ON GROUPS OF EXPONENT FOUR WITH GENERATORS OF ORDER TWO

C. R. B. Wright

1. If $x, y, \cdots$ are elements of a group G, we define the *commutator* (x, y) of x and y by $(x, y) = x^{-1}y^{-1}xy$. More generally, we define *extended commutators* inductively by $(x, \cdots, y, z) = ((x, \cdots, y), z)$. In this paper we shall also be concerned with higher commutators of type $((a_1, \cdots, a_s), (b_1, \cdots, b_t), \cdots, (c_1, \cdots, c_r))$ which we denote by $(a_1, \cdots, a_s; b_1, \cdots, b_t; \cdots; c_1, \cdots, c_r)$. If we let G_i be the subgroup of G which is generated by all extended commutators of length i, (i.e., with i entries), then G_i is a characteristic subgroup of G, and the series $G = G_1 \supset G_2 \supset \cdots$ is called the *lower central series* of G.[1]

Let $G(n)$ $(n = 1, 2, \cdots)$ be the freest group of exponent 4 on n generators of order 2. That is, $G(n)$ is a group in which the fourth power of every element is the identity, 1, $G(n)$ is generated by n elements of order 2, and if H is any other group with these properties, then H is a homomorphic image of $G(n)$.

We prove $G(n)_{n+2} = 1$. For this purpose it may be assumed, since $G(n)$ is finite[2] and hence nilpotent, that $G(n)_{n+3} = 1$. Moreover, it will be enough to show $(x_1, \cdots, x_{n+2}) = 1$ for all choices of $x_1, \cdots, x_{n+2}$ from among the generators of $G(n)$.

2. Lemma 2.1. *If $x, y, \cdots, z$ are elements of order 2 in a group of exponent 4, then $(x, y)^2 = 1, (x, y, \cdots, z)^2 = 1$, and $(x, y, x) = 1$.*

Proof. Since $(x, y) = xyxy = (xy)^2$, $(x, y)^2 = 1$. By induction, $(x, y, \cdots, z)^2 = 1$, while $(y, x) = yxyx = x(x, y)x = (x, y)(x, y, x)$, so that $(x, y, x) = (y, x)^2 = 1$.

The relation $(x, y, \cdots, z)^2 = 1$ will be the justification for future substitutions and will be used without specific mention.

Theorem 2.1. $G(2)_3 = 1$.

Proof. By Lemma 2.1, if the generators of $G(2)$ are a and b, then $(a, b, a) = (b, a, a) = (a, b, b) = (b, a, b) = 1$.

3. Lemma 3.1. *If a, b and c are elements of order 2 in a group G of exponent 4, then*

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[1] For properties of commutators and the lower central series see Hall, [1], Ch. 10.

[2] See Sanov, [2], or Hall, [1], pp. 324–325.

$$(1)\qquad (a, b, c) \equiv (b, c, a)(c, a, b) \bmod G_5$$

$$(2)\qquad (a, b; c, a) = (a, c; b, a) \equiv (a, c, b, a) \bmod G_5$$

$$(3)\qquad (a, b, c, a) \equiv (b, c, a, b)(c, a, b, c) \bmod G_5 .$$

Proof. We may assume that a, b and c generate G. Now

$$abcabc = aba\,(a, c)b(b, c) = (a, b)(a, c)(a, c, b)(b, c) .$$

Thus, modulo G_5, $(abc)^2 = (a, b)(a, c)(b, c)(a, c, b)$. Hence

$1 \equiv [(a, b)(a, c)(b, c)]^2 \bmod G_5$, so that, modulo G_5,
$1 = (a, b)(a, c)(b, c)(a, b)(a, c)(b, c) = (a, b)(a, c)(a, b)(a, b; b, c)(a, c)(a, c; b, c)$,

$$(4)\qquad 1 \equiv (a, b; a, c)(a, b; b, c)(a, c; b, c) \bmod G_5 .$$

But also

$$\begin{aligned} abc &= ca(a, c)b(b, c) \\ &= bc(c, b)a(a, b)(a, c)(a, c, b)(b, c) \\ &= ab(b, a)c(c, a)(c, b)(c, b, a)(a, b)(a, c)(a, c, b)(b, c) , \end{aligned}$$

so that $1 = (b, a)(b, a, c)(c, a)(c, b)(c, b, a)(a, b)(a, c)(a, c, b)(b, c)$, and hence, modulo G_5,

$$\begin{aligned} 1 &= (b, a)(c, a)(c, b)(a, b)(a, c)(b, c)(b, a, c)(c, b, a)(a, c, b) \\ &= [(a, b)(a, c)(b, c)]^2(a, b, c)(b, c, a)(c, a, b) . \end{aligned}$$

Thus (1) is proved. Replacing b by (a, b) in (1) gives $(a, b, c, a)(c, a; a, b) \equiv 1 \bmod G_5$ or (2). And (2) and (4) together give (3).

LEMMA 3.2. *If $x_1, \cdots, x_k$ and a are elements of order 2 in a group G of exponent 4, then $(x_1, \cdots, x_k, a) \equiv X \bmod G_{k+2}$, where X is a product of commutators of form $(a, y_1, \cdots, y_k)$ with $y_1, \cdots, y_k$ from among $x_1, \cdots, x_k$.*

COROLLARY. *If $x_1, \cdots, x_k, z_1, \cdots, z_s$ and a are elements of order 2 in a group G of exponent 4, then*

$$(x_1, \cdots, x_k, a, z_1, \cdots, z_s) \equiv X \bmod G_{k+s+2}$$

where X is a product of commutators of form $(a, y_1, \cdots, y_k, z_1, \cdots, z_s)$ with $y_1, \cdots, y_k$ from among $x_1, \cdots, x_k$.

Proof of Lemma 3.2. Certainly the lemma and corollary are true if $k = 1$. Assume for induction that both are true for $k = n - 1 \geq 1$.

Now by (1), modulo G_{n+2}, $(x_1, \cdots, x_{n-1}, x_n, a) = (x_1, \cdots, x_{n-1}, a, x_n)(x_1, \cdots, x_{n-1}; a, x_n)$. But by the inductive assumption $(x_1, \cdots, x_{n-1}, a, x_n)$ is a product of terms $(a, y_1, \cdots, y_{n-1}, x_n)$, and $(x_1, \cdots, x_{n-1}; a, x_n)$ is a product of terms $(a, x_n, y_1, \cdots, y_{n-1})$. The lemma and its immediate corollary follow by induction.

THEOREM 3.1. $G(3)_5 = 1$.

Proof. Let a, b and c be the generators of $G(3)$. Consider any commutator $C = (x_1, x_2, x_3, x_4, x_5)$ in arguments a, b and c. We show $C = 1$. There is no loss of generality in taking $x_5 = a$. If a does not appear again in C, then by Theorem 2.1, $C = (1, x_5) = 1$. If a appears again, then by Lemma 3.2 and the assumption that $G(3)_6 = 1$, we may suppose $C = (a, x_2, x_3, x_4, a)$. By Lemma 2.1, if a appears a third time, then $C = 1$. Thus we may take $C = (a, b, c, b, a)$. Now $(a, b, c, b, a) = (b, c, a, b, a)(c, a, b, b, a) = (b, c, a, b, a)$ by (1). Replacing c by (b, c) in (3) gives $(a, b; b, c,; a) = (b; b, c; a; b) = 1$, while replacing c by (b, c) in (2) gives $(a, b; b, c; a) = (b, c, a, b, a)$. Hence, $C = (a, b, c, b, a) = (b, c, a, b, a) = (a, b;, b, c; a) = 1$, and the theorem is proved.

COROLLARY 1. *If a, b and c are elements of order 2 in a group of exponent 4, then*

$$(a, b, c) = (b, c, a)(c, a, b) \tag{1'}$$

$$(a, b; c, a) = (a, b, c, a) \tag{2'}$$

$$(a, b, c, a) = (b, c, a, b)(c, a, b, c) \tag{3'}$$

Proof. These follow from Lemma 3.1.

COROLLARY 2. *If $x_1, \cdots, x_k, y_1, \cdots, y_s, z_1, \cdots, z_t$ $(s \geq 2)$ are elements of order 2 in a group G of exponent 4, then*

$$(x_1, \cdots, x_k; y_1, \cdots, y_s; z_1; \cdots; z_t) \equiv AB \bmod G_{k+s+t+1}$$

where

$$A = (x_1, \cdots, x_k; y_1, \cdots, y_{s-1}; y_s; z_1; \cdots; z_t)$$
$$B = (x_1, \cdots, x_k, y_s; y_1, \cdots, y_{s-1}; z_1; \cdots; z_t) .$$

Proof. This follows from (1').

The following corollary lists some relations for future use.

COROLLARY 3. *If a, b, c, d and f are elements of order 2 in a group G of exponent 4, then*

$$(5) \qquad (a, b, c, d, c) \equiv (a, b, d, c, d) \bmod G_6$$

$$(6) \qquad (b, c, a; d, f, a) \equiv 1 \bmod G_7$$

$$(7) \qquad (a, f; b, d, c) \equiv (a, f, c; b, d)(a, f; b, d; c)$$

$$(8) \qquad (b, f, d; a, c)(d, f, b; a, c) \equiv (b, d, f; a, c) \bmod G_8 .$$

Proof. By (3′), with a replaced by (a, b) and b replaced by d, $(a, b, d, c; a, b) = (d, c; a, b; d)(c; a, b; d; c) = (a, b; d, c; d)(a, b, c, d, c)$, so that, since $(a, b; d, c; d) = (a, b, d, c, d)$, (5) is true. By (2′) and (3′) with b replaced by (b, c) and c replaced by (d, f), $(b, c, a; d, f, a) = (a; b, c; d, f; a) = (b, c; d, f; a; b, c)(d, f; b, c; a; d, f)$, so that (6) is true. Finally, (7) and (8) are obvious from (1′).

4. LEMMA 4.1. *If a, b, c and d are elements of order 2 in a group G of exponent 4, then*

$$(9) \qquad (a, b; c, d) \equiv (a, c; b, d)(a, d; b, c) \bmod G_5 .$$

Proof. First, working modulo G_5 and collecting as we did in the proof of Lemma 3.1 we obtain $(abcd)^2 = T_2T_3T_4$ where

$$T_2 = (a, b)(a, c)(b, c)(a, d)(b, d)(c, d)$$
$$T_3 = (a, c, b)(a, d, c)(a, d, b)(b, d, c)$$
$$T_4 = (a, d, b, c) .$$

Note that modulo G_5, T_2, T_3 and T_4 commute, and $T_3^2 = T_4^2 = 1$. Hence, modulo G_5, $1 = (abcd)^4 = T_2^2$. Collecting the (a, d)'s in T_2^2 we obtain $1 \equiv XABCY \bmod G_5$, where

$$X = [(a, b)(a, c)(b, c)]^2$$
$$A = (b, c; b, d)(b, c; c, d)(b, d; c, d)$$
$$B = (a, c; a, d)(a, c; c, d)(a, d; c, d)$$
$$C = (a, b; a, d)(a, b; b, d)(a, d; b, d)$$
$$Y = (a, b; c, d)(a, c; b, d)(a, d; b, c) .$$

Now modulo G_5, $X = 1$, while $A = B = C = 1$ by (2′) and (3′). Hence, $1 \equiv (a, b; c, d)(a, c; b, d)(a, d; b, c) \bmod G_5$, which is (9).

COROLLARY 1. *If $x_1, \cdots, x_k$ and a are elements of order 2 in a group G of exponent 4, then for $i = 2, \cdots, k$,*

$$(x_1, a, x_2, a, \cdots, x_i, \cdots, x_k) \equiv (x_1, x_2, \cdots, a, x_i, a, \cdots, x_k) \bmod G_{k+3} .$$

Hence, if two of $x_1, \cdots, x_k, a$ are equal, $(x_1, a, x_2, a, \cdots, x_k) \equiv 1 \bmod G_{k+3}$.

Proof. Let a, b, c and d be elements of order 2 in G. Then modulo G_6,

$$\begin{aligned}(b, a, c, a, d) &= (b, a;, c, a; d)\\ &= (b, a, d; c, a)(c, a, d; b, a)\\ &= (b, a, c; d, a)(c, a, b; d, a)\\ &= (b, c, a; d, a)\\ &= (b, c, a, d, a)\ .\end{aligned}$$

The first statement follows. Now the second statement is clearly true if a appears a third time, since then $(x_1, a, x_2, a, \cdots, a, \cdots, x_k) = (x_1, x_2, \cdots, a, a, a, \cdots, x_k) = 1$. If some x_i appears twice, then modulo $G_{k+3}(x_1, a, x_2, a, \cdots, x_i, \cdots, x_k) = (x_1, \cdots, a, x_i, a, \cdots, x_k) = (x_1, x_2, \cdots, x_i, a, x_i, \cdots, x_k) = (x_1, x_i\, x_2, x_i, \cdots, a, \cdots, x_k)$ (the second step following from (5)), and we are back to the case of three appearances of a. Thus the corollary is proved.

COROLLARY 2. *If a, b, c, d and f are elements of order 2 in a group G of exponent 4, then*

$$1 \equiv (a, f, b; c, d)(a, f, c; b, d)(a, f, d; b, c) \bmod G_6 \tag{10}$$

$$(a, c; d, f; b)(a, d; c, f; b) \equiv (c, d; a, f; b) \bmod G_6\ . \tag{11}$$

Proof. These follow from (9).

THEOREM 4.1. $G(4)_6 = 1$.

Proof. Let the generators of $G(4)$ be a, b, c and d and consider any commutator $C = (x_1, x_2, x_3, x_4, x_5, x_6)$ in a, b, c and d. It will be sufficient to prove $C = 1$ under the assumption that $G(4)_7 = 1$. As in the proof of Theorem 3.1, we may suppose that $C = (a, x_2, x_3, x_4, x_5, a)$. Moreover, if x_2, x_3, x_4 or x_5 is a, then by Theorem 2.1 or Corollary 1 of Lemma 4.1, $C = 1$. It will thus be sufficient to prove $(a, b, c, b, d, a) = 1$, $(a, b, c, d, b; a) = 1$, and $(a, c, b, d, b, a) = 1$. Now by Corollary 1 of Lemma 4.1, $(a, b, c, b, d, a) = (a, c, b, d, b, a) = 1$, while by (1′), $(a, b, c; b, d, a) = (a, c, b; b, d; a)(b, c, a; b, d; a)$, so that by (6) $(a, b, c; b, d; a) = 1$. Thus $(a, b, c, d, b, a) = (a, b, c, b, d, a)(a, b, c; b, d; a) = 1$, and the theorem is proved.

5. The main result, that $G(n)_{n+2} = 1$, has now been proved for $n = 2, 3$ and 4. In this section we derive an identity analogous to (1) and (9) for five generators. This identity enables us to prove, in § 6, that $G(n)_{n+2} = 1$ for $n \geq 5$.

LEMMA 5.1. *If a, b, c, d and f are elements of order 2 in a group G of exponent 4, then*

$$(12) \qquad (a, b; c, d; f) \equiv (c, b; f, d; a)(f, b; a, d; c) \bmod G_6 .$$

COROLLARY. *If* $(x_1, \cdots, x_k)$, $(y_1, \cdots, y_j)$, $(z_1, \cdots, z_m)$, a *and* b $(k, j, m \geq 1)$ *are elements of order* 2 *in a group* G *of exponent* 4, *then*

$$(13) \qquad (x_1, \cdots, x_k, a; y_1, \cdots, y_j, b; z_1, \cdots, z_m) \equiv C_1C_2 \bmod G_{k+j+m+3} ,$$

where

$$\begin{aligned} C_1 &= (y_1, \cdots, y_j; z_1, \cdots, z_m; x_1, \cdots, x_k, b; a) \\ C_2 &= (x_1, \cdots, x_k; z_1, \cdots, z_m; y_1, \cdots, y_j, a; b) . \end{aligned}$$

Proof of Lemma 5.1. First, working modulo G_5, we collect f's in the expression $(abcdf)^2$ to get $(abcdf)^2 = (abcd)a(a, f)b(b, f)c(c, f)d(d, f)$. Then collecting b, c and d in that order we obtain $(abcdf)^2 = (abcd)^2S_2S_3S_4$ where

$$\begin{aligned} S_2 &= (a, f)(b, f)(c, f)(d, f) \\ S_3 &= (a, f, d)(a, f, c)(a, f, b)(b, f, d)(b, f, c)(c, f, d) \\ S_4 &= (a, f, c, d)(a, f, b, d)(a, f, b, c)(b, f, c, d) . \end{aligned}$$

But as in the proof of Lemma 4.1, $(abcd)^2 \equiv T_2T_3T_4 \bmod G_5$, where

$$\begin{aligned} T_2 &= (a, b)(a, c)(a, d)(b, d)(c, d) \\ T_3 &= (a, c, b)(a, d, c)(a, d, b)(b, d, c) \\ T_4 &= (a, d, b, c) . \end{aligned}$$

Thus, modulo G_5, $(abcdf)^2 = T_2T_3T_4S_2S_3S_4$. But then, modulo G_6,

$$\begin{aligned} 1 = (abcdf)^4 &= T_2T_3T_4S_2S_3T_2T_3T_4S_2S_3 \\ &= T_2T_3T_4T_2S_2(S_2, T_2)S_3(S_3, T_2)T_3T_4S_2S_3 \\ &= (T_2T_3T_4)^2S_2(S_2, T_3)(S_2, T_2)S_3(S_3, T_2)S_2S_3 \\ &= S_2(S_2, T_3)(S_2, T_2)S_3(S_3, T_2)S_2S_3 \\ &= S_2^2(S_2, T_3)(S_2, T_2)S_3(S_3, S_2)(S_3, T_2)S_3 \\ &= S_2^2(S_2, T_3)(S_2, T_2)S_3^2(S_3, S_2)(S_3, T_2) . \end{aligned}$$

But modulo G_6, $S_3^2 = 1$, while S_2^2 is a product of commutators of weight 4. Thus the last relation may be rewritten as $1 \equiv A \bmod G_6$ where A is a product of commutators in a, b, c, d and f of weight 4 or 5; hence the factors of A commute modulo G_6. Let A'_a be the product of all factors of A which do *not* contain a as argument, and let A_a be the product of the remaining factors of A. Then $1 \equiv A'_aA_a \bmod G_6$, so that, setting $a = 1, 1 \equiv A'_a \bmod G_6$, and hence $1 \equiv A_a \bmod G_6$. Continuing this argument we finally arrive at $1 \equiv A_{abcdf} \bmod G_6$, where A_{abcdf} is the product of all factors of A which contain each of a, b, c, d and f. But what are these factors? Clearly S_2^2 and (S_2, T_2) do not contain any such factors; and since each factor of S_2 and S_3 contains f, (S_3, S_2) cannot contain any such factors. We are left with (S_2, T_3) and (S_3, T_2). The product of the desired factors of (S_2, T_3) is clearly

$$(a, f; b, d, c)(b, f; a, d, c)(c, f; a, d, b)(d, f; a, c, b) ,$$

while the product of the desired factors of (S_3, T_2) is

$$(a, f, d; b, c)(a, f, c; b, d)(a, f, b; c, d)(b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) .$$

Hence, modulo G_6,

$$\begin{aligned} 1 = (a, f; b, d, c)(b, f; a, d, c)(c, f; a, d, b)(d, f; a, c, b) \\ \cdot (a, f, d; b, c)(a, f, c; b, d)(a, f, b; c, d) \\ \cdot (b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) . \end{aligned}$$

so that by (10)

$$\begin{aligned} 1 = (a, f; b, d, c)(b, f; a, d, c)(c, f; a, d, b)(d, f; a, c, b) \\ \cdot (b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) . \end{aligned}$$

Using (7) on the first four factors gives, modulo G_6,

$$\begin{aligned} 1 &= (a, f, c; b, d)(a, f; b, d; c)(b, f, c; a, d)(b, f; a, d; c) \\ &\quad \cdot (c, f, b; a, d)(c, f; a, d; b)(d, f, b; a, c)(d, f; a, c; b) \\ &\quad \cdot (b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) \\ &= (a, f, c; b, d)(a, f; b, d; c)(b, f; a, d; c)(c, f, b; a, d)(c, f; a, d; b) \\ &\quad \cdot (d, f, b; a, c)(d, f; a, c; b)(b, f, d; a, c)(c, f, d; a, b) \\ &= (a, f, c; b, d)(a, f; b, d; c)(b, f; a, d; c)(c, f, b; a, d) \\ &\quad \cdot (c, f; a, d; b)(d, f; a, c; b)(b, d, f; a, c)(c, f, d; a, b) , \end{aligned}$$

where the last step follows from (8). Now applying (11) twice gives

$$\begin{aligned} 1 = (a, f, c; b, d)(a, b; d, f; c)(c, f, b; a, d)(a, f; c, d; b) \\ \cdot (b, d, f; a, c)(c, f, d; a, b) , \end{aligned}$$

so that by (10)

$$1 = (a, f, c; b, d)(a, b; d, f; c)(a, f; c, d; b)(b, d, f; a, c)(c, f, a; b, d)$$

and hence by (8)

$$1 = (a, b; d, f; c)(a, f; c, d; b)(b, d, f; a, c)(a, c, f; b, d) .$$

Thus, by (7)

$$1 \equiv (a, b; d, f; c)(a, f; c, d; b)(a, c; b, d; f) \bmod G_6 ,$$

so that interchanging a with b and c with f we get

$$1 \equiv (a, b; c, d; f)(c, b; f, d; a)(f, b; a, d; c) \bmod G_6$$

which is (12). Thus the lemma is proved.

The corollary follows immediately.

6. Having proved the crucial relation (12), we are now in a position to prove the main theorem.

THEOREM 6.1. *Let $G(n)$, $(n = 1, 2, \cdots)$ be the freest group of exponent 4 generated by n elements of order 2. Then $G(n)_{n+2} = 1$.*

Proof. The proof is by induction on n. We have the result for $n = 1, 2, 3$ and 4. Assuming the result true for n we now prove it for $n + 1$. As before, we may assume $G(n + 1)_{n+4} = 1$. Consider a commutator $C = (y_1, y_2, \cdots, y_{n+3})$ in the generators $x_1, \cdots, x_n$, a and b of $G(n + 1)$. As before, we may restrict attention to the case $C = (a, y_2, \cdots, y_{n+2}, a)$. There are two possibilities to consider—*Case* 1: a appears again; *Case* 2: b appears twice. In either case we may assume that every x_i appears once, since otherwise, by the inductive assumption, $C = 1$.

Case 1. The proof in this case is by induction on the position of the middle a. Clearly $(a, y_2, a, \cdots, a) = 1$. Assume that for some $i \geq 3$, $(a, y_2, \cdots, y_{i-1}, a, \cdots, a) = 1$. Then

$$\begin{aligned}(a, y_2, \cdots, y_i, a, y_{i+1}, \cdots, y_{n+2}, a) &\\ &= (a, y_2, \cdots, y_{i-1}; y_i, a; y_{i+1}; \cdots; y_{n+2}; a)\\ &= (a, y_2, \cdots, y_{i-1}; y_i, a; a, y_{n+2}, \cdots, y_{i+1}) ,\end{aligned}$$

where the last step follows from $G(n)_{n+2} = 1$. But by (13),

$$(a, y_2, \cdots, y_{i-1}; y_i, a; a, y_{n+2}, \cdots, y_{i+1}) = C_1 C_2$$

where

$$\begin{aligned}C_1 &= (a, y_2, \cdots, y_{i-2}, y_i; a, y_{n+2}, \cdots, y_{i+1}, a; y_{i-1})\\ C_2 &= (a, y_2, \cdots, y_{i-2}; a, y_{n+2}, \cdots, y_{i+1}; y_{i-1}, a; y_i) .\end{aligned}$$

Since y_i and y_{i-1} appear only once, by the assumption that $G(n)_{n+2} = 1$ we have $C_1 = C_2 = 1$. Hence, by induction, $C = 1$ if a appears three times.

Case 2. In this case also the proof is by induction, this time on the distance between the b's. Let

$$C = (a, z_1, \cdots, z_i, b, z_{i+1}, \cdots, z_j, b, z_{j+1}, \cdots, z_{n-1}, a) ,$$

where $0 \leq i < j \leq n - 1$ (that is, there might be no entries between the a's and the b's). If $j - i = 1$, then clearly $C = 1$. Assume that $C = 1$ for $j - i = k \geq 1$. Then as in Case 1,

$$\begin{aligned}
&(a, z_1, \cdots, z_i, b, z_{i+1}, \cdots, z_{i+k+1}, b, z_{i+k+2}, \cdots, z_{n-1}, a) \\
&\qquad = (a, z_1, \cdots, z_i, b, z_{i+1}, \cdots, z_{i+k}; z_{i+k+1}, b; a, z_{n-1}, \cdots, z_{i+k+2}) \\
&\qquad = C_1 C_2
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= (a, \cdots, b, \cdots, z_{i+k-1}, z_{i+k+1}; a, z_{n-1}, \cdots, z_{i+k+2}, b; z_{i+k}) = 1 \\
C_2 &= (a, \cdots, b, \cdots, z_{i+k-1}; a, z_{n-1}, \cdots, z_{i+k+2}; z_{i+k}, b; z_{i+k+1}) = 1
\end{aligned}$$

Thus $C = 1$ for $j - i = k + 1$, so that by induction $C = 1$ if b appears twice.

Since $C = 1$ in both cases, we conclude that $G(n+1)_{n+3} = 1$, so that by induction $G(n)_{n+2} = 1$ for $n = 1, 2, \cdots$.

7. The author conjectures that the class of $G(n)$ is precisely $n + 1$ for $n > 2$. As supporting evidence, he has constructed $G(n)/G(n)''$ and shown that its class is exactly n. Moreover, for $n = 3$ and $n = 4$, $G(n)''$ is fairly large, and $G(n)_{n+1} \neq 1$.

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University of Wisconsin
California Institute of Technology