ASYMPTOTICS III: STATIONARY PHASE FOR TWO PARAMETERS WITH AN APPLICATION TO BESSEL FUNCTIONS

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1. Introduction. The method of stationary phase has long been a valuable analytical tool for investigating the asymptotic behavior as $p \to \infty$ of integrals of the form

$$I(p) = \int_0^a Q(t) \exp(ipF(t))dt .$$

As a natural generalization of the method of stationary phase involving one parameter we will investigate the asymptotic behavior of an integral of the form

$$I(h, k) = \int_0^a t^{\gamma-1}q(t) \exp[i(ht^\lambda f(t) + kt^\nu g(t))]dt$$

where h and k tend to infinity independently.

It will be shown that under certain restrictions between the real numbers λ, ν and γ that the asymptotic form of $I(h, k)$ is determined by the behavior of the ratio $kh^{-\nu/\lambda}$ as $h, k \to \infty$ and by the character of f and g in a neighborhood of $t = 0$. For example, if $\gamma < \nu < \lambda$, $\gamma > 0$, $f(0) > 0$, $g(0) > 0$ and $kh^{-\nu/\lambda} \to \infty$ then

$$I(h, k) \backsim \frac{q(0)\Gamma\left(\frac{\gamma}{\nu}\right)\exp\left(\frac{i\pi\gamma}{2\nu}\right)}{\nu[kg(0)]^{\gamma/\nu}} .$$

As an immediate application of our results we will determine the asymptotic behavior of the Bessel function $J_\nu(x)$ in Watson's transition region, i.e. when ν, x and $|\nu - x|$ are large and ν/x is nearly equal to 1. In particular, we will obtain a simple rigorous proof of Nicholson's formulas under the restriction that $0 < \limsup\ x^{-1/3}|\nu - x| < \infty$.

2. General assumptions. Throughout the paper we shall use $A \backsim B$ to mean $\lim A/B = 1$, and all limits will mean the limit as h and k tend to infinity. A similar remark applies to order symbols.

We shall consider $I(h, k)$ under the following general assumptions:

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(i) $k = o(h)$,
(ii) $\lambda > 0$, $\nu > 0$, and $\gamma > 0$,
(iii) $q(0) \neq 0$ and $g(0) \neq 0$,
(iv) f, g and q are real valued functions such that $f \in C^2$, $g \in C^2$ and $q \in C$ on $[0, a]$,
(v) $\lambda f(t) + t f'(t) > 0$ on $[0, a]$.

For convenience we shall consider here only the case $f(0) > 0$. If $f(0) < 0$ and $-f$ satisfies certain obvious conditions one obtains analogous results with $-i$ and $-g$ replacing i and g, respectively.

3. Preliminary lemmas. We shall first establish the following lemmas.

LEMMA I. *Consider* $I(p) = \int_0^b \omega(t)\Psi(t) \exp(ip\Phi(t))dt$. *Suppose* d *is a nonnegative constant and* p, α and μ *are functions of* h *and* k *such that* $p \to \infty$, $\mu \to 0$ *and* α *is bounded as* $h, k \to \infty$.

(i) $\Phi(t) = t^r\phi(\alpha t)$, $\Psi(t) = t^{s-1}\psi(\mu(t + d))$, *the functions* ϕ *and* ψ *are real with* $\psi(0) \neq 0$, $r > 0$, $0 < s < r$, $\phi(\alpha t) > 0$ *for* $0 \leqq t \leqq c'$, $c' > 0$ *and* $\phi \in C^{n+2}$ *and* $\psi \in C^n$ *for* $0 \leqq t \leqq c'$ *where* m *and* n *are the least integers such that* $mr > 1$ *and* $n \geqq m(r - s) + 1$, *respectively.*

(ii) b *is a constant such that* $0 < b < c'$ *and* $bMK/m_0r < 1$ *where* $M = \text{maximum}_{0 \leqq t \leqq c'} |\phi'(t)|$, $m_0 = \text{minimum}_{0 \leqq t \leqq c'} \phi(t)$ *and* $K \geqq \alpha$ *when* h, k *are sufficiently large.*

(iii) $\omega = u + iv$ *is a complex valued function such that* $u(0) = 1$, $v(0) = 0$ *and* $u, v \in C^n$ *for* $0 \leqq t \leqq c'$. *Then*

$$I(p) \backsim \frac{\psi(0)\Gamma\left(\frac{s}{r}\right)\exp\left(\frac{i\pi s}{2r}\right)}{r[p\phi(0)]^{s/r}}\,.$$

Proof. We may set $v \equiv 0$ since it will be seen that the contribution from v to $I(p)$ is negligible because $v(0) = 0$. Let $x = t[\phi(\alpha t)]^{1/r}$. Since $x'(t) > 0$ for $0 \leqq t \leqq b$ and $x \in C^{n+2}$ there exists a unique inverse function, say $t(x)$, such that $t \in C^{n+2}$ for $0 \leqq x \leqq b[\phi(\alpha b)]^{1/r} = a$, $t(0) = 0$ and $t'(0) = a_1 = [\phi(0)]^{-1/r}$. Hence we may write $t(x) = a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + A(x)x^n$ where $A \in C^2$ and a_l is bounded as $h, k \to \infty$ for $2 \leqq l \leqq n - 1$. We may assume that c' is sufficiently small such that if $t(x) = a_1x(1 + w(x))$ then $|w(x)| < 1$ for $0 \leqq x \leqq a$. This implies that

$$(t(x))^{s-1} = a_1^{s-1}(1 + b_1x + \cdots + b_{n-2}x^{n-2} + z(x)x^{n-1})x^{s-1}$$

where $z \in C$ and b_l is independent of x for $1 \leqq l \leqq n - 2$. If we now expand ψ and ω about $t = 0$ and substitute $t(x)$, and let $B(x) = \omega(t(x))(t(x))^{s-1}\psi(\mu(t(x) + d))$ we have

$$B(x)\,\frac{dt(x)}{dx} = a_1^s u(\mu d)x^{s-1} + c_0x^s + \cdots + c_{n-3}x^{s+n-3} + D(x)x^{n+s-2}$$

where h and k are sufficiently large such that $\mu d < b$, $D \in C$ and c_l is bounded as $h, k \to \infty$ and independent of x for $0 \leqq l \leqq n-3$. Therefore,

$$I(p) = a_1^s \psi(\mu d) \int_0^a x^{s-1} \exp(ipx^r) dx + J(p) + \sum_{l=0}^{n-3} c_l \int_0^a x^{s+l} \exp(ipx^r) dx$$

where $J(p) = \int_0^a D(x) x^{n+s-2} \exp(ipx^r) dx$. Since $\int_0^\infty e^{it} t^{\beta-1} dt = \exp(i\pi\beta/2)\Gamma(\beta)$ for $0 < \beta < 1$ and $r \leqq n+s-1 < r+1$ when $r > 1$ we have

$$\begin{aligned} I(p) &= \frac{a_1^s \psi(\mu d) \exp\left(\frac{i\pi s}{2r}\right) \Gamma\left(\frac{s}{r}\right)}{rp^{s/r}} + J(p) + o(p^{-s/r}) \\ &= \frac{a_1^s \psi(0) \exp\left(\frac{i\pi s}{2r}\right) \Gamma\left(\frac{s}{r}\right)}{rp^{s/r}} + J(p) + o(p^{-s/r}) . \end{aligned}$$

Finally an integration by parts yields $J(p) = 0(1/p)$ since $n-(r+1-s) \geqq 0$ by the choice of n and $D \in C$. This completes the proof of Lemma I for the case $r > 1$. For $0 < r \leqq 1$ one makes the change of variable $t = x^m$ and the desired result follows from the case $r > 1$.

LEMMA II. *Suppose that in addition to the assumptions of Lemma I that r is an even integer, $s = 1$, $\phi(\alpha t) > 0$ for $-c' \leqq t \leqq c'$, b satisfies the same conditions as in Lemma I except that M and m_0 are now determined for $-c' \leqq t \leqq c'$, and ω, ψ and ϕ are now in their respective differentiability classes given in Lemma I for $-c' \leqq t \leqq c'$. Then*

$$\int_{-b}^{b} \omega(t)\Psi(t) \exp(ip\Phi(t)) dt \backsim \frac{2\psi(0)\Gamma\left(\frac{1}{r}\right)\exp\left(\frac{i\pi}{2r}\right)}{r[p\phi(0)]^{1/r}} .$$

The proof follows immediately from Lemma I.

We will introduce the following functions which will be used throughout the remainder of the paper:

$$F(t) = t^\lambda f(t), \; G(t) = t^\nu g(t) \text{ and } Q(t) = t^{\gamma-1} q(t).$$

LEMMA III. *Under the general assumptions on F, G and Q we have for each arbitrarily small but fixed positive constant $c < a$ that*

$$L(h, k) = \int_c^a Q(t) \exp[i(hF(t) + kG(t))] dt = 0(1/h) .$$

Proof. Let $H(t) = F(t) + (k/h)G(t)$. Then $H'(t) > 0$ for $c \leqq t \leqq a$ and h, k sufficiently large since $\lambda f(t) + tf'(t) > 0$ by hypothesis and

$k = o(h)$. Hence an integration by parts implies $L(h, k) = 0(1/h)$.

This completes the necessary lemmas and the main results of the paper will now be presented.

4. The asymptotic evaluation of $I(h, k)$. We shall first consider the case where $kh^{-\nu/\lambda} \to 0$ so that $I(h, k)$ is almost completely determined by the character of hf at the origin.

THEOREM I. *Suppose that*

1. $f \in C^{n+2}$ and $q \in C^n$ for $0 \leqq t \leqq c$, $c > 0$, *where m and n are the least integers such that* $m\lambda > 1$ and $n \geqq m(\lambda - \gamma) + 1$, *respectively,*

2. *if $0 \leqq t \leqq c$ then $t^\nu g(\beta t) = b_0 + b_1 t + \cdots + b_{n-2}t^{n-2} + B(t)t^{n-1}$ where $B \in C$ and b_l is bounded as $\beta \to 0$ for $0 \leqq l \leqq n-2$,*

3. *$k^\lambda = o(h^\nu)$ and $\gamma < \lambda$. Then*

$$I(h, k) \backsim \frac{q(0)\Gamma\left(\frac{\gamma}{\lambda}\right)\exp\left(\frac{i\pi\gamma}{2\lambda}\right)}{\lambda[hf(0)]^{\gamma/\lambda}} .$$

Proof of Theorem I. For c as given we have

$$I(h,k) = \int_0^c + \int_c^a = I'(h, k) + 0(1/h)$$

by Lemma III. Let $t = xk^{-1/\nu}$, $\tilde{f}(x) = f(xk^{-1/\nu})$, $\tilde{g}(x) = g(xk^{-1/\nu})$, $\tilde{Q}(x) = Q(xk^{-1/\nu})$ and $p = hk^{-\lambda/\nu}$. For any b such that $0 < b < c$ we have

$$I'(h, k) = k^{-1/\nu}\int_0^b [\tilde{Q}(x)\exp(i\tilde{g}(x)x^\nu)]\exp(ip\tilde{f}(x)x^\lambda)dx$$
$$+ k^{-1/\nu}\int_b^{ck^{1/\nu}} = I''(h, k) + J(h, k), \text{ respectively.}$$

Set $\mu = k^{-1/\nu}$, $\psi = q$, $\phi = f$, $\lambda = r$, $\gamma = s$, $\omega(x) = \exp(ix^\nu\tilde{g}(x))$ and note that $f(0) > 0$ implies that $\tilde{f}(x) > 0$ for $0 \leqq x \leqq c'$, $c' > 0$, so that b may be chosen to satisfy the requirements of Lemma I. Hence by Lemma I

$$I''(h, k) \backsim \frac{q(0)\Gamma(\gamma/\lambda)\exp(i\pi\gamma/2\lambda)}{\lambda[hf(0)]^{\gamma/\lambda}} .$$

Therefore to complete the proof of Theorem I it is sufficient to show that $h^{\gamma/\lambda}J(h, k) = o(1)$. Let $d = bk^{-1/\nu}$, $H(t) = F(t) = + (k/h)G(t)$ and $P(t) = \lambda f(t) + tf'(t) + kt^{\nu-\lambda}/h[\nu g(t) + g'(t)t]$. Note that $P(d) \to \lambda f(0) = 2B > 0$ as $h, k \to \infty$ since $k^\lambda = 0(h^\nu)$ and $P(t)$ is continuous for $0 < d \leqq t \leqq a$. We may assume that c is such that for h, k sufficiently large, $P(t) \geqq B$ for the entire closed interval $d \leqq t \leqq c$. This implies $H'(t) \geqq Bt^{\lambda-1} > 0$ for $0 < d \leqq t \leqq c$ and hence we can integrate $J(h, k)$ by parts as follows;

$$J(h, k) = \int_d^c Q(t) \exp(ihH(t))dt = \frac{Q(c)\exp(ihH(c))}{ihH'(c)} - \frac{Q(d)\exp(ihH(d))}{ihH'(d)}$$
$$- \frac{1}{ih}\int_d^c \frac{Q'(t)\exp(ihH(t))dt}{H'(t)} + \frac{1}{ih}\int_d^c \frac{Q(t)H''(t)\exp(ihH(t))dt}{(H'(t))^2}$$
$$= 0(1/h) + A + J'(h, k), \text{ respectively.}$$

Using the estimates $H'(t) \geqq Bt^{\lambda-1}$ and $|H''(t)| \leqq Kt^{\lambda-2}$ for some K we see immediately that $J = 0(k^{(\lambda-\gamma)\nu}/h)$. Since $k^\lambda = o(h^\nu)$ this implies $h^{\gamma/\lambda}J(h, k) = o(1)$ which completes the proof of Theorem I.

We state the following corollary to Theorem I which may apply when $t^\nu g(\beta t)$ does not have the required smoothness at the origin but f, g and q are highly differentiable on $[0, c]$, $c > 0$.

Corollary. Suppose that $\nu + \gamma > \lambda$ *and*

1. $f \in C^{n+2}$, $g \in C^n$ and $q \in C^n$ for $0 \leqq t \leqq c$, $c > 0$ where m and n are the least integers such that $m(\nu+\gamma-\lambda) \geqq 2$, $m\lambda > 1$ and $n \geqq m(\lambda-\gamma)+1$,
2. $k^\lambda = o(h^\nu)$ and $\gamma < \lambda$. Then

$$I(h, k) \backsim \frac{q(0)\Gamma\left(\frac{\gamma}{\lambda}\right)\exp\left(\frac{i\pi\gamma}{2\lambda}\right)}{\lambda[hf(0)]^{\gamma/\lambda}}.$$

Proof. Note that $m\nu \geqq m(\lambda - \gamma) + 2 > n$ by the definition of n and hence $x^{m\nu} \in C^n$. The change of variable $t = x^m$ and the use of Theorem I completes the proof.

We shall next consider the case where the behavior of kg at the origin becomes a significant factor in the asymptotic evaluation of $I(h, k)$.

THEOREM II. *Suppose that*

1. $q \in C^n$ *and* $g \in C^{n+2}$ *for* $0 \leqq t \leqq c$, $c > 0$, *where* m *and* n *are the least integers such that* $m\nu > 1$ *and* $n \geqq m(\nu - \gamma) + 1$, *respectively,*
2. *if* $0 \leqq t \leqq c$ *then* $t^\lambda f(\beta t) = b_0 + b_1 t + \cdots + b_{n-2}t^{n-2} + B(t)t^{n-1}$ *where* $B \in C$ *and* b_l *is bounded as* $\beta \to 0$ *for* $0 \leqq l \leqq n - 2$,
3. $g(0) > 0$, $h^\nu = o(k^\lambda)$ *and* $\gamma < \nu < \lambda$. *Then*

$$I(h, k) \backsim \frac{q(0)\Gamma\left(\frac{\gamma}{\nu}\right)\exp\left(\frac{i\pi\gamma}{2\nu}\right)}{\nu[kg(0)]^{\gamma/\nu}}.$$

Proof of Theorem II. The proof of Theorem II follows from the proof of Theorem I with the roles of f and g, λ and ν, h and k intercharged.

COROLLARY. *Suppose that*

1. $f \in C^n$, $q \in C^n$ *and* $g \in C^{n+2}$ *for* $0 \leqq t \leqq c$, $c > 0$, *where* m *and* n *are the least integers such that* $m(\lambda + \gamma - \nu) \geqq 2$, $m\nu > 1$ *and* $n \geqq m(\nu - \gamma) + 1$,
2. $g(0) > 0$, $h^\nu = o(k^\lambda)$ *and* $\gamma < \nu < \lambda$. *Then*

$$I(h, k) \backsim \frac{q(0)\Gamma\left(\frac{\gamma}{\nu}\right)\exp\left(\frac{i\pi\gamma}{2\nu}\right)}{\nu[kg(0)]^{\gamma/\nu}} .$$

When $kh^{-\nu/\lambda} \longrightarrow \infty$ and $g(0) < 0$ the character of both F and G in a neighborhood of $t = 0$ becomes important since for h and k sufficiently large they determine uniquely in some $(0, c_0)$ a number τ such that $hF'(\tau) + kG'(\tau) = 0$ and in terms of which the asymptotic form of $I(h, k)$ may be expressed.

THEOREM III. *Suppose that* $g(0) < 0$, $\nu < \lambda$, $\gamma < \lambda$, $h^\nu = o(k^\lambda)$, $f \in C^6$, $g \in C^6$ *and* $q \in C^2$ *for* $0 \leqq t \leqq c$, $c > 0$, *and hypothesis* 1 *and* 2 *Theorem* II *are satisfied when* $\nu \geqq \gamma$.

A. *If* $\nu < 2\gamma$ *then*

$$I(h, k) \backsim \frac{\sqrt{2}\, q(0)\Gamma\left(\frac{1}{2}\right)\exp\left(\frac{i\pi}{4}\right)}{(\lambda - \nu)^{1/2}} \left[\frac{(\lambda h f(0))^{\nu - 2\gamma}}{(-\nu k g(0))^{\lambda - 2\gamma}}\right]^{1/2(\lambda - \nu)}$$
$$\times \exp\left[i(hF(\tau) + kG(\tau))\right] .$$

B. *If* $\nu = 2\gamma$ *then*

$$I(h, k) \backsim \frac{q(0)\Gamma\left(\frac{1}{2}\right)\exp\left(\frac{i\pi}{4}\right)}{(-\nu k g(0))^{1/2}} \left\{\frac{\sqrt{2}\exp\left[i(hF(\tau) + kG(\tau))\right]}{(\lambda - \nu)^{1/2}} - i\nu^{-1/2}\right\} .$$

C. *If* $\nu > 2\gamma$ *then*

$$I(h, k) \backsim \frac{q(0)\Gamma\left(\frac{\gamma}{\nu}\right)\exp\left(\frac{\gamma\pi}{2\nu i}\right)}{\nu[-kg(0)]^{\gamma/\nu}} .$$

Proof of Theorem III. We may assume that c is such that $G'(t) < 0$ and $f(t) > 0$ for $0 < t \leqq c$. For $0 < t \leqq c$ let $D(t) = F'(t)/-G'(t)$ with $D(0) = 0$. Then $D'(t) = t^{\lambda+\nu-\xi}/(G'(t))^2[\nu\lambda f(t)g(t)(\nu - \lambda) + tE(t)]$ for $0 < t \leqq c$ where E is continuous on $[0, c]$. Hence there exists c_0 such that $0 < c_0 < c$, $D'(t) > 0$ for $0 < t \leqq c_0$, $D(c_0) > 0$ and for h and k sufficiently large $k/h < D(c_0)$. This implies that there exists a unique $\tau \in (0, c_0)$ such that $D(\tau) = k/h$ which is equivalent to $hF'(\tau) + kG'(\tau) = 0$. Moreover from the definition of D we have

$$\tau = \left(\frac{-\nu k g(0)}{\lambda h f(0)}\right)^{1/\lambda - \nu} (1 + o(1))$$

which implies that $\tau^{\lambda-\nu} = o(k/h) = o(1)$.

If we now let $H(t) = F(t) + (k/h)G(t)$ and expand $h(H(t) - H(\tau))$ about $t = \tau$ we have using the integral form of the remainder

$$h(H(t) - H(\tau)) = h\int_\tau^t (t-y)F''(y)dy + k\int_\tau^t (t-y)G''(y)dy$$
$$= hR(t, \tau) + kS(t, \tau), \text{ respectively.}$$

We may further assume that c_0 is so small that f, f', f'', g, g' and g'' are of constant sign for $0 \leqq t \leqq c_0$. If we apply the mean value theorem for integrals and substitute $t = \tau(x+1)$ we have for $-1 < x < 1$ that

$$T(x, \tau) = R(\tau(x+1), \tau) = \frac{\tau^\lambda x^2}{2}[\lambda(\lambda-1)f(\tau_0(x))\alpha_0(x)$$
$$+ \tau(\lambda+1)\alpha_1(x)f'(\tau_1(x)) + \tau^2 f''(\tau_2(x))\alpha_2(x)] = \frac{\tau^\lambda x^2}{2}P_1(x)$$

where $\alpha_0, \alpha_1, \alpha_2 \in C^\infty$, $\alpha_0(0) = \alpha_2(0) = \alpha_1(0) = 1$, $P_1 \in C^4$ and $P_1(0) = \lambda(\lambda-1)f(0) + o(1)$. Similarly

$$W(x, \tau) = S(\tau(x+1), \tau) = \tfrac{1}{2}\tau^\nu x^2 P_2(x)$$

where $P_2 \in C^4$ and $P_2(0) = \nu(\nu-1)g(0) + o(1)$. Let $d_0 = (c_0/\tau) - 1$, $I'(h, k) = \exp(-ihH(\tau))I(h, k)$ and choose b such that $\tau(b+1) < c_0$ and $0 < b < 1$.

$$I'(h, k) = \int_0^{c_0} + 0\left(\frac{1}{h}\right) = I''(h, k) + 0\left(\frac{1}{h}\right).$$

$$I''(h, k) = \tau\int_{-1}^{-b} + \tau\int_{-b}^{b} + \tau\int_b^{d_0} Q(\tau(x+1))\exp[i(hT(x, \tau) + kW(x, \tau))]dx$$
$$= L(h, k) + I'''(h, k) + J(h, k), \text{ respectively.}$$

Let

$$\mu = \tau,\ p = (1/2)h\tau^\lambda,\ \psi = q,\ \omega(x) = (1+x)^{\gamma-1} \text{ and } \phi(x) = P_1(x) + (k\tau^{\nu-\lambda}/h)P_2(x).$$

Then $\phi(0) = \lambda(\lambda-\nu)f(0) + o(1)$ implies for h, k sufficiently large that $\phi(x) > 0$ for $-c' \leqq x \leqq c'$, $c' > 0$. Hence b may be chosen small enough that the conditions on b in Lemma II are satisfied. Therefore

$$I'''(h, k) \backsim \frac{\sqrt{2}\, q(0)\Gamma\left(\frac{1}{2}\right)\exp\left(\frac{i\pi}{4}\right)}{(\lambda-\nu)^{1/2}}\left[\frac{(\lambda h f(0))^{\nu-2\gamma}}{(-\nu k g(0))^{\lambda-2\gamma}}\right]^{1/2(\lambda-\nu)}.$$

The contribution of $L(h, k)$ to $I(h, k)$ may be determined by considering

$$L'(h, k) = \int_0^{\tau(1-b)} Q(t)\exp(ihH(t))dt.$$

We note that the uniqueness of τ in $[\varepsilon, c_0]$, $\varepsilon > 0$, implies that $H'(t) \neq 0$ on $\varepsilon \leqq t \leqq \tau(1-b)$ for every $\varepsilon > 0$. In fact, there exists a number $K > 0$ which is independent of ε and for which we have $|H'(t)| \geqq K(k/h)t^{\nu-1}$ for $\varepsilon \leqq t \leqq \tau(1-b)$.

(i) For $\nu < \gamma$ the usual integration by parts together with the above inequality for $H'(t)$ yields that $L'(h, k) = o(1/k)$. Hence $L'(h, k) = o(h^{\nu-2\gamma}/k^{\lambda-2\gamma})^{1/2(\lambda-\nu)}$ since $h^\nu = o(k^\lambda)$.

(ii) For $\nu > \gamma$ we rewrite $L'(h, k)$ as

$$L'(h, k) = \int_0^{\tau(1-b)} Q(t) \exp\{-i[kt^\nu(-g(t)) + ht^\lambda(-f(t))]\}dt$$

and apply Theorem II with $-g$ playing the role of f. Hence

$$L'(h, k) \backsim \frac{q(0)\Gamma\left(\frac{\gamma}{\nu}\right)\exp\left(\frac{-i\pi\gamma}{2\nu}\right)}{\nu[-kg(0)]^{\gamma/\nu}} .$$

(iii) Finally for $\nu = \gamma$ a closer examination of the proof of Lemma I together with the change of variable $t = xh^{-1/\lambda}$ implies for $p' = k/h^{\nu/\lambda}$ that $L'(h, k) = h^{-\gamma/\lambda}0(1/p') = 0(1/k)$.

The given relation $h^\nu = o(k^\lambda)$ and the calculation

$$k^{\gamma/\nu}0\left(\left(\frac{h^{\nu-2\gamma}}{k^{\lambda-2\gamma}}\right)^{1/2(\lambda-\nu)}\right) = 0\left(\left(\frac{h}{k^{\lambda/\nu}}\right)^{\nu-2\gamma/2(\lambda-\nu)}\right)$$

then imply that $I'''(h, k) = o(k^{-\gamma/\nu})$ if $\nu > 2\gamma$ and $L'(h, k) = o((h^{\nu-2\gamma}/k^{\lambda-2\gamma})^{1/2(\lambda-\nu)})$ if $\gamma \leqq \nu < 2\gamma$. When $\nu = 2\gamma$ we note that both $L'(h, k)$ and $I'''(h, k)$ are of the same order so that both terms contribute to $I(h, k)$.

To complete the proof of Theorem III we need only show that $J(h, k)$ is negligible compared to $I'''(h, k)$. For $P(t)$ defined as in the proof of Theorem I and $d = \tau(b+1)$ we have

$$P(d) = \lambda f(0)[1 - (b+1)^{\nu-\lambda}](1 + o(1)) .$$

Then $P(d) > 0$ for h and k sufficiently large and hence proceeding exactly as in the proof of Theorem I we obtain $H'(t) \geqq Bt^{\lambda-1} > 0$ for $0 < d \leqq t \leqq c_0$ and $2B = \lambda f(0)[1 - (1+b)^{\nu-\lambda}]$. We now write

$$J(h, k) = \int_d^{c_0} Q(t) \exp(ihH(t))dt$$

and integrate by parts as in Theorem I to obtain $J(h, k) = 0((h^{\nu-\gamma}/k^{\lambda-\gamma})^{1/\lambda-\nu})$. Hence $h^\nu = o(k^\lambda)$ implies that

$$\left(\frac{k^{\lambda-2\gamma}}{h^{\nu-2\gamma}}\right)^{1/2(\lambda-\nu)} J(h, k) = 0\left(\left(\frac{h^\nu}{k^\lambda}\right)^{1/2(\lambda-\nu)}\right) = o(1) .$$

To obtain the value of $\exp[i(hF(\tau) + kG(\tau))]$ in a more explicit form we need to know more about the exact relation between h and k. For example we shall state the following corollary under more stringent assumptions.

COROLLARY. *If in addition to the above assumptions in Theorem* III *we have* $k^{\lambda+1} = o(h^{\nu+1})$ *then*

$$\exp[i(hF(\tau) + kG(\tau))] \backsim \exp\left\{\frac{i(\nu-\lambda)}{\lambda\nu}\left[\frac{(-\nu k g(0))^{\lambda}}{(\lambda h f(0))^{\nu}}\right]^{1/\lambda-\nu}\right\}.$$

Proof of the Corollary to Theorem III. We will use the same notation as in the proof of Theorem III. If we expand $hH(\tau)$ about the origin and substitute for τ we have

$$\begin{aligned}\exp((ihH(\tau)) &= \exp\left\{i\left[hf(0)\left(\frac{-\nu k g(0)}{\lambda h f(0)}\right)^{\lambda/\lambda-\nu}\right.\right.\\ &\quad \left.\left.+ kg(0)\left(\frac{-\nu k g(0)}{\lambda h f(0)}\right)^{\nu/\lambda-\nu}\right]\left[1 + 0\left(\left(\frac{k}{h}\right)^{1/\lambda-\nu}\right)\right]\right\}\\ &= \exp\left\{\frac{i(\nu-\lambda)}{\lambda\nu}\left[\frac{(-\nu k g(0))^{\lambda}}{(\lambda h f(0))^{\nu}}\right]^{1/\lambda-\nu} + 0\left(\left(\frac{k^{\lambda+1}}{h^{\nu+1}}\right)^{1/\lambda-\nu}\right)\right\}.\end{aligned}$$

Hence if $k^{\lambda+1} = o(h^{\nu+1})$ the Corollary is established.

Finally, we shall consider the case where lim sup $kh^{-\nu/\lambda}$ is bounded away from both 0 and ∞.

THEOREM IV. *Suppose that* $\gamma < \lambda$, $\nu < \lambda$ *and* $0 < \limsup\ p < \infty$ *where* $p = kh^{-\nu/\lambda}$. *Then*

$$\begin{aligned} I(h,k) &\backsim q(0)h^{-\gamma/\lambda}\int_0^\infty x^{\gamma-1}\exp[i(f(0)x^\lambda + pg(0)x^\nu)]dx\\ &= q(0)\left(\frac{k}{h}\right)^{\gamma/\lambda-\nu}\int_0^\infty x^{\gamma-1}\exp[ip^{\lambda/\lambda-\nu}(f(0)x^\lambda + g(0)x^\nu)]dx\,.\end{aligned}$$

Proof of Theorem IV. We will consider only values of $c > 0$ such that (i) $\nu g(t) + tg'(t)$ is of constant sign for $0 \leqq t \leqq c$ and (ii) for each $\varepsilon > 0$ we have $|q(t) - q(0)| < \varepsilon$, $|f(t) - f(0)| < \varepsilon$ and $|g(t) - g(0)| < \varepsilon$ for $0 \leqq t \leqq c$. Set $H(t) = F(t) + (k/h)G(t)$ and $I'(h,k) = \int_0^c$ as usual. Let $m = \text{minimum}_{0\leqq t\leqq a}\lambda f(t) + tf'(t) > 0$, $\omega = \limsup p$ and $M = \text{maximum}_{0\leqq t\leqq a}(1, |f^{(l)}|, |q|, |q'|, |g^{(l)}|, |\nu g(t) + tg'(t)|)$ for $l = 0, 1, 2$. Consider a number $b > 1$ chosen such that $b > N = (4M\omega/m)^{1/(\lambda-\nu)}$. If $d = bh^{-1/\lambda} < c$ then for $0 < d \leqq t \leqq c$ we have for $g(0) < o$

$$H'(t) \geqq t^{\lambda-1}\left(m - \frac{kM}{hd^{\lambda-\nu}}\right) \geqq t^{\lambda-1}\left(m - \frac{mk}{4\omega h^{\nu/\lambda}}\right) \geqq \frac{1}{2}mt^{\lambda-1}$$

since $2\omega > kh^{-\nu/\lambda}$ for h and k sufficiently large. Hence $H'(t) \geqq (1/2)mt^{\lambda-1} > 0$ for $0 < d \leqq t \leqq c$. Let $t = xh^{-1/\lambda}$, $\tilde{q}(x) = q(xh^{-1/\lambda})$, $\tilde{f}(x) = f(xh^{-1/\lambda})$ and $\tilde{g}(x) = g(xh^{-1/\lambda})$. Then

$$\begin{aligned} h^{\gamma/\lambda} I'(h, k) &= \int_0^{ch^{1/\lambda}} x^{\gamma-1}\tilde{q}(x) \exp\,[i(\tilde{f}(x)x^\lambda + p\tilde{g}(x)x^\nu)]dx \\ &= \int_0^b + \int_b^{ch^{1/\lambda}} = I''(h, k) + J(h, k), \text{ respectively.} \end{aligned}$$

We will first estimate $J(h, k)$ in terms of the number b. Since $H'(t) \geqq (1/2)mt^{\lambda-1} > 0$ for $0 < d \leqq t \leqq c$ we may integrate $J(h, k)$ by parts as follows:

$$\begin{aligned} J(h, k) = h^{\gamma/\lambda} \Bigg\{ &\frac{Q(c)\exp(ihH(c))}{ihH'(c)} - \frac{Q(d)\exp(ihH(d))}{ihH'(d)} \\ &- \frac{1}{ih}\int_d^c \frac{Q'(t)\exp(ihH(t))dt}{H'(t)} + \frac{1}{ih}\int_d^c \frac{Q(t)F''(t)\exp(ihH(t))dt}{[H'(t)]^2} \\ &+ \frac{k}{ih^2}\int_d^c \frac{Q(t)G''(t)\exp(ihH(t))dt}{[H'(t)]^2} \Bigg\} \\ = 0(h^{\gamma-\lambda/\lambda}) &+ A + J'(h, k) + J''(h, k) + J'''(h, k), \text{ respectively.} \end{aligned}$$

Hence $|A| \leqq 2M/mb^{\lambda-\gamma} = Bb^{\gamma-\lambda}$,

$$|J'(h, k)| \leqq \frac{2Mh^{\gamma/\lambda}}{mh}\int_d^c t^{\gamma-\lambda-1}dt < \frac{2M}{m(\lambda-\gamma)b^{\lambda-\gamma}} = B'b^{\gamma-\lambda},$$

$$|J''(h, k)| \leqq \frac{4M^2}{m^2(\lambda-\gamma)b^{\lambda-\gamma}} = B''b^{\gamma-\lambda}, \text{ and}$$

$$|J'''(h, k)| \leqq \frac{4M^2kh^{-\nu/\lambda}}{m^2(2\lambda-\gamma-\nu)b^{2\lambda-\gamma-\nu}} \leqq \frac{8M^2\omega}{m^2(2\lambda-\gamma-\nu)b^{\lambda-\gamma}} = B'''b^{\gamma-\lambda}.$$

Define

$$h^{\gamma/\lambda} I_0(h, k) = \int_0^\infty x^{\gamma-1}q(0)\exp\,[i(f(0)x^\lambda + pg(0)x^\nu)]dx = \int_0^b + R(b).$$

Then there exists a number K which is independent of h, k and ε and for which $|J(h, k)| \leqq Kb^{\gamma-\lambda}$ and $|R(b)| \leqq Kb^{\gamma-\lambda}$. Consider

$$\begin{aligned} (*) h^{\gamma/\lambda}(I_0 - I) &= \int_0^b x^{\gamma-1}(q(0) - \tilde{q}(x))\exp\,[i(\tilde{f}(x)x^\lambda + p\tilde{g}(x)x^\nu)]dx \\ &\quad + q(0)\int_0^b x^{\gamma-1}(1 - P(x))\exp\,[i(f(0)x^\lambda + pg(0)x^\nu)]dx \\ &\quad + R(b) + 0(h^{\gamma-\lambda/\lambda}) - J(h, k) \\ &= L(h, k) + L'(h, k) + R(b) + 0(h^{\gamma-\lambda/\lambda}) - J(h, k), \\ &\qquad\qquad \text{respectively,} \end{aligned}$$

where $P(x) = \exp\{i[\tilde{f}(x) - f(0))x^\lambda + p(\tilde{g}(x) - g(0))x^\nu]\}$. By the choice of c for each $\varepsilon > 0$ we have $|L(h, k)| \leqq M\varepsilon b^{\lambda+\gamma}$, $|L'(h, k)| < 2M\varepsilon b^{\lambda+\gamma} + 2M\varepsilon\omega b^{\nu+\gamma}$. If we take lim sup of both sides of (*) as $h, k \to \infty$ we obtain

$$0 \leqq \limsup h^{\gamma/\lambda} | I_0 - I | \leqq 3M\varepsilon b^{\lambda+\gamma} + 2M\varepsilon b^{\nu+\gamma} + 2Kb^{\gamma-\lambda}$$

which is true for $\varepsilon > 0$ and $b > N$. Since $\limsup h^{\gamma/\lambda} | I_0 - I |$ is independent of both ε and b we first let $\varepsilon \to 0$ and then $b \to \infty$. Hence $h^{\gamma/\lambda}(I_0 - I) = o(1)$ which implies that $I(h, k) \backsim I_0(h, k)$. To obtain the alternate form of $I_0(h, k)$ we let $x = h^{1/\lambda}(k/h)^{1/(\lambda-\nu)}t$.

5. Discussion of the suggested application. Consider for $x > 0$ Schlafli's generalization of Bessel's integral:

$$\begin{aligned} J_\nu(x) &= \frac{1}{\pi}\int_0^\pi \cos(\nu t - x \sin t)dt - \frac{\sin \nu\pi}{\pi}\int_0^\infty \exp[-\nu t - x \sin h\, t]dt \\ &= \frac{1}{\pi} R \int_0^\pi \exp[i(\nu t - x \sin t)]dt + 0\left(\frac{1}{\nu}\right). \end{aligned}$$

Let $F(t) = t - \sin t$ and $|G(t)| = t$. We rewrite $F(t)$ as $F(t) = (1/6)t^3 \cos(r(t))$ and let $h = x$, $k = |\nu - x|$, $q(t) \equiv 1$ and $f(t) = 1\backslash 6 \cos(r(t))$. It follows that the condition $3f(t) + tf'(t) > 0$ for $0 \leqq t \leqq \pi$ is satisfied since $F'(t) = 1 - \cos t > 0$ for $0 < t \leqq \pi$.

We note that our Theorem I and III yield the dominant terms of some well known complete asymptotic expansions for $J_\nu(x)$ with τ=Arccos ν/x in Theorem III[1]. For the case $0 < \limsup x^{-1/3} | \nu - x | < \infty$ we have by Theorem IV with $p = x^{-1/3} | (\nu - x)x^{-1/3} |$ that

$$J_\nu(x) \backsim \frac{1}{\pi x^{1/3}} \int_0^\infty \cos\left(\frac{1}{6} t^3 + pt\right)dt$$

where the expression on the right is one of Airy's integrals[2], whose evaluation for $p > 0$ and $p < 0$ yields precisely Nicholson's formulas when ν is an integer[3].

[1] See W. Magnus and F. Oberhettinger, "Formeln und Satze fur die Speziellen Funktionen der Mathematischen Physik," Springer-Verlag, Berlin, 1948, pp. 33-34. Our theorems I and III give results which are equivalent to the dominant terms of the expansions (b_3) and (b_1), respectively.

[2] See, for example, G. N. Watson, "Theory of Bessel Functions," Cambridge, 1944, pp. 188-190.

[3] See G. N. Watson, op. cit., pp. 248-249.

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