VON NEUMANN DIFFERENCE APPROXIMATION TO HYPERBOLIC EQUATIONS

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1. Introduction.

Consider the finite difference equation

$$v_{t\bar{t}} = v_{x\bar{x}} + \alpha k^2 v_{x\bar{x}t\bar{t}} \tag{1.1}$$

(1.[1])

devised by von Neumann for the numerical solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \,. \tag{1.2}$$

An essential property of any finite difference approximation to a differential equation is the convergence of solutions of the difference equation to the solution of the corresponding differential equation as $h, k \to 0$. It turns out that a sufficient condition for a difference approximation to be convergent is that it be consistent and stable (cf. Lax and Richtmyer [3]). Roughly speaking, a difference approximation to a differential equation is *consistent* when the difference equation converges to the differential equation, and it is *stable* when the solutions of the difference equation can be estimated (in a suitable norm) in terms of the prescribed data. A finite difference approximation to a hyperbolic differential equation is said to be *unconditionally stable* when it is stable for *all* positive values of the mesh ratio $R = k/h$; it is said to be *conditionally stable* when it is stable for some values of R, but not unconditionally stable.

O'Brien, Hyman and Kaplan [5] determined completely the stability properties of the difference equation (1.1). They showed that (1.1) is unconditionally stable when $\alpha \geq 1/4$, and conditionally stable when $\alpha < 1/4$ (the mesh ratio limitation being $R \leq (1 - 4\alpha)^{-\frac{1}{2}}$). This reduces to a classical result of Courant, Friedrichs and Lewy [1] when $\alpha = 0$.

It is well known that the *implicit backward* difference approximation

$$v_{\bar{t}\bar{t}} = v_{x\bar{x}} \tag{1.3}$$

to the wave equation (1.2) is unconditionally stable (cf. [5]). However, the unconditionally stable approximation (1.3) involves an error of approximation which is of order $k + k^2$, while the conditionally stable approximation

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[1] See section 2 for notation.

(1.4) $$v_{t\bar{t}} = v_{x\bar{x}}$$

leads to an error of order $k^2 + h^2$. The error incurred when the wave equation (1.2) is approximated by the von Neumann difference equation (1.1) is of order $k^2 + h^2$ for all values of the parameter α.

The difference approximation (1.4) was extended to a difference analogue of the hyperbolic differential equation

(1.5) $$\frac{\partial^2 u}{\partial t^2} = a(x, t)\frac{\partial^2 u}{\partial x^2} + b(x, t)\frac{\partial u}{\partial x} + c(x, t)\frac{\partial u}{\partial t} + d(x, t)u + e(x, t)$$

by Courant, Friedrichs and Lewy, and shown to be conditionally stable, the mesh ratio limitation being $1 - aR^2 > 0$. The implicit backward difference approximation (1.3) has been extended to difference analogues of equations of the form (1.5) by Lees [4], and shown to be unconditionally stable. Both extensions preserve the order of magnitude of the error of approximation.

There are two natural extensions of the von Neumann difference equation (1.1) to a difference analogue of the hyperbolic differential equation (1.5). We consider both of these extensions of (1.1), and give sufficient conditions that they be unconditionally stable. Both of these extensions lead to an error of approximation of order $k^2 + h^2$ for all values of the parameter α. Unfortunately, our method of proving stability gives no information about the conditional stability of the von Neumann difference approximations to (1.5).

We establish the stability theorems by showing that the solutions of the von Neumann difference approximations to (1.5) satisfy an *energy inequality* similar to the classical energy inequality of Friedrichs and Lewy [2]. It is the energy inequality which allows us to handle differential equations with variable coefficients; the case of constant coefficients can be treated by Fourier analysis (cf. [3]).

2. Preliminary remarks. Let $\bar{\Omega}_h$ be a rectangular lattice with mesh widths h and $k = Rh$ fitted to the region

$$\bar{\Omega}: 0 \leq x \leq 1, 0 \leq t \leq T .$$

More precisely, $\bar{\Omega}_h$ is the set of all points of intersection of the coordinate lines

$$x = nh, n = 0, 1, \cdots, N ,$$
$$t = mk, m = 0, 1, \cdots, M ,$$

where $Nh = 1$ and $Mk = T$. The quantity R is called the *mesh ratio* of the lattice.

Let

$$\Delta^i(m, l) = \{(x, t) \in \bar{\Omega}_h \,|\, x = ih \text{ and } mk \leq t \leq lk\}$$

and put

$$\Omega_h = \bigcup_{i=1}^{N-1} \Delta^i(k, M - k) \ .$$

Denote by $T_{\pm nh}$ and $T_{\pm mk}$ the translation operators defined as follows:

$$T_{\pm nh} v(x, t) = v(x \pm nh, t) \ ,$$
$$T_{\pm mk} v(x, t) = v(x, t \pm mk) \ .$$

For the first order partial difference quotients of functions $v(x, t)$ we employ the following notation:

$$v_x = \frac{1}{h}(T_h - 1)v, \qquad v_t = \frac{1}{k}(T_k - 1)v \ ,$$

$$v_{\bar{x}} = \frac{1}{h}(1 - T_{-h})v, \qquad v_{\bar{t}} = \frac{1}{k}(1 - T_{-k})v \ ,$$

$$v_{\hat{x}} = \frac{1}{2h}(T_h - T_{-h})v, \qquad v_{\hat{t}} = \frac{1}{2k}(T_k - T_{-k})v \ .$$

Difference quotients of order higher than the first are fo
peated application of the above formulas, for example,

$$v_{x\bar{x}} = \frac{1}{h^2}(T_h - 2 + T_{-h})v \ .$$

We shall use von Neumann's finite difference method to approximate the sufficiently smooth solutions of the following mixed initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = a(x, t)\frac{\partial^2 u}{\partial x^2} + b(x, t)\frac{\partial u}{\partial x} + c(x, t)\frac{\partial u}{\partial t} + d(x, t)u + e(x, t), \tag{2.1}$$
$$(0 < x < 1, 0 < t \leq T) \ ,$$

$$\begin{aligned} u(x, 0) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= g(x) \\ u(0, t) &= h_0(t) \\ u(1, t) &= h_1(t) \ . \end{aligned} \tag{2.2}$$

We assume that the functions b, c, d and e are continuous in $\bar{\Omega}$, and that there exist constants c_i, $(i = 0, 1, 2, 3)$ such that

$$0 < c_0 \leq a(x, t) \leq c_1 \text{ in } \bar{\Omega} \ , \tag{2.3}$$

$$|a(x, t) - a(\bar{x}, \bar{t})| \leq c_2|x - \bar{x}| + c_3|t - \bar{t}| \; . \tag{2.4}$$

We also assume that the functions g, h_0 and h_1 are continuous and that f is twice continuously differentiable.

The first finite difference approximation of (2.1) is

$$v_{t\bar{t}} = a(x, t)v_{x\bar{x}} + \alpha k^2 v_{x\bar{x}t\bar{t}} + b(x, t)v_{\hat{x}} + c(x, t)v_{\hat{t}} + d(x, t)v + e(x, t) \; , \tag{2.5}$$

where α is a parameter to be specified later.

The initial and boundary conditions (2.2) are approximated as follows:

$$\begin{aligned} v(x, 0) &= f(x) \\ v(x, k) &= f(x) + kg(x) + \frac{k^2}{2}[a(x, 0)f''(x) + b(x, 0)f'(x) \\ &\qquad + c(x, 0)g(x) + d(x, 0)f(x) + e(x, 0)]^{[2]} \\ v(0, t) &= h_0(t) \\ v(1, t) &= h_1(t) \; . \end{aligned} \tag{2.6}$$

As a second approximation to (2.1), we take

$$\begin{aligned} v_{t\bar{t}} = a(x, t)v_{x\bar{x}} &+ a(x, t)\alpha k^2 v_{x\bar{x}t\bar{t}} + b(x, t)v_{\hat{x}} \\ &+ c(x, t)v_{\hat{t}} + d(x, t)v + e(x, t) \; . \end{aligned} \tag{2.7}$$

Both of these difference equations are of the implicit type when $\alpha > 0$, i.e., their solution, subject to the auxilliary conditions (2.6), requires the inversion of a linear system of $(N - 1)$ algebraic equations in the same number of unknowns at each time step.

3. **Energy inequalities.** In this section, we derive sufficient conditions for the solutions of the difference equations (2.5) and (2.7) to satisfy an energy inequality. As a corollary of the energy inequalities, we prove the existence of a unique solution to the systems (2.5), (2.6), and (2.7), (2.6).

Before giving the energy inequalities, we prove several preliminary results.

LEMMA 1. *The function* $E(t) = (1 + \beta k)^{-t/k}$, $(\beta > 0)$ *satisfies the following conditions*:

(i) $E_{\bar{t}} + \beta E = 0$,

(ii) $T_k E = (1 + \beta k)^{-1}E \leq E$,

(iii) $E^{-1}(t) \leq e^{\beta t}$.

[2] This approximation is taken to insure that the approximation error is of uniform order of magnitude over the region $\bar{\Omega}_h$.

Proof. Properties (i) and (ii) are readily verified, and property (iii) follows by exponentiating the inequality

$$t/k \log(1+\beta k) \leq \beta t \ .$$

The next three lemmas give finite difference analogues of certain differential identities employed in the proof of the classical energy inequality of Friedrichs and Lewy.

LEMMA 2. *If $E(t) = (1+\beta k)^{-t/k}$, $(\beta > 0)$, and $v(x, t)$ is any function defined on $\bar{\Omega}_h$, then the following identity holds:*

$$Ev_{\hat{t}}v_{t\bar{t}} = \tfrac{1}{2}[(T_k E)v_{\hat{t}}^2]_{\bar{t}} + \tfrac{1}{2}\beta(T_k E)v_{\hat{t}}^2 \tag{3.1}$$

Proof. We have that

$$[v_t^2]_{\bar{t}} = v_t v_{t\bar{t}} + (T_{-k}v_t)v_{t\bar{t}} = 2v_t v_{t\bar{t}} - (v_t - T_{-k}v_t)v_{t\bar{t}} = 2v_t v_{t\bar{t}} - k(v_{t\bar{t}})^2 \ .$$

Similarly,

$$[v_{\bar{t}}^2]_{\bar{t}} = 2v_{\bar{t}}v_{t\bar{t}} + k(v_{t\bar{t}})^2 \ .$$

Hence,

$$v_{\hat{t}}v_{t\bar{t}} = \tfrac{1}{2}[v_{\hat{t}}^2]_{\bar{t}} \ .$$

Therefore

$$Ev_{\hat{t}}v_{t\bar{t}} = \tfrac{1}{2}[(T_k E)v_{\hat{t}}^2] - \tfrac{1}{2}(T_k E)_{\bar{t}}v_{\hat{t}}^2$$

which reduces to (3.1) in view of property (i) of Lemma 1.

LEMMA 3. *Let $E(t) = (1+\beta k)^{-t/k}$, and let $v(x, t)$ be any function defined on $\bar{\Omega}_h$. Then the following identity holds:*

$$\begin{aligned} Ev_{\hat{t}}av_{x\bar{x}} = {} & (v_{\hat{t}}Eav_x)_{\bar{x}} - v_{\bar{x}}Ea_{\bar{x}}(T_{-h}v_{\hat{t}}) \\ & - \frac{1}{4}[(T_{-k}Ea)v_{\bar{x}}^2]_t - \frac{1}{4}[(T_k Ea)v_{\bar{x}}^2]_{\bar{t}} \\ & + \frac{1}{2}(Ea)_{\hat{t}}v_{\bar{x}}^2 + \frac{k^2}{4}[(T_k Ea)v_{\bar{x}t}^2]_{\bar{t}} - \frac{k^2}{4}(Ea)_t v_{\bar{x}t}^2 \ . \end{aligned} \tag{3.2}$$

Proof. We have

$$\begin{aligned} Ev_t av_{x\bar{x}} &= (v_t Eav_x)_{\bar{x}} - (v_t Ea)_{\bar{x}}v_{\bar{x}} \\ &= (v_t Eav_x)_{\bar{x}} - Eav_{\bar{x}}v_{\bar{x}t} - (T_{-k}v_t)v_{\bar{x}}Ea_{\bar{x}} \\ &= (v_t Eav_x)_{\bar{x}} - (T_{-h}v_t)v_{\bar{x}}Ea_{\bar{x}} - \tfrac{1}{2}[(T_{-k}Ea)v_{\bar{x}}^2]_t \\ &\quad + \frac{1}{2}(Ea)_{\bar{t}}v_{\bar{x}}^2 + \frac{k}{2}Eav_{\bar{x}t}^2 \ . \end{aligned}$$

Combining this identity with a similar representation for $Ev_{\bar{t}}av_{x\bar{x}}$ we obtain (3.2).

LEMMA 4. *Let $E(t) = (1 + \beta k)^{-t/k}$, and let $v(x, t)$ be any function defined on $\bar{\Omega}_h$. Then the following identity holds:*

$$(3.3) \quad Ev_{\bar{t}}av_{x\bar{x}t\bar{t}} = (Ev_{\hat{t}}av_{xt\bar{t}})_{\bar{x}} - Ea_{\bar{x}}(T_{-h}v_{\hat{t}})v_{\bar{x}t\bar{t}} - \tfrac{1}{2}[(T_kEa)v_{\bar{x}t}^2]_{\bar{t}} + \tfrac{1}{2}(Ea)_tv_{\bar{x}t}^2 .$$

The proof of this lemma is similar to the proof of Lemma 3, and will be omitted.

In order to present the energy inequalities in a convenient form, we introduce several norms. If $v(x, t)$ is defined on $\bar{\Omega}_h$, then

$$||v||_{0,t}^2 = h\sum_{n=1}^{N} v^2(nh, t)$$

$$^*||v||_{0,t}^2 = h\sum_{n=1}^{N-1} v^2(nh, t)$$

$$||v||_{1,t}^2 = ||v_{\bar{t}}||_{0,t}^2 + \tfrac{1}{2}\{||v_{\bar{x}}||_{0,t}^2 + ||v_{\bar{x}}||_{0,t-k}^2\} .$$

For functions defined on $\bar{\Omega}$ we introduce the maximum norm

$$|v|_{\bar{\Omega}} = \max_{\bar{\Omega}} |v(x, t)| .$$

THEOREM 1. (Energy Inequality) *Let $v(x, t)$ be a solution of the difference equation* (2.5) *in Ω_h. Assume that $v(x, t)$ vanishes $\Delta^0(0, M)$ and $\Delta^N(0, M)$. If the quantity*

$$(3.5) \qquad 4\alpha - a(x, t)$$

is bounded away from zero in $\bar{\Omega}$, then there exists a constant c depending only on α, T, c_i, $(i = 0, 1, 2, 3)$ and the coefficients of (2.1) *such that for all sufficiently small k*

$$(3.6) \qquad ||v||_{1,t}^2 \leq c[||v||_{1,k}^2 + k\sum_{\tau=k}^{t-k} {}^*||e||_{0,\tau}^2] .$$

Proof.[3] We have

$$hk\sum_{\Omega_h} Ev_{\hat{t}}[v_{t\bar{t}} - av_{x\bar{x}} - \alpha k^2v_{x\bar{x}t\bar{t}} - bv_{\hat{x}} - cv_{\hat{t}} - dv - e] = 0 .$$

Let $\Delta_h = \Omega_h \cup \Delta^N(k, M - k)$. Since v vanishes on $\Delta^N(0, M)$, we can write the preceding equation in the form

$$(3.7) \qquad hk\sum_{\Lambda_h} Ev_{\hat{t}}[v_{t\bar{t}} - av_{x\bar{x}} - \alpha k^2v_{x\bar{x}t\bar{t}}] = \tfrac{1}{2}B(v) ,$$

where

[3] The basic idea in the proof is made more transparent by taking $b = c = d = e = 0$.

$$(3.8) \qquad B(v) = 2hk\sum_{\Omega_h} Ev_{\hat{t}}[bv_{\hat{x}} + cv_{\hat{t}} + dv + e] \ .$$

Using the identities of Lemmas 2, 3 and 4[4], we can write (3.7) as

$$(3.9) \qquad \begin{aligned} hk\sum_{\Lambda_h} [(T_kE)v_{\bar{t}}^2]_{\bar{t}} &+ \tfrac{1}{2}[(T_{-k}Ea)v_{\bar{x}}^2]_t + \tfrac{1}{2}[(T_kEa)v_{\bar{x}}^2]_{\bar{t}} \\ &- \tfrac{1}{2}k^2[(T_kEa)v_{\bar{x}t}^2] + \alpha k^2[(T_kEa)v_{\bar{x}t}^2]_{\bar{t}} = R(v) + B(v) \ , \end{aligned}$$

where

$$(3.10) \qquad \begin{aligned} R(v) = hk\sum_{\Lambda_h} \{&-\beta(T_kE)v_{\bar{t}}^2 - 2Ea_{\bar{x}}v_{\bar{x}}(T_{-h}v_{\hat{t}}) \\ &+ (Ea)_{\hat{t}}v_{\bar{x}}^2 - \tfrac{1}{2}Eak^2v_{\bar{x}t}^2 + \alpha k^2E_tv_{\bar{x}t}^2\} \ . \end{aligned}$$

In deriving (3.9), we have used the fact that

$$\sum_{\Lambda_h} (Ev_{\hat{t}}av_x)_{\bar{x}} = \sum (Ev_{\hat{t}}v_{xt\bar{t}})_{\bar{x}} = 0$$

which follows from our assumption that $v(x, t)$ vanishes on $\Delta^0(0, M)$ and $\Delta^N(0, M)$.

Summation with respect to t yields the following formulas.

$$(3.11) \qquad hk\sum_{\Lambda_h} [(T_kE)v_{\bar{t}}^2]_{\bar{t}} = E(T)\|v_{\bar{t}}\|_{0,T}^2 - \|v_{\bar{t}}\|_{0,k}^2$$

$$(3.12) \qquad \begin{aligned} &hk\sum_{\Lambda_h} [(T_{-k}Ea)v_{\bar{x}}^2 \\ &\quad = h\sum_{x=h}^{Nh}(T_{-k}E(T)a(x, T)v_{\bar{x}}^2(x, T) - h\sum_{x=h}^{Nh} a(x, 0)v_{\bar{x}}^2(x, k) \end{aligned}$$

$$(3.13) \qquad \begin{aligned} &hk\sum_{\Lambda_h} [(T_kEa)v_{\bar{x}}^2]_{\bar{t}} \\ &\quad = h\sum_{x=h}^{Nh} E(T)a(x, T)v_{\bar{x}}^2(x, T-k) - h\sum_{x=h}^{Nh} E(k)a(x, k)v_{\bar{x}}^2(x, 0) \end{aligned}$$

$$(3.14) \qquad \begin{aligned} &hk\sum_{\Lambda_h} k^2[(T_kEa)v_{\bar{x}t}^2]_{\bar{t}} \\ &= h\sum_{x=h}^{Nh} E(T)a(x, T)[v_{\bar{x}}^2(x, T) - 2v_{\bar{x}}(x, T)v_{\bar{x}}(x, T-k) + v_{\bar{x}}^2(x, T-k)] \\ &\quad - h\sum_{x=h}^{Nh} E(k)a(x, k)[v_{\bar{x}}^2(x, k) - 2v_{\bar{x}}(x, k)v_{\bar{x}}(x, 0) + v_{\bar{x}}^2(x, 0)] \end{aligned}$$

$$(3.15) \qquad \begin{aligned} &hk\sum_{\Lambda_h} \alpha k^2[(T_kE)v_{\bar{x}t}^2]_{\bar{t}} \\ &= \alpha h\sum_{x=h}^{Nh} E(T)[v_{\bar{x}}^2(x, T) - 2v_{\bar{x}}(x, T)v_{\bar{x}}(x, T-k) + v_{\bar{x}}^2(x, T-k)] \\ &\quad - \alpha h\sum_{x=h}^{Nh} E(k)[v_{\bar{x}}^2(x, k) - 2v_{\bar{x}}(x, k)v_{\bar{x}}(x, 0) + v_{\bar{x}}^2(x, 0)] \ . \end{aligned}$$

It follows now from (3.9), (3.11)–(3.15) that

[4] In Lemma 4, we take $a \equiv 1$.

$$
\begin{aligned}
& E(T)\| v_{\bar{t}}\|^2_{0,T} + \frac{h}{2}\sum_{x=h}^{Nh}[2\alpha E(T)v_{\bar{x}}^2(x, T-k) \\
& \quad + \{2\alpha E(T) - k(E(T)a(x, T))_{\bar{t}}\} v_{\bar{x}}^2(x, T) \\
& \quad + \{2E(T)a(x, T) - 4\alpha E(T)\} v_{\bar{x}}(x, T)v_{\bar{x}}(x, T-k)] \\
& = \| v_{\bar{t}}\|^2_{0,k} + \frac{h}{2}\sum_{x=h}^{Nh}[2\alpha E(k)a(x, k)v_{\bar{x}}^2(x, 0) \\
& \quad + \{2\alpha E(k) - k(E(k)a(x, k))_{\bar{t}}\} v_{\bar{x}}^2(x, k) \\
& \quad + \{2E(k)a(x, k) - 4\alpha E(k)\} v_{\bar{x}}(x, k)v_{\bar{x}}(x, 0)] + R(v) + B(v) .
\end{aligned} \tag{3.16}
$$

Consider the real quadratic form

$$
Q(\xi, \iota) = 2\alpha E\xi^2 + (2Ea - 4\alpha E)\xi\eta + (2\alpha E - k(Ea)_{\bar{t}})\eta^2 . \tag{3.17}
$$

Now,

$$
(Ea)_{\bar{t}} = E_{\bar{t}}(T_{-k}a) + Ea_{\bar{t}} = E(-\beta(T_{-k}a) + a_{\bar{t}})
$$

by property (i) of Lemma 1. Therefore

$$
2\alpha E - k(Ea)_{\bar{t}} \geq E2\alpha \tag{3.18}
$$

if $\beta a \geq a_{\bar{t}}$.

It follows from (3.17) and (3.18) that by choosing β large enough

$$
Q(\xi, \eta) \geq E(2\alpha\xi^2 + 2\{a - 2\alpha\}\xi\eta + 2\alpha\eta^2) . \tag{3.19}
$$

In view of our assumption concerning the expression (3.5), we see that the right side of (3.19) is a positive definite quadratic form, and

$$
Q(\xi, \eta) \geq E\mu_0(\xi^2 + \eta^2) , \tag{3.20}
$$

where

$$
\mu_0 = \min [4\alpha - a, a] .
$$

Also, there is a constant μ_1 such that

$$
Q(\xi, \eta) \leq E\mu_1(\xi^2 + \eta^2) . \tag{3.21}
$$

Hence, combining (3.16), (3.20) and (3.21) we fined that

$$
\begin{aligned}
& E(T)\| v_{\bar{t}}\|^2_{0,T} + E(T)\mu_0\tfrac{1}{2}\{\| v_{\bar{x}}\|^2_{0,T} + \| v_{\bar{x}}\|^2_{0,T-k}\} \\
& \quad \leq \| v_{\bar{t}}\|^2_{0,k} + \mu_1\tfrac{1}{2}\{\| v_{\bar{x}}\|^2_{0,0} + v_{\bar{x}}\|^2_{0,k}\} + R(v) + B(v) .
\end{aligned} \tag{3.22}
$$

If we shown that there is a constant B such that

$$
R(v) + B(v) \leq kB\| v_{\bar{t}}\|^2_{0,k} + k\sum_{\tau=k}^{T-k}{}^{*}\| e\|^2_{0,\tau}E(\tau) \tag{3.23}
$$

for all sufficiently small k, then (3.6) will follow.

It is readily verified that

$$\begin{aligned} B(v) \leq hk\sum_{\Omega_h} E(|b|_{\bar{\Omega}} + |c|_{\bar{\Omega}} + |d|_{\bar{\Omega}} + 1)v_t^2 \\ + hk\sum_{\Omega_h} E(|b|_{\bar{\Omega}} + |d|_{\bar{\Omega}})v_x^2 + k\sum_{\tau-k}^{T-k}{}^{*}\|e\|_{0,\tau}^2 E(\tau) \;. \end{aligned}$$

Since

$$\sum_{\Omega_h} Ev_x^2 \leq \sum Ev_{\bar{x}}^2$$

and

$$\sum_{\Omega_h} Ev_t^2 \leq \sum_{\Omega_h} Ev_{\bar{t}}^2 + k\frac{h}{2}\sum_{x=h}^{Nh-h} E(k)v_t^2(x, k)$$

it follows that there are constants B_1 and B_2 such that

$$\begin{aligned} B(v) \leq B_1hk\sum_{\Lambda_h} v_{\bar{x}}^2 E + B_2hk\sum_{\Omega_h} v_{\bar{t}}^2 E + B_2E(k)\frac{k}{2}\|v_{\bar{t}}\|_{0,k}^2 \\ + k\sum_{\tau=k}^{T-k}{}^{*}\|e\|_{0,\tau}^2 E(\tau) \;. \end{aligned} \tag{3.24}$$

Using (3.10), it is not difficult to show that

$$\begin{aligned} R(v) \leq \sum_{\Lambda_h} E\Big\{\frac{-\beta}{1+\beta k}v_t^2 + \left|\frac{\partial a}{\partial x}\right|_{\bar{\Omega}} v_t^2 + \left|\frac{\partial a}{\partial x}\right|_{\bar{\Omega}} v_{\bar{x}}^2 \\ + \frac{1}{2}|a|_{\bar{\Omega}}k^2 v_{\bar{x}t}^2 - \alpha k^2\frac{2}{1+\beta k}v_{\bar{x}t}^2 \\ + \Big(-\frac{\beta}{2}(T_{-k}a) + a_t - \frac{\beta}{2(1+\beta k)}(T_k a)\Big)v_{\bar{x}}^2\Big\} + kB_3\|v_{\bar{t}}\|_{0,k}^2 \end{aligned} \tag{3.25}$$

for a suitable constant B_3. It follows now from (3.24) and (3.25) that we can choose β such that for all sufficiently small k (3.23) holds. This completes the proof of Theorem 1.

Restating Theorem 1, we have

COROLLARY 1. *If* $4\alpha - a$ *is bounded away from zero in* $\bar{\Omega}$, *then the von Neumann difference equation* (2.5) *is unconditionally stable for all sufficiently small* k.

THEOREM 2. *If* $4\alpha - a$ *is bounded away from zero in* $\bar{\Omega}$, *then for all sufficiently small* k, *the finite difference equation* (2.5) *with the auxiliary conditions* (2.6) *possesses a unique solution.*

Proof. At the end of §2, we remarked that the system (2.5), (2.6) is equivalent to a system of $(N-1)(M-2)$ linear equations in the same number of unknowns. It is sufficient to prove that the associated homogeneous system of equations has only the trivial solution. The homogeneous system is obtained by putting e, $v(x, 0)$, $v(x, k)$, $v(0, t)$ and $v(1, t)$ equal to zero. From Theorem 1 we conclude that $\|v\|_{1,t} = 0$, which implies that $v = 0$ in Ω_h. Hence, the associated homogeneous system has only the trivial solution.

We now state without proof the following approximation theorem.

THEOREM 3. *Let $u(x, t)$ be of class C^4 in $\bar{\Omega}$ and satisfy the mixed initial boundary value problem* (2.1), (2.2). *Let $4\alpha - a$ be bounded away from zero in $\bar{\Omega}$ and let $v(x, t)$ denote the solution of the von Neumann approximation* (2.5), (2.6). *Then for all sufficiently small k there exists a constant B_4 independent of h and k such that*

$$\max_{\bar{\Omega}_h} |u(x, t) - v(x, t)| \leq B_4(h^2 + k^2) .$$

We now consider the finite difference equation (2.7). The preceding theorems can all be extended to this difference equation provided that we modify the range of the parameter α.

THEOREM 4. (Energy Inequality) *Let $v(x, t)$ be a solution of the difference equation* (2.7) *in Ω_h. Assume that $v(x, t)$ vanishes on $\Delta^0(0, M)$ and $\Delta^N(0, M)$. If*

$$4\alpha - 1 > 0 , \tag{3.26}$$

then there exists a constant c independent of h and k such that

$$\|v\|^2_{1,t} \leq c\left[\|v\|^2_{1,k} + k\sum_{\tau=k}^{t-k}{}^{*}\|e\|^2_{0,\tau}\right] . \tag{3.27}$$

The proof of Theorem 4 is quite similar to the proof of Theorem 1. Only slight changes in the proof are required due to the fact that we must use the full form of Lemma 4.

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