CHARACTERISTIC SUBGROUPS OF MONOMIAL GROUPS

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1. Introduction. Let U be a set, $o(U) = B = \aleph'_u$, $u \geqq 0$, where $o(U)$ means the number of elements of U. Let H be a fixed group. A monomial substitution y is a transformation that maps every x of U in a one-to-one fashion into an x of U multiplied on the left by an element h_x of H. Multiplication of substitutions means successive applications. The set of all monomial substitutions forms the monomial group Σ. Ore [5] has studied this group for finite U, and some of his results have been generalized to general U in [2], [3], and [4].

This paper determines the structure of the characteristic subgroups for the case when U is infinite in the cases where normal subgroups and automorphisms are known. The method used makes clear how corresponding theorems for the case where U is finite might be proved but does not list these results.

2. Definitions and preliminaries. Let d be the cardinal of the integers. Let B be an infinite cardinal; B^+, the successor of B; U, a set such that $o(U) = B$; and C such that $d \leqq C \leqq B^+$. Let H be a fixed group and e the identity of H. Denote by $\Sigma = \Sigma(H; B, d, C)$ the monomial group of U over H whose elements are of the form

$$(1) \qquad y = \begin{pmatrix} \cdots, & x_\varepsilon, & \cdots \\ \cdots, & h_\varepsilon x_{i_\varepsilon}, & \cdots \end{pmatrix}$$

where only a finite number of the h_ε are not e and the number of x not mapped into themselves is less than C. Any element of Σ may be written in the form

$$y = \begin{pmatrix} \cdots, & x_\varepsilon, & \cdots \\ \cdots, & h_\varepsilon x_\varepsilon, & \cdots \end{pmatrix} \begin{pmatrix} \cdots, & x_\varepsilon, & \cdots \\ \cdots, & e x_{i_\varepsilon}, & \cdots \end{pmatrix}$$

or $y = vs$ where v sends every x into itself and every h of s is e. Elements of the form of

$$v = \begin{pmatrix} \cdots, & x_\varepsilon, & \cdots \\ \cdots, & h_\varepsilon x_\varepsilon, & \cdots \end{pmatrix} = [\cdots, h_\varepsilon, \cdots]$$

are *multiplications* and all such elements form a normal subgroup, *the basis groups* $V(B, d) = V$ of Σ. The h_ε of y are called the *factors* of y. Elements of the form of s are *permutations* and all such elements form a subgroup, the *permutation group*, $S(B, C) = S$ of $\Sigma(H; B, d, C)$. Cycles

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of s will also be written as $(x_1, \cdots, x_n)$ and $(\cdots, x_{-1}, x_0, x_1, \cdots)$. Baer [1] has shown that the normal subgroups of $S(B, C)$ are the alternating group, $A = A(B, d)$, and $S(B, D)$ where $d \leqq D \leqq C$. Let E be the identity of Σ, I the identity of S.

3. Characteristic subgroups of $\Sigma(H; B, d, C)$, $d \leqq C < B^+$.

The normal subgroups of $\Sigma(H; B, d, C)$ are known [2], [3]. They are classified first as to whether or not they are contained in the basis group V.

If N is normal in Σ and $N \subset V$ its elements are multiplications with only a finite number of non-identity factors which are contained in a normal subgroup G of H. The set of all possible products of factors of all elements of N form a normal subgroup G_1 of H. The group G/G_1 is Abelian and G/G_1 is in the center of H/G_1.

If M is normal in Σ and $M \not\subset V$ then $M \cap S = P \neq E$ is a normal subgroup of S. The group $N = M \cap V$ is as above except that $G = H$. It becomes necessary to consider the cases where $P = S(B, D)$ with $d \leqq D \leqq C$ and $P = A(B, d)$. When $P = S(B, D)$ then $M = N \cup P$.

If M is normal in Σ, $M \not\subset V$, $P = A(B, d)$, $M \cap V = N$, $M/N \cong A(B, d)$ then $M = N \cup A(B, d)$.

If M is normal in Σ, $M \not\subset V$, $P = A(B, d)$, $M \cap V = N$, $M/N \ncong A(B, d)$ then $M = N \cup A(B, d) \cup L$ where L is the cyclic group generated by $[e, a](1, 2)$ with $a^2 \in G_1$, $a \notin G_1$.

The converses of these theorems are true. That is, if one starts with the proper subgroups of H and constructs N or M as above the resulting group is normal in Σ.

The automorphisms of $\Sigma(H; B, d, C)$ are known [4]. A mapping θ is an automorphism of $\Sigma(H; B, d, C)$ if and only if $\theta = T^+ I_{(s^+)} I_{(v^+)}$ where T^+, $I_{(s^+)}$, $I_{(v^+)}$ are automorphisms of Σ defined as follows. Let T be any automorphism of H. Then

$$yT^+ = vst^+ = [h_1, \cdots, h_\varepsilon, \cdots]sT^+ = [h_1^T, \cdots, h_\varepsilon^T, \cdots]s .$$

Let $s^+ \in S(B, B^+)$. Then $I_{(s^+)}$ is defined by $yI_{(s^+)} = s^+ y (s^+)^{-1}$. Let $v^+ \in V(B, B^+)$ if $C = d$, $v^+ \in V(B, d)$ if $d < C$ then $I_{(v^+)}$ is defined by $yI_{(v^+)} = v^+ y (v^+)^{-1}$.

THEOREM 1. *If N is a subgroup of $\Sigma(H; B, d, C)$ contained in the basis group then N is characteristic in Σ if and only if N is normal in Σ, (hence is as described above) and G, G_1 are characteristic in H.*

Proof. Assume N is characteristic in Σ. Then N is normal in Σ and its structure is known. Choose $\theta = T^+$ with T arbitrary in the automorphism group of H and v arbitrary in N. Then

$$\begin{aligned} v\theta &= [e, \cdots, e, e, g_{i_1}, e, \cdots, e, g_{i_n}, e, \cdots]T^+ \\ &= [e, \cdots, g_{i_1}^T, e, \cdots, e, g_{i_n}^T, e, \cdots] . \end{aligned}$$

The elements $g_{i_1}^T$ must be in G. This shows G is characteristic in H. Furthermore $g_{i_1}^T g_{i_2}^T \cdots g_{i_n}^T = (g_{i_1} \cdots g_{i_n})^T$ must be in G_1 and since $g_{i_1} \cdots g_{i_n}$ is arbitrary in G_1, G_1 is characteristic in H.

Conversely, if $N \subset V(B, d)$, N is normal in Σ, G, G_1 are characteristic in H then N is characteristic in Σ. To see this let v_1 be arbitrary in N. Then $v_1\theta = v_1 T I_{(s^+)} I_{(v^+)} = v_2 I_{(s^+)} I_{(v^+)}$. The non-identity factors of v_2 are in G and their product in G_1 by G, G_1 characteristic in H. Now $v_2 I_{(s^+)} I_{(v^+)} = (v^+)(s^+)v_2(s^+)^{-1}(v^+)^{-1}$. The effect of $I_{(s^+)}$ on v_2 is to permute the non-identity factors so $(v^+)(v_3)(v^+)^{-1}$ is now to be considered with v_3 in N. Since G is normal in H in G/G_1 is in the center of H/G_1, $(v^+)v_3(v^+)^{-1}$ will be in N.

THEOREM 2. *Let $M = N \cup P$ be a normal subgroup of $\Sigma(H; B, d, C)$, $d \leqq C < B^+$, where N is as described above, $P = S(B, D)$. Then M is characteristic in Σ if and only if G_1 is characteristic in H.*

Proof. By an argument similar to that used in Theorem 1, G_1 is characteristic in H.

Conversely, if $y = v_1 s_1$ is arbitrary in M then

$$v_1 s_1 \theta = v_1 s_1 T^+ I_{(s^+)} I_{(v^+)} = v_2 s_1 I_{(s^+)} I_{(v^+)} .$$

Since G_1 is characteristic in H, v_2 belongs to N. Now consider

$$(v^+)(s^+)v_2 s_1 (s^+)^{-1}(v^+)^{-1} = (v^+)v_3(s^+)s_1(s^+)^{-1}(v^+)^{-1} = (v^+)v_3 s_2 (v^+)^{-1}.$$

The multiplication v_3 is in N since the factors are still in H, and the product of the factors is still in G_1 since H/G_1 is Abelian. The permutation s_2 is in P since P is normal in $S(B, B^+)$. It is now convenient to consider two cases. If $C = d$ the permutation s_2 is finite and $(v^+)v_3 s_2 (v^+)^{-1} = (v^+)v_3 v_4 s_2$ where the factors of v_4 differ from the inverse of those in(v^+) in only a finite number of places. Therefore $(v^+)v_3 v_4$ will have a finite number of factors of the form $k_\varepsilon h_\varepsilon k_{i_\varepsilon}^{-1}$. If $k_\varepsilon \neq k_{i_\varepsilon}$ then $k_{i_\varepsilon} h_{i_\varepsilon} k_\alpha$, $k_{i_\varepsilon} \neq k_\alpha$, will be a factor of $(v)v_3 v_4$. Since H/G_1 is Abelian the product of the factors is in G_1. Therefore, $(v^+)v_3 v_4 s_2 = v_5 s_2$ belongs to M. If $C > d$ then (v^+), v_4 have only a finite number of non-identity factors and the same argument holds. Therefore $(v^+)v_3 v_4 s_2$ belongs to M.

THEOREM 3. *Let $M = N \cup A(B, d)$ be a normal subgroup of $\Sigma(H; B, d, C)$, $d \leqq C < B^+$. Then M is characteristic in Σ if and only if G_1 is characteristic in H.*

Proof. The argument used in the proof of Theorem 1 may be used to show that G_1 is characteristic in H if M is characteristic in Σ.

Conversely, if $y = v_1s_1$ is arbitrary in M then

$$\begin{aligned} y\theta = v_1s_1\theta &= v_1s_1T^+I_{(s^+)}I_{(v^+)} = v_2s_1I_{(s^+)}I_{(v^+)} = (v^+)(s^+)v_2s_1(s^+)^{-1}(v^+)^{-1} \\ &= (v^+)v_3(s^+)s_1(s^+)^{-1}(v^+)^{-1} = (v^+)v_3s_2(v^+)^{-1} = (v^+)v_3v_4s_2 \,. \end{aligned}$$

Now $v_2 \in N$ by G_1 characteristic in H and v_3 will be in N by H/G_1 Abelian. Since $A(B, d)$ is normal in $S(B, B^+)$, s_2 belongs to $A(B, d)$. The factors of v_4 differ from the inverse of those in v in only a finite number of places since s_2 moves only a finite number of x. Therefore, $(v^+)v_3v_4 \in N$, $s_2 \in A(B, d)$ and M is characteristic in Σ.

THEOREM 4. *Let $M_1 = N \cup A \cup L$ be a normal subgroup of $\Sigma(H;$ $B, d, C)$, $d \leqq C < B^+$. Let L be generated by $y = [e, a](1, 2)$ with $a^2 \in G_1$, $a \notin G_1$. Then M_1 is characteristic in Σ if and only if G_1 is characteristic in H, and a^T belongs to the coset aG_1 for all automorphisms T of H.*

Proof. By considering $v \in N$ and $\theta = T^+$ we see that G_1 is characteristic in H. By considering $y = [e, a]\,(1, 2)$ of M_1 and $\theta = T^+$ we see that $[e, a^T](1, 2)$ must belong to M_1. This means a^T belongs to aG.

Conversely, if $v_1s_1 \in M_1$ then

$$\begin{aligned} v_1s_1\theta &= v_1s_1T^+I_{(s^+)}I_{(v^+)} = v_2s_1I_{(s^+)}I_{(v^+)} = (v^+)(s^+)v_2s_1(s^+)^{-1}(v^+)^{-1} \\ &= (v^+)v_3(s^+)s_1(s^+)^{-1}(v^+)^{-1} = (v^+)v_3s_2(v^+)^{-1} = (v^+)v_3v_4s_2 \,. \end{aligned}$$

Now v_2s_1 is in M_1 by G_1 characteristic if the product of the factors of v_1 is in G_1 and by a^T in aG_1 if the product of the factors is in aG_1. The multiplication v_3 has only a finite number of non-identity factors because v_2 has only a finite number of non-identity factors. Since s_1 is finite, s_2 is a finite permutation and is even or odd as s_1 is even or odd. Therefore, v_4 has only a finite number of factors different from the inverse of the factors of (v^+). The factors of $(v^+)v_3v_4$ have their product in G_1 or aG_1 according as v_3 has its product in G_1 or aG_1. Therefore, if s_1 was even s_2 is even, v_1 had the product of its factors in G_1 and so does $(v^+)v_3v_4$. If s_1 was odd so is s_2 and v_1 had the product of its factors in aG_1 and so does $(v^+)v_3v_4$. That is, M_1 is characteristic.

4. Characteristic subgroups of $\Sigma_A(H; B, d, d)$. The normal subgroups of $\Sigma_A(H; B, d, d)$ are precisely those of $\Sigma(H; B, d, d)$ that are contained in $\Sigma_A(H; B, d, d)$ [**2**, p. 210]. The automorphism of $\Sigma_A(H; B, d, d)$ are those of $\Sigma(H; B, d, d)$ restricted to $\Sigma(H; B, d, d)$ [**4**, p. 84].

THEOREM 5. *Let N be a subgroup of $\Sigma_A(H; B, d, d)$ contained in the basis group. Then N is characteristic in Σ_A if and only if N is normal in Σ_A and G, G_1 are characteristic in H.*

THEOREM 6. *Let M be a subgroup of $\Sigma_A(H; B, d, d)$, $M \not\subset V(B, d)$. Then M is characteristic in Σ_A if and only if M is normal,* i.e. $M = N \cup A$, *and G_1 is characteristic in H.*

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