X \longrightarrow Y$ and $f_0(x), f_1(x)$ are in a common $T(y) \cap Y$ for each x, they can be joined by a line segment that lies in $T(y)$ and therefore lies in V. Letting $\phi(x,t)$ be the point $tf_0(x) + (1-t)f_1(x)$, we see that $r\phi(x,t)$, $0 \leq t \leq 1$, gives the required homotopy.

It is not known whether this property implies that Y is an ANR. It does follow readily, however, from 7.2, that an ANR is locally contractible. The theorem also holds for LC^n metric spaces, provided $\dim X \leq n$; the property is in fact equivalent to LC^n. It should be noted that Lemma 7.2 holds also if X is any CW polytope, since then ϕ is still continuous (Whitehead [14]).

Our second lemma requires the following definition (Lefschetz [11]): Let Y be a space, and $\{U\}$ a covering of Y. Let P be a CW polytope, and Q a subpolytope of P containing all the vertices of P. An $f: Q \longrightarrow Y$ is called a *partial realization of* P *relative to* $\{U\}$ if, for every cell $\sigma \subset P$, we have $f(Q \cap \bar{\sigma}) \subset$ some U.

7.3 LEMMA. *Let Y be an ANR. Then given any covering $\{U\}$ of Y, there exists a refinement $\{V\}$ with the property that any partial realization of any CW polytope P relative to $\{V\}$ extends to a full realization of P relative to $\{U\}$.*

The proof given by Lefschetz [11, 10.2, p. 89] can easily be applied to yield this result, after a preliminary embedding of Y in $H(Y)$. This property is in fact equivalent to ANR; when we restrict P so that $\dim P \leq n+1$, this property characterizes the LC^n spaces.

The final lemma required is a covering lemma.

7.4 LEMMA. *Let Y be a metric space, and $\{U\}$ a covering of Y. There exists a refinement $\{V\}$ of $\{U\}$ with the property that whenever $\bigcap_\alpha V_\alpha \neq 0$, then $\bigcup_\alpha V_\alpha \subset$ some U. The covering $\{V\}$ is called a barycentric refinement of $\{U\}$* (cf. also Dowker [4]).

Proof. Let $\{U'\}$ be a locally finite refinement of $\{U\}$, and $N(U')$ the nerve of $\{U'\}$, K the barycentric mapping (2.31) $K: Y \longrightarrow N(U')$. Let N' be the barycentric subdivision of the polytope $N(U')$ and $\{p'\}$ its vertices. We take stars in N' (the CW topology of N is a subdivision invariant); then the open sets $V = K^{-1}$ (star p') form the required covering.

We now prove the "factorization" theorem:

7.5 THEOREM. *Let Y be an ANR; then there exists a polytope P and a continuous $g: P \longrightarrow Y$ with the property that, if X is any metric space, and $f: X \longrightarrow Y$, there exists a $\mu: X \longrightarrow P$ such that $g\mu$ is homotopic to f.*

Proof. Let us take the covering of Y by Y alone, and obtain a refinement $\{W\}$ satisfying Lemma 7.2. Let $\{V'\}$ be a refinement of $\{W\}$ satisfying Lemma 7.3 relative to $\{W\}$, and $\{V\}$ a locally finite refinement of a barycentric refinement of $\{V'\}$. We now construct a mapping $g: N(V) \longrightarrow Y$, as follows: if p_v is the vertex of $N(V)$ corresponding to $V \in \{V\}$, select $y_v \in V$ and set $g(p_v) = y_v$. This is clearly a partial realization of $N(V)$. If $(p_{v_1}, \cdots, p_{v_n})$ is a cell of $N(V)$, then $V_1 \cap \cdots \cap V_n \neq 0$ so that $\bigcup_{i=1}^n V_i \subset$ some V'; thus all vertices are sent into a set of V'. Hence (7.3), the mapping g extends to a $g: N(V) \longrightarrow Y$. This map g and polytope $N(V)$ are those required.

Now, for any metric space X and $f: X \longrightarrow Y$, construct the covering $\{f^{-1}(V)\}$ of X, and let $\{U\}$ be a barycentric nbd-finite refinement of $\{f^{-1}(V)\}$. We take $K: X \longrightarrow N(U)$ and define $g': N(U) \longrightarrow Y$ as follows: if p_U is a vertex of $N(U)$, select $x_U \in U$ and set $g'(p_U) = f(x_U)$.

Again, as before, g' extends to a mapping of $N(U)$ into Y.

We shall first show that f is homotopic to $g'K$ by showing that for each $x, f(x)$ and $g'K(x)$ are in a common W (7.2). If $x \in U_1 \cap \cdots \cap U_n$ and $x \in$ only these sets, then $K(x) \in (p_{U_1}, \cdots, p_{U_n})$; since $g'(p_{U_i}) = f(x_{U_i}) \in f(U_i)$ we have $\bigcup_{i=1}^n g'(p_{U_i}) \subset \bigcup_{i=1}^n f(U_i) \subset V$, so that $g'K(x)$ is in some $W \supset V$. On the other hand, $f(x) \in f(U_1 \cap \cdots \cap U_n) \subset \bigcup_{i=1}^n f(U_i) \subset V$ also; this shows that $g'K(x)$ and $f(x)$ are in a common set W for each x, and hence are homotopic.

Next, we map $N(U)$ into $N(V)$ simplicially as follows: if p_U is a vertex of $N(U)$, select some V with $U \subset f^{-1}(V)$ and set $\pi(p_U) = p_V$. It is easy to verify that π is simplicial. Extending linearly, we have $\pi: N(U) \longrightarrow N(V)$. Again it is simple to verify that $g\pi(x)$ and $g'(x)$ are in a common set W for every $x \in N(U)$, and hence are homotopic.

Thus we see that f is homotopic to $g\pi K$, so that, with $\pi K = \mu$, the theorem is proved.

The property is not known to be equivalent with ANR. The theorem also holds for LC^n spaces, if $\dim X \leq n$; the polytope P can be chosen so that $\dim P \leq n$ in this case. We have the trivial consequence:

7.6 COROLLARY. *If Y is an ANR, and P is the polytope of the theorem, then the continuous homology groups of Y are direct summands of the corresponding groups of P.*

Proof. By taking $X = Y$ and $i: Y \longrightarrow Y$ the identity map, we have i homotopic to $g\mu$; hence, for each n, the homorphism $H_n(Y) \longrightarrow H_n(Y)$ induced by $g\mu$ is the identity automorphism. The result now follows from the trivial group theoretic result:

7.7 THEOREM. *If A, B are two abelian groups and $\mu: A \longrightarrow B$, $g: B \longrightarrow A$ homomorphisms such that $g\mu(a) = a$ for each $a \in A$, then A is isomorphic to a direct summand of B.*

Proof. Since $g\mu(a) = a$ for every $a \in A$, it follows at once that $\mu A \longrightarrow B$ is an isomorphism into. Furthermore, $\mu(A)$ is a retract of B. In fact, defining $r = \mu g$ we see that $r: B \longrightarrow \mu(A)$; further, for each $b = \mu(a)$, we have $r(b) = \mu g \mu(a) = \mu(a) = b$. Since $\mu(A)$ is a retract of B, it is a direct summand of B, and $B = \mu(A) \oplus \text{Kernel } \mu g$.

In the case that Y is a compactum, all coverings involved can be chosen finite, and 7.6 yields known results (Lefschetz [11; p. 109]). If the Y is a separable metric ANR, the coverings can be so chosen (Kaplan [7]) so that the polytope P is a locally finite one.

It should further be remarked that the method of proof used in Theorem 6.2 is a completely general procedure to prove that an ANR which is connected in all dimensions is in fact an absolute retract.

REFERENCES

1. P. Alexandroff and H. Hopf, *Topologie I.*, Springer, Berlin, 1935.

2. S. Banach, *Theòrie des opérations linéaires*, Hafner, Lwow, 1932.

3. K. Borsuk, *Über Isomorphie der Funktionalräume*, Bull. Acad. Polonaise, 1933, 1-10.

4. C. H. Dowker, *An extension of Alexandroff's mapping theorem*, Bull. Amer. Math. Soc. 54 (1945), 386-391.

5. R. H. Fox, *A characterization of absolute nbd retracts*, Bull. Amer. Math. Soc. 48 (1942), 271-275.

6. S. Kakutani, *Topological properties of the unit sphere of a Hilbert space*, Proc. Imp. Acad. Tokyo 19 (1943), 269-271.

7. S. Kaplan, *Homology properties of arbitrary subsets of Euclidean spaces*, Trans. Amer. Math. Soc. 62 (1947), 248-271.

8. C. Kuratowski, *Quelques problèmes concernant les espaces metriques non-separables*, Fund. Math. 25 (1935), 543.

9. ———, *Prolongement des fonctions continues et les transformations en polytopes*, Fund. Math. 24 (1939), 259-268.

10. S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27; American Mathematical Society, New York, 1942.

11. ———, *Topics in topology*, Princeton University Press, Princeton, 1942.

12. A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. 54 (1948), 969-977.

13. A. Tychonoff, *Ein Fixpunktsatz*, Math. Ann. 111 (1935), 767-776.

14. J. H. C. Whitehead, *Combinatorial homotopy I.*, Bull. Amer. Math. Soc. 55 (1949),

15. M. Wojdyslawski, *Retractes absolus et hyperespaces des continus*, Fund. Math. 32 (1939), 184-192.

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