PHYSICAL INTERPRETATION AND STRENGTHENING OF M. H. PROTTER'S METHOD FOR VIBRATING NONHOMOGENEOUS MEMBRANES; ITS ANALOGUE FOR SCHRÖDINGER'S EQUATION

JOSEPH HERSCH

The origin of this work lies partly in *M. H. Protter*'s method [7], [8], partly in two papers [3], [5], developing the idea, found in *Payne-Weinberger* [6], of auxiliary one-dimensional problems; the physical interpretation in § 3 rejoins that of [2] and [4].

1. We consider the first eigenvalue λ_1 of a nonhomogeneous membrane with specific mass $\rho(x, y) \geqq 0$ covering a plane domain D and elastically supported (elastic coefficient $k(s)$) along its boundary Γ:

$$\Delta u + \lambda\rho(x, y)u = 0 \text{ in } D\,, \quad \frac{\partial u}{\partial n} + k(s)u = 0 \text{ along } \Gamma\,,$$

where $\vec{n}$ is the outward normal.

Every continuous and piecewise smooth function $v(x, y)$ furnishes an upper bound for λ_1: By Rayleigh's principle

$$\lambda_1 = \mathrm{Min}_v \frac{D(v) + \oint_\Gamma k(s)v^2 ds}{\iint_D \rho v^2 dA}\,,$$

where ds is the length element, dA the element of area, and $D(v)$ the Dirichlet integral $\iint_D \mathrm{grad}^2\, v\, dA$. *The Minimum is realized if* $v = u_1(x, y)$ (*first eigenfunction*, satisfying $\Delta u_1 + \lambda_1\rho u_1 = 0$).

In the opposite direction, we are here in search of a Maximum principle for λ_1, from which we could calculate lower bounds.

2. Let us consider in D a sufficiently regular vector field $\vec{p}$ (we shall discuss presently what discontinuities are allowed), satisfying the *condition*

$$\vec{p} \cdot \vec{n} \leqq k(s) \qquad \text{along } \Gamma\,. \tag{1}$$

$$\begin{aligned}\mathrm{grad}^2\, u_1 + (\vec{p}^2 - \mathrm{div}\,\vec{p})\, u_1^2 &= -\mathrm{div}\,(u_1^2\vec{p}) + \mathrm{grad}^2\, u_1 + u_1^2\vec{p}^2 + 2u_1\,\mathrm{grad}\, u_1 \cdot \vec{p} \\ &= -\mathrm{div}(u_1^2\vec{p}) + (\mathrm{grad}\, u_1 + u_1\vec{p})^2 \geqq -\mathrm{div}(u_1^2\vec{p})\,.\end{aligned}$$

Let us integrate this inequality:

Received October 10, 1960. Battelle Memorial Institute, Geneva, and Swiss Federal Institute of Technology, Zürich.

$$0 \leqq D(u_1) + \oint u_1^2 \vec{p} \cdot \vec{n} ds + \iint (\vec{p}^2 - \operatorname{div} \vec{p}) u_1^2 dA$$
$$\leqq D(u_1) + \oint k(s) u_1^2 ds + \iint (\vec{p}^2 - \operatorname{div} \vec{p}) u_1^2 dA = \iint (\lambda_1 \rho + \vec{p}^2 - \operatorname{div} \vec{p}) u_1^2 dA ,$$

whence the lower bound

$$\lambda_1 \geqq \inf_D \left(\frac{\operatorname{div} \vec{p} - \vec{p}^2}{\rho} \right) . \tag{2}$$

We have equality if (and only if) $\vec{p} = -\operatorname{grad} u_1 / u_1$, whence the Maximum principle

$$\boxed{\lambda_1 = \operatorname{Max}_{\vec{p} \cdot \vec{n} \leqq k(s) \text{ along } \Gamma} \inf_D \left(\frac{\operatorname{div} \vec{p} - \vec{p}^2}{\rho} \right) .} \tag{3}$$

Allowed discontinuities (see also [5]): the same as in Thomson's principle for boundary value problems. — If D is cut into subdomains $D_1, D_2, \cdots, D_n$ by analytic arcs, it is sufficient that the vector field $\vec{p}$ be continuous and differentiable in each D_i *and that its normal component be continuous across all those analytic arcs*; the tangential component need *not* be continuous. — Other sufficient condition: $\vec{p} = \{p_1, p_2\}$, p_1 continuous in x and differentiable with respect to x, p_2 continuous in y and differentiable with respect to y.

Two properties of a "good" concurrent vector field: One should try to construct $\vec{p}$ such that $\vec{p} \cdot \vec{n} = k(s)$ along Γ and $(\operatorname{div} \vec{p} - \vec{p}^2)/\rho =$ const in D (such is the case for the extremal field -grad u_1/u_1); the examples calculated in [5] show that such a "good" field may be easy to construct.

REMARK. For a fixed boundary ($u = 0$ along Γ), $k \equiv \infty$ and condition (1) falls off. — A "good" field will then be singular along Γ.

3. A physical interpretation.

3.1. One verifies immediately that *the nonhomogeneous membrane upon D, with specific mass $= \lambda_1 \rho(x, y)$ and elastic coefficient $k(s)$, vibrates with ground eigenfrequency* 1: $\Delta u_1 + 1 \cdot (\lambda_1 \rho) u_1 = 0$.

We shall presently establish the following *theorem*: Given an admissible vector field $\vec{p}$ in D, *the nonhomogeneous membrane with specific mass $\tilde{\rho}(x, y) = \operatorname{div} \vec{p} - \vec{p}^2$ in D and elastic coefficient $\tilde{k}(s) = \vec{p} \cdot \vec{n}$ along Γ, vibrates with ground frequency* $\geqq 1$.

The inequality (2) *follows as a corollary*: according to two general principles regarding vibrating systems (cf. [1], pp. 354 and 357), a homogeneous membrane with specific mass $\leqq \tilde{\rho}$ and elastic coefficient $k(s) \geqq \tilde{k}(s)$ vibrates *a fortiori* with ground frequency $\geqq 1$; whence (2).

3.2. The above theorem will be established by proving the following statement to be true: If we *cut* the membrane (specific mass $\tilde{\rho}(x, y) = \operatorname{div} \vec{p} - \vec{p}^2$, elastic coefficient $\tilde{k}(s) = \vec{p} \cdot \vec{n}$) *into slices* D_j of infinitesimal breadth along all trajectories of $\vec{p}$, it *then* vibrates with ground frequency 1.

Indeed: Each slice D_j has the first eigenfrequency 1: Call s the arc length along the trajectory (measured from an arbitrary origin on D_j); we define in D_j a function $f(x, y) = f(s) = c_j \exp\left\{-\int_{s=0}^{s} \vec{p} \cdot \overrightarrow{ds}\right\}$, $c_j > 0$ arbitrary. Then $\operatorname{grad} f = -f\vec{p}$;

$$\Delta f = -f \operatorname{div} \vec{p} - \vec{p} \cdot \operatorname{grad} f = (\vec{p}^2 - \operatorname{div} \vec{p}) f = -\tilde{\rho} f\ ,$$

$$\frac{\partial f}{\partial n} = -(\vec{p} \cdot \vec{n}) f = \begin{cases} -\tilde{k} f \text{ on } \Gamma_j \text{ (infinitesimal part of } \Gamma \text{ bounding } D_j); \\ 0 \text{ along the cuts;} \end{cases}$$

$f > 0$ in D. Thus, our function f is the first eigenfunction of the vibrating slice D_j with specific mass $\tilde{\rho}$, free along the cuts and with elastic coefficient $\tilde{k}$ on Γ_j; its first eigenfrequency is 1, because $\Delta f + 1 \cdot \tilde{\rho} f = 0$: this proves the theorem and justifies our physical interpretation of (2).—The light in which the Maximum principle is viewed here, is in agreement with [2] and [4].

4. An inequality of M. H. Protter.

Let $\vec{p} = \dfrac{\vec{t}}{a} - \dfrac{\operatorname{grad} a}{2a}$, where $\vec{t}(x, y)$ is a vector field and $a(x, y) > 0$ a scalar field. Then

$$\operatorname{div} \vec{p} - \vec{p}^2 = \frac{\operatorname{div} \vec{t}}{a} - \frac{\vec{t}^2}{a^2} - \frac{\Delta a}{2a} + \frac{\operatorname{grad}^2 a}{4a^2} \geqq \frac{\operatorname{div} \vec{t}}{a} - \frac{\vec{t}^2}{a^2} - \frac{\Delta a}{2a}\ .$$

For a membrane with fixed boundary, Condition (1) falls off, so we have by (2)

$$\lambda_1 \geqq \inf_D \left[\frac{\operatorname{div} \vec{t} - \dfrac{\vec{t}^2}{a} - \dfrac{\Delta a}{2}}{a\rho} \right]. \tag{4}$$

This is M. H. Protter's inequality [7], [8] (if we write $\vec{t} = \{P, Q\}$) —although he requires $P(x, y)$ and $Q(x, y)$ to be C^1 in D, which is unnecessarily restrictive (cf. also [5] and [3]): P might be discontinuous in y and Q in x.

M. H. Protter indicates in [8] very interesting applications of (4) to comparison theorems between ground eigenfrequencies of two non-homogeneous membranes spanning the same domain D.

Critical remark.—In the proof of (4) we neglected the positive term $\operatorname{grad}^2 a/4a^2$: *equality is impossible in* (4) *unless* $a(x, y) = \text{const}$, in which case (4) reduces back to (2) with $\vec{p} = \vec{t}/a$.

5. Strengthening of Protter's inequality. Let first (a little more generally) $\vec{p} = \dfrac{\vec{t}}{a} + \vec{v}$ with $\vec{t}(x, y), \vec{v}(x, y), a(x, y) > 0$; $\operatorname{div}\vec{p} - \vec{p}^2 = \dfrac{\operatorname{div}\vec{t}}{a} - \dfrac{\vec{t}^2}{a^2} + \operatorname{div}\vec{v} - \vec{v}^2 - \dfrac{\operatorname{grad} a}{a^2}\cdot\vec{t} - 2\dfrac{\vec{v}}{a}\cdot\vec{t}$; in order that the two last terms may cancel everywhere, let (with Protter) $\vec{v} = -\dfrac{\operatorname{grad} a}{2a} = -\dfrac{\operatorname{grad}\sqrt{a}}{\sqrt{a}}$; then $\operatorname{div}\vec{v} - \vec{v}^2 = -\dfrac{\Delta\sqrt{a}}{\sqrt{a}}$; let $\sqrt{a(x, y)} = b(x, y) > 0$ in D, i.e. $\vec{p} = \dfrac{\vec{t}}{b^2} - \dfrac{\operatorname{grad} b}{b}$; $\operatorname{div}\vec{p} - \vec{p}^2 = \dfrac{\operatorname{div}\vec{t}}{b^2} - \dfrac{\vec{t}^2}{b^4} - \dfrac{\Delta b}{b}$.—Under the *condition*

$$\frac{\vec{t}\cdot\vec{n}}{b^2} - \frac{1}{b}\frac{\partial b}{\partial n} \leqq k(s) \qquad \text{(identically satisfied if } k \equiv \infty) \,, \tag{5}$$

we have the lower bound

$$\boxed{\lambda_1 \geqq \inf_D\left[\frac{1}{\rho}\left(\frac{\operatorname{div}\vec{t}}{b^2} - \frac{\vec{t}^2}{b^4} - \frac{\Delta b}{b}\right)\right]} \,, \tag{6}$$

with equality whenever $\dfrac{\vec{t}}{b^2} - \dfrac{\operatorname{grad} b}{b} = -\dfrac{\operatorname{grad} u}{u}$, as no term has been neglected.—If, for example, we take $\vec{t} \equiv 0$, we get an inequality of *Barta-Pólya* $\lambda_1 \geqq \inf_D\left(-\dfrac{\Delta b}{\rho b}\right)$.—$\Big[$In fact, if $\dfrac{\partial b}{\partial n} + k(s)b = 0$ on Γ, λ_1 is comprised between the two Barta-Pólya bounds

$$\inf_D\left(-\frac{\Delta b}{\rho b}\right) \leqq \lambda_1 \leqq \sup_D\left(-\frac{\Delta b}{\rho b}\right).\Big]$$

The expression in square brackets in (6) is larger than that in (4), because

$$-\frac{\Delta a}{2a} = -\frac{\Delta(b^2)}{2b^2} = -\frac{\operatorname{div}(b \operatorname{grad} b)}{b^2} = -\frac{\Delta b}{b} - \frac{\operatorname{grad}^2 b}{b^2}\,;$$

this does not diminish M. H. Protter's merit, as his inequality (4) contains (2) as a special case, whence (6) follows.

6. Applications.

6.1. The inequalities obtained by M. H. Protter in [8] may be sharpened by using (6) instead of (4).

6.2. *Small variation of the elastic coefficient along the boundary.*
First case: $\rho(x, y)$, $k(s)$; λ_1, $u_1(x, y)$.
Second case: $\tilde{\rho}(x, y) = \rho(x, y)$, $\tilde{k}(s) = k(s) + \varepsilon K(s)$; $\tilde{\lambda}_1$, $\tilde{u}_1(x, y)$.
By Rayleigh's principle,

$$\tilde{\lambda}_1 \leqq \frac{D(u_1) + \oint \tilde{k} u_1^2 ds}{\iint \rho u_1^2 dA} = \lambda_1 + \varepsilon Q \ , \quad \text{where } Q = \frac{\oint K u_1^2 ds}{\iint \rho u_1^2 dA} \ . \tag{7}$$

We now introduce $b = u_1(x, y)$ into (6):

$$\tilde{\lambda}_1 \geqq \lambda_1 + \inf_D \left\{ \frac{1}{\rho} \left(\frac{\operatorname{div} \vec{t}}{u_1^2} - \frac{\vec{t}^2}{u_1^4} \right) \right\} \text{ under the condition } \frac{\vec{t} \cdot \vec{n}}{u_1^2} \leqq \varepsilon K(s),$$

whence $\iint \operatorname{div} \vec{t} dA = \oint \vec{t} \cdot \vec{n} ds \leqq \varepsilon \oint K u_1^2 ds = \varepsilon Q \iint \rho u_1^2 dA$.—There exists a vector field $\vec{t}$ such that

div $\vec{t} = \varepsilon Q \rho(x, y) u_1^2$ in D and $\vec{t} \cdot \vec{n} = \varepsilon K(s) u_1^2$ along Γ: indeed, we can even impose the supplementary condition rot $\vec{t} = 0$, $\vec{t} =$ grad v; v (determined up to an additive constant) is the solution of the Poisson-Neumann problem

$$\Delta v = \varepsilon Q \rho(x, y) u_1^2 \text{ in } D \text{ and } \frac{\partial v}{\partial n} = \varepsilon K(s) u_1^2 \text{ along } \Gamma \ .$$

Clearly, v and $\vec{t}$ are proportional to ε. Thus,

$$\tilde{\lambda}_1 \geqq \lambda_1 + \varepsilon Q - \sup_D \left(\frac{\vec{t}^2}{\rho u_1^4} \right) = \lambda_1 + \varepsilon Q - O(\varepsilon^2) \ . \tag{7'}$$

(7) and (7′) give

$$\tilde{\lambda}_1 = \lambda_1 + \varepsilon Q - O(\varepsilon^2) \ . \tag{7''}$$

The *first perturbation calculus* gives $\tilde{\lambda}_1 = \lambda_1 + \varepsilon Q$; we thus verify that this is *the tangent* to the exact curve $\tilde{\lambda}_1 = \tilde{\lambda}_1(\varepsilon)$.

6.3. *Small variation of the specific mass* $\rho(x, y)$.
First case: $\rho(x, y)$, $k(s)$; λ_1, $u_1(x, y)$.
Second case: $\tilde{\rho}(x, y) = \rho(x, y) + \varepsilon\sigma(x, y)$, $\tilde{k}(s) = k(s)$; $\tilde{\lambda}_1$, $\tilde{u}_1(x, y)$.
By Rayleigh's principle,

$$\tilde{\lambda}_1 \leqq \frac{D(u_1) + \oint k(s) u_1^2 ds}{\iint \tilde{\rho} u_1^2 dA} = \frac{\lambda_1}{1 + \varepsilon R}\,, \text{ where } R = \frac{\iint \sigma u_1^2 dA}{\iint \rho u_1^2 dA}\,. \tag{8}$$

We now introduce again $b = u_1(x, y)$ into (6):

$\tilde{\lambda}_1 \geqq \inf_D \left\{ \dfrac{1}{\tilde{\rho}} \left(\dfrac{\operatorname{div} \vec{t}}{u_1^2} - \dfrac{\vec{t}^2}{u_1^4} + \lambda_1 \rho \right) \right\}$ under the condition $\vec{t} \cdot \vec{n} \leqq 0$ along Γ; we want to use a vector field $\vec{t}$ such that $\vec{t} \cdot \vec{n} = 0$ along Γ and $\dfrac{1}{\tilde{\rho}} \left(\dfrac{\operatorname{div} \vec{t}}{u_1^2} + \lambda_1 \rho \right) = c = \text{const in } D$, so $\operatorname{div} \vec{t} = u_1^2 (c\tilde{\rho} - \lambda_1 \rho)$; the constant c is determined by the condition

$$0 = \oint \vec{t} \cdot \vec{n} ds = \iint \operatorname{div} \vec{t} dA = c \iint \tilde{\rho} u_1^2 dA - \lambda_1 \iint \rho u_1^2 dA\,,$$

whence

$$c = \frac{\lambda_1}{1 + \varepsilon R}\,; \quad \operatorname{div} \vec{t} = \lambda_1 u_1^2 \left(\frac{\rho + \varepsilon\sigma}{1 + \varepsilon R} - \rho \right) = \varepsilon \lambda_1 u_1^2 \frac{\sigma - \rho R}{1 + \varepsilon R}\,;$$

such a vector field $\vec{t}$ exists: we can even request that it be a gradient field; $\vec{t} = O(\varepsilon)$.

$$\tilde{\lambda}_1 \geqq \frac{\lambda_1}{1 + \varepsilon R} - \sup_D \left(\frac{\vec{t}^2}{\tilde{\rho} u_1^4} \right) = \frac{\lambda_1}{1 + \varepsilon R} - O(\varepsilon^2)\,. \tag{8'}$$

(8) and (8′) give

$$\tilde{\lambda}_1 = \frac{\lambda_1}{1 + \varepsilon R} - O(\varepsilon^2)\,. \tag{8''}$$

7. Schrödinger's equation.

7.1. We consider an equation of Schrödinger's type in 3-space:

$$\Delta u + [\lambda - W(x, y, z)]u = 0 \tag{9}$$

with some boundary conditions not specified here, but which must permit partial integrations analogous to those of § 2; $W = \dfrac{2m}{\hbar^2} V(x, y, z)$, $\lambda_1 = \dfrac{2m}{\hbar^2} E_1$, where V is the potential, and E_1 the lowest energy level.

Rayleigh's principle:

(10) $$\lambda_1 = \mathrm{Min}_v \frac{D(v) + \iiint W(x, y, z)v^2 d\tau}{\iiint v^2 d\tau},$$

with, possibly, a supplementary term at the numerator, owing to the boundary conditions; $d\tau$ is the volume element.—The Minimum is realized for the first eigenfunction $u_1(x, y, z)$ of the differential equation.

7.2. An argument almost identical to that of § 2 (cf. also [5]) gives the Maximum principle:

(11) $$\boxed{\lambda_1 = \mathrm{Max}_{\vec{p}} \inf_D \{W(x, y, z) + \mathrm{div}\, \vec{p} - \vec{p}^2\}},$$

where the concurrent vector fields $\vec{p}$ must satisfy corresponding boundary conditions.—The Maximum is realized for $\vec{p} = -\mathrm{grad}\, u_1/u_1$.—Allowed discontinuities: cf. § 2 (continuity of the normal derivative, etc.).—To get a good lower bound, one should try to construct a vector field $\vec{p}$ such that $W(x, y, z) + \mathrm{div}\ \vec{p} - \vec{p}^2 = \mathrm{const.}$

7.3. *A physical interpretation.*—For expository purposes, we shall consider here equation (9) for *2 dimensions* only.—This is exactly the equation of a *vibrating homogeneous membrane* covering a plane domain D, *on which each area element* $dxdy$ (at the point (x, y)) *is pulled towards its equilibrium position* $u = 0$ *by a weak spring of infinitesimal elastic coefficient* $W(x, y)dxdy$.—We suppose that the membrane's boundary Γ is also elastically supported with elastic coefficient $k(s)$: $\partial u/\partial n + k(s)u = 0$ along Γ.

Analogously to § 3.1, we verify immediately: *The homogeneous membrane covering D, with specific mass $\equiv \lambda_1$ and "interior springs" $W(x, y)$, vibrates with the ground eigenfrequency* 1.

Let us now consider another vibrating system: Given in D an admissible vector field $\vec{p}$ with $\vec{p} \cdot \vec{n} \leqq k(s)$, we study the system formed by:

(a) A nonhomogeneous membrane covering a copy D_a of D, with specific mass $= (\mathrm{div}\, \vec{p} - \vec{p}^2)$ and elastic coefficient $= \vec{p} \cdot \vec{n}$ along Γ;

(b) Another copy D_b of D, without any "transversal elasticity", where every area element $dxdy$ contains a mass $W(x, y)dxdy$ vibrating independently under the action of a spring with elastic coefficient $W(x, y)dxdy$.

According to § 3, the nonhomogeneous membrane (a) has ground eigenfrequency $\geqq 1$; each infinitesimal mass of the system (b) vibrates with the exact frequency $\omega = 1$, as this mass is equal to the spring coefficient.—Therefore 1 *is* the ground eigenfrequency of the system $(a) + (b)$.

By superposing D_a and D_b and *welding*, in each point (x, y), the two masses there placed, we synthesize a nonhomogeneous membrane with specific mass $W(x, y) + \operatorname{div} \vec{p} - \vec{p}^2$, elastic coefficient $= \vec{p} \cdot \vec{n}$ along Γ, and "interior springs" $W(x, y)$.—As the addition of supplementary constraints (welding!) can only make the ground eigenfrequency higher ([1], p. 354), our "synthetic" membrane vibrates with a ground frequency $\geqq 1$.

Consider now the homogeneous membrane with specific mass $\equiv \inf_D[W(x, y) + \operatorname{div} \vec{p} - \vec{p}^2]$, elastic coefficient $k(s)$ along Γ, and the same "interior springs" $W(x, y)$; this membrane has smaller masses and greater constraints: therefore ([1], pp. 354 and 357), its ground frequency is *a fortiori* $\geqq 1$.

As our initial membrane [specific mass $\equiv \lambda_1$; elastic coefficient $= k(s)$; interior springs $W(x, y)$] has ground eigenfrequency 1, its specific mass λ_1 must be $\geqq \inf_D[W(x, y) + \operatorname{div} \vec{p} - \vec{p}^2]$, which explains (11).

7.4. (Analogous to § 5): Let $\vec{p} = \dfrac{\vec{t}}{b^2} - \dfrac{\operatorname{grad} b}{b}$; we get

$$\boxed{\lambda_1 \geqq \inf_D\left\{W(x, y, z) + \frac{\operatorname{div} \vec{t}}{b^2} - \frac{\vec{t}^2}{b^4} - \frac{\Delta b}{b}\right\}}\ , \tag{12}$$

where adequate boundary restrictions must be imposed on the concurrent vector fields $\vec{t}(x, y, z)$ and scalar fields $b(x, y, z)$.

7.5. *An application.—Small variation of the potential;* boundary conditions on the surface Γ of D: $\partial u/\partial n + k(X)u = 0$ $(X \in \Gamma)$.

Boundary conditions to be satisfied by $\vec{t}$ and b:

$$\frac{\vec{t} \cdot \vec{n}}{b^2} - \frac{1}{b}\frac{\partial b}{\partial n} \leqq k(X) \text{ on } \Gamma\ . \tag{5'}$$

First case:

$$W(x, y, z)\ , \quad k(X);\ \lambda_1,\ u_1(x, y, z).$$

Second case:

$$\tilde{W}(x, y, z) = W(x, y, z) + \varepsilon w(x, y, z)\ , \quad \tilde{k}(X) = k(X);\ \tilde{\lambda}_1,\ \tilde{u}_1(x, y, z).$$

By Rayleigh's principle (10),

$$(13) \qquad \tilde{\lambda}_1 \leqq \frac{D(u_1) + \iiint \tilde{W} u_1^2 d\tau}{\iiint u_1^2 d\tau} = \lambda_1 + \varepsilon U\,, \quad \text{where } U = \frac{\iiint w u_1^2 d\tau}{\iiint u_1^2 d\tau}\,.$$

Now let $b = u_1(x, y, z)$ into (12):

$\tilde{\lambda}_1 \geqq \lambda_1 + \inf_D\left[\varepsilon w + \dfrac{\operatorname{div}\vec{t}}{u_1^2} - \dfrac{\vec{t}^2}{u_1^4}\right]$ under the condition $\vec{t}\cdot\vec{n} \leqq 0$ on Γ. We want to use a vector field $\vec{t}$ such that $\vec{t}\cdot\vec{n} = 0$ and $\dfrac{\operatorname{div}\vec{t}}{u_1^2} + \varepsilon w = c =$ const, $\operatorname{div}\vec{t} = u_1^2(c - \varepsilon w)$; the constant c is determined by the condition $0 = \oint\oint \vec{t}\cdot\vec{n}dS = \iiint \operatorname{div}\vec{t}d\tau = c\iiint u_1^2 d\tau - \varepsilon\iiint w u_1^2 d\tau$, where dS is the surface element; hence, $c = \varepsilon U$; $\operatorname{div}\vec{t} = \varepsilon u_1^2(U - w)$; there exists such a vector field $\vec{t}$: we can even impose that it be a gradient field; $\vec{t}$ is proportional to ε.

$$(13') \qquad \tilde{\lambda}_1 \geqq \lambda_1 + \varepsilon U - \sup_D(\vec{t}^2/u_1^4) = \lambda_1 + \varepsilon U - O(\varepsilon^2)\;;$$

(13) and (13′) give

$$(13'') \qquad \tilde{\lambda}_1 = \lambda_1 + \varepsilon U - O(\varepsilon^2).$$

The *first* approximation $\tilde{\lambda}_1 = \lambda_1 + \varepsilon U$ of the *perturbation calculus* is, as we see, the *tangent* to the exact curve $\tilde{\lambda}_1 = \tilde{\lambda}_1(\varepsilon)$.

Post-scriptum. For the case $k \equiv \infty$ and $\rho \equiv 1$, the inequality (2), written for the components $\vec{p} = \{\varphi(x, y), \psi(x, y)\}$ instead of vectorially, was known (except for the allowed discontinuities) to E. Picard *as early as 1893*: *Traité d'Analyse*, t. II, p. 25–26, and to T. Boggio: *Sull'equazione del moto vibratorio delle membrane elastiche*, Atti Accad. Lincei, ser. 5, vol. **16** (2° sem., 1907), 386–393, especially p. 390.—They also chose φ and ψ to be continuous in the domain, which is criticized here and in [5] as an unnecessary restriction.—In contrast with M. H. Protter, both Picard and Boggio seem to have under-estimated the importance of inequality (2): it just incidentally appears (in the quoted places) in the course of demonstrations for very simple monotony properties.

Bibliography

1. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. I, Springer, Berlin (1931).

2. J. Hersch, *Une interprétation du principe de Thomson et son analogue pour la fréquence fondamentale d'une membrane. Application.* C. R. Acad. Sci. Paris, **248** (1959), 2060.

3. *Un principe de maximum pour la fréquence fondamentale d'une membrane,* C. R. Acad. Sci. Paris, **249** (1959), 1074.
4. *Sur quelques principes extrémaux de la physique mathématique,* L'Enseignement Math., 2e série, **5** (1959), 249-257.
5. *Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défaut et principe de maximum,* ZAMP, **11** (1960), 387-413.
6. L. E. Payne and H. F. Weinberger, *Lower bounds for vibration frequencies of elastically supported membranes and plates,* J. Soc. Indust. Appl. Math., **5** (1957), 171-182.
7. M. H. Protter, *Lower bounds for the first eigenvalue of elliptic equations and related topics,* Tech. Report No. 8, AFOSR, University of California, Berkeley (1958).
8. *Vibration of a nonhomogeneous membrane,* Pacific J. Math., **9** (1959), 1249-1255.

INSTITUT BATTELLE, GENEVA SWITZERLAND