PACIFIC JOURNAL OF MATHEMATICS
Vol. 18, No. 2, 1966

ON CLOSED MAPPINGS, BICOMPACT SPACES, AND A PROBLEM OF P. ALEKSANDROV

A. ARHANGEL'SKII

The purpose of this paper is to show, under very general circumstances, that if $f: X \to Y$ is a closed map, then $f^{-1}y$ must be bicompact for "most" $y \in Y$. Two theorems of this sort are obtained, one of which is then used to answer a question of P. Alexandroff on the effect of closed maps on countable-dimensional spaces.

If $f: X \to Y$ is a closed map, then it is known that, under suitable assumptions, $f^{-1}y$ has a bicompact boundary for all $y \in Y$ (see I. Vaĭnšteĭn [19], A. H. Stone [18], and K. Morita and S. Hanai [12]), and $f^{-1}y$ itself is bicompact for "most" $y \in Y$ (see K. Morita [11] and the author [4]). In §§1 and 2 of this paper, we prove two theorems of the latter kind, whose main feature is that they require minimal restrictions on X and no restriction at all (other than being T_1) on Y.

In §3, we give some applications of the results from §2. The most interesting among them is the following, which gives a complete answer to a question of P. Alexandroff, (Terminology is defined below).

THEOREM (3.1). *Let X be a countable-dimensional space with a countable net, and let $f: X \to Y$ be a closed mapping of X onto some uncountable-dimensional space Y. Then $Y_1 = \{y \in Y \mid \text{card}\,(f^{-1}y) \geqq \mathfrak{c}\}$ is uncountable-dimensional.*

Observe that Theorem 3.1 is new even in case X is compact metric. In that case, E. Skljarenko [15] has shown that Y_1 is not void, but his proof gives no further information about Y_1. Our proof is based on entirely different ideas.

Let me say here that I am very grateful to P. Alexandroff for valuable discussions about this question and to E. Michael for helping with the translation of this paper.

Notation and terminology. All spaces are completely regular (often is it sufficient to suppose T_1); all mappings are continuous, and all coverings are open. We call a family $\gamma = \{S\}$ of sets $S \subseteqq X$ a *net* in X, if, for every $x \in X$ and each open U containing x, there exists an $S \in \gamma$ with $x \in S \subseteqq U$ (see [3]). We write card A for the cardinality of A, and $\mathfrak{c}$ for the cardinality of the continuum. If γ is a family of subsets of a space X, and if $x \in X$, then γx denotes the union of all elements of γ containing x. As usual, we call a space *countable-dimensional* if it is a countable union of subspaces with dim $= 0$;

otherwise, we call the space *uncountable-dimensional*. We write βX for the Stone-Čech bicompactification of X, and we call X a *G_δ-space* if it is a G_δ in βX. We call X *point-paracompact* if every (open) covering of X has an (open) point-finite refinement. Finally, X is called a *k-space* if a subset $U \subseteqq X$ is open whenever its intersection with every compact $K \subseteqq X$ is open in K.

1. Closed mappings of point-paracompact G_δ-spaces.

THEOREM (1.1). *Let X be a point-paracompact G_δ-space, and let $f: X \to Y$ be a closed map. Then*

$$Y = Y_0 \cup (\bigcup_{n=1}^{\infty} Y_n) ,$$

where Y_n is discrete in Y for all n, and $f^{-1}y$ is bicompact for all $y \in Y_0$.

For the proof of this theorem, we need

LEMMA (1.2). *Let X be a k-space, let γ be a point-finite covering of X, and let $f: X \to Y$ be a closed mapping of X onto some Y. Then*

$$N = \{y \in Y \mid \text{no finite } \gamma' \subseteqq \gamma \text{ covers } f^{-1}y\}$$

is discrete in Y.

Proof. Suppose that some $y \in Y$ is an accumulation point for N. Then the set $N_1 = N \backslash y$ is not closed. Since X is a k-space and f a closed mapping, Y is a k-space [8]. Therefore there exists a bicompact $F \subseteqq Y$ such that $F \cap N_1$ is not closed, and hence is infinite. Let $\{y_n\}$ be a sequence of distinct points from $F \cap N_1$. Since F is bicompact, there exists an accumulation point y' for this sequence, which we may suppose different from all y_n. We let $A_n = f^{-1}y_n$ for $n = 1, 2, \cdots$. Next we shall define a sequence $\{x_n\}$, with $x_n \in A_n$, where for x_1 we take any point from A_1. Suppose the points x_k are defined for all $k < m$. We take for x_m any point of the set $A_m \backslash \bigcap_{i=1}^{m-1} \gamma x_i$; this set is not empty by the very definition of N.

Now we prove that the sequence $\{x_n\}$ is discrete. Consider any point $x \in X$. We only have to consider the case where $\gamma x \cap \{x_n\} \neq \varnothing$. Let $x_m \in \gamma x$; then $x \in \gamma x_m$ and $U = \gamma x_m$ is a neighborhood of x; by the definition of x_n, this U can contain only points x_n with $n \leqq m$. Thus the discreteness of $\{x_n\}$ is proved.

It follows that $P = \bigcup_{n=1}^{\infty} x_n$ is closed, while the set $Q = fP = \bigcup_{n=1}^{\infty} y_n$ is not (since $y' \in \bar{Q} \backslash Q$). This contradiction completes the proof of the lemma.

We now proceed to the

Proof of (1.1). Let $\{G_n\}$ be a countable family of open subsets of βX such that $X = \bigcap_{n=1}^{\infty} G_n$. We write γ_n for some covering of X (by open sets in X) such that the closure in βX of each element of γ is contained in G_n. We take a point-finite refinement λ_n of γ_n. For $n = 1, 2, \cdots$, let

$$Y_n = \{y \in Y \mid \text{no finite } \lambda'_n \subseteqq \lambda_n \text{ covers } f^{-1}y\} .$$

It is known [5] that every G_δ-space is a k-space. Thus, by Lemma (1.2), Y_n is discrete in Y. We write $Y_0 = Y \backslash \bigcup_{n=1}^{\infty} Y_n$.

We now prove that $f^{-1}y$ is bicompact for every $y \in Y_0$. For each n there exists a finite $\lambda'_n \subseteqq \lambda_n$, say $\lambda'_n = \{V_i^n \mid i = 1, \cdots, k(n)\}$, such that $f^{-1}y \subseteqq \bigcup_{i=1}^{k(n)} V_i^n$. Then F_n, the closure of $\bigcup_{i=1}^{k(n)} V_i^n$ in βX, is bicompact, and $f^{-1}y \subseteqq F_n \subseteqq G_n$. Therefore

$$f^{-1}y \subseteqq F = \bigcap_{n=1}^{\infty} F_n \subseteqq \bigcap_{n=1}^{\infty} G_n = X ,$$

where F is bicompact. As $f^{-1}y$ is closed in X, it follows that $f^{-1}y$ is bicompact too. This completes the proof of (1.1).

REMARK. As the proof shows, a result analogous to (1.1) could be obtained for k-spaces X which, for some cardinal τ, are $G_{\delta(\tau)}$-spaces (i.e. an intersection of τ open subsets of βX). In particular, taking $\tau = 1$, we conclude: *If $f: X \to Y$ is a closed mapping, and if X is point-paracompact and locally bicompact, then the set of all $y \in Y$ such that $f^{-1}y$ is not bicompact is discrete in Y.* This is a slight generalization of a theorem of K. Morita [11], whose proof needs the assumption that X is *paracompact* and locally bicompact.

In case X is a Lindelöf space, the conclusion in (1.1) can be simplified.

COROLLARY (1.3). *If X is a Lindelöf G_δ-space, and if $f: X \to Y$ is a closed map, then $f^{-1}y$ is bicompact for all but countably many $y \in Y$.*

Proof. Since a (regular) Lindelöf space is paracompact, and hence surely point-paracompact, Theorem (1.1) is applicable. Now Y, as the continuous image of the Lindelöf space X, is itself Lindelöf, and hence all its discrete subsets are countable. Hence the set $\bigcup_{n=1}^{\infty} Y_n$ in (1.1) is countable, and that proves the corollary.

2. Closed mappings of spaces with countable net. The class of spaces with countable net (see the introduction for definition of net) contains all separable metric spaces and all their continuous images. Spaces with countable net are Lindelöf, and hence paracompact.

The main result of this section (Theorem (2.1)) is similar to Corollary (1.3), but the hypotheses are different. Note that the hypotheses of (1.3) are satisfied by complete separable metric spaces, while the hypotheses of (2.1) are satisfied by *all* separble metric spaces.[1)]

THEOREM (2.1). *If X is a space with countable net, and $f: X \to Y$ is a closed mapping, then $f^{-1}y$ is bicompact for all but countably many $y \in Y$.*

The proof is based on Lemma (2.2) below, which will also be used in the proof of Theorem (2.3). We will use the following terminology: If X is a space with a net γ, and if $x \in X$, then an *x-sequence* is a sequence $\{S_n(x)\}$, with $x \in S_n(x) \in \gamma$ for all n, such that any sequence $\{x_n\}$ with $x_n \in S_n(x)$ for all n has an accumulation point in X.

LEMMA (2.2). *Let $f: X \to Y$ be a closed mapping of a normal space X with net γ, such that for each $y \in Y$ there exists an $x \in f^{-1}y$ possessing an x-sequence $\{S_n(x)\}$. Let Y_1 be the set of all $y \in Y$ such that $f^{-1}y$ is not countably compact. Then* card $Y_1 \leqq$ card γ.

Proof. Without loss of generality, we may suppose that $S_1 \cap S_2 \in \gamma$ for every pair $S_1, S_2 \in \gamma$. We define Y_0 as the set of all $y \in Y$ such that $y = \bigcap \{f(S) \mid S \in \gamma'\}$ for some finite subcollection $\gamma' \subseteqq \gamma$. Clearly card $Y_0 \leqq$ card γ (if γ is infinite). Consider any point $y \in Y \backslash Y_0$. Our purpose is to show that $f^{-1}y$ is countably compact. Then the conclusion of the theorem will follow.

Let $x \in f^{-1}y$ be a point with an x-sequence $\{S_n(x)\}$. Suppose that $F = f^{-1}y$ is not countably compact, and pick $x_1, x_2 \cdots$ is F such that x_i is discrete. By an obvious induction (using the normality of X), we can construct a discrete sequence $\{U_i\}$ of open sets such that $x_i \in U_i$. Now, again by induction, we define three sequences $\{x_n\}$, $\{x'_n\}$, and $\{0_n x\}$, where x_n, $x'_n \in X$ and $0_n x$ is a neighborhood of x, such that

(a) for $n > k$, $0_k x$ contains x_n;
(b) for $n \leqq k$, $X \backslash (0_k x)^- \supseteqq f^{-1}fx_n$;
(c) for all n, $x_n \in S_n(x)$;
(d) for all $n > 1$, $x'_n \in U_n$;
(e) for all n, $fx_n = fx'_n$.

We take x_1 to be any point of $X \backslash F$, and for $0_1 x$ any neighborhood of

[1] For separable metric X, (2.1) also follows from a recent result of N. Lashnev.

x which satisfies (b) (with $n = k = 1$). Suppose that x_n, x'_n, and $0_n x$ are already defined for $n \leqq k$ so as to satisfy (a)-(e). Since γ is a net, there exist S, $S' \in \gamma$ such that $x_{k+1} \in S \subseteqq 0_k x$ and $x_{k+1} \in S' \subseteqq U_{k+1}$. Let $S^* = S \cap S_{k+1}(x)$. Then $S^* \in \gamma$ and $(fS^* \cap fS')\backslash y \neq \varnothing$.

Let $x_{k+1} \in S^*$, $x'_{k+1} \in S'$ be points such that $fx_{k+1} = fx'_{k+1} \neq y$. Finally, we take for $0_{k+1}x$ some neighborhood of x such that

$$x \in 0_{k+1}x \subseteqq (0_{k+1}x)^- \subseteqq X \backslash \bigcup_{i=1}^{k+1} f^{-1}fx_i \ .$$

Clearly conditions (a)-(e) are satisfied, and we can go further in our induction.

Since $\{S_n(x)\}$ is an x-sequence, the set $P = \{x_n\}$ has in X an accumulation point, say x^*. The conditions imply that $x^* \notin f^{-1}fP$. Then $fx^* \in (fP)^- \backslash fP$, and hence fP is not closed. On the other hand, $fP = fQ$, where $Q = \{x'_n\}$. Since $x'_n \in U_n$, and the family U_n is discrete, Q is closed in X. Thus fQ is closed in Y, and we have a contradiction which completes the proof of (2.2).

Proof of (2.1). In a space X with a *countable* net γ, every $x \in X$ has an x-sequence, namely all elements of γ containing x. Also, as observed earlier, such an X is paracompact, and hence it is normal and all closed, countably compact subsets are bicompact. We therefore see that (2.1) follows from (2.2).

In the following theorem, a space X is of *point-countable type* (see [5]) if it is the union of bicompact subsets K having a countable base of neighborhoods $\{U_n\}$ (i.e., if $V \supseteqq K$ is open, then $K \subseteqq U_n \subseteqq V$ for some n).[2] All first-countable spaces, and all p-spaces (in the sense of [5]) are spaces of point-countable type.

THEOREM (2.3). *Let X be a normal point-paracompact space of point-countable type, with a net γ of cardinality $\leqq \tau$. If $f\colon X \to Y$ is a closed mapping, and*

$$Y_1 = \{y \in Y \mid f^{-1}y \textit{ is not bicompact}\} \ ,$$

then card $Y_1 \leqq \tau$.

Proof. Let us show that every $x \in X$ has a γ-sequence $\{S_n(x)\}$, so that (2.2) applies. Pick a compact $K \subseteqq X$ such that $x \in K$ and K has a countable base of neighborhoods $\{U_n\}$. Clearly, if we pick $S_n(x)$ so that $x \in S_n(x) \subseteqq U_n$, then $\{S_n(x)\}$ is an x-sequence. Applying (2.2), we have the conclusion of the theorem since, in a point-paracompact space, closed countably compact subsets are bicompact.

[2] For point-paracompact spaces, the spaces of point-countable type are the same as the q-spaces in the sense of E. Michael [**10**].

3. An application.

THEOREM (3.1). *If X is a countable-dimensional space with countable net γ, and $f: X \to Y$ is a closed mapping onto an uncountable-dimensional space Y, then*

$$Y_1 = \{y \in Y \mid \text{card } (f^{-1}y) \geqq \mathfrak{c}\}$$

is uncountable dimensional.

The proof is based on two lemmas.

LEMMA (3.2). *Let f: $X \to Y$ be a closed mapping of a space X with a countable net $\gamma = \{S_i\}$ such that, for each $y \in Y$, the set $f^{-1}y$ contains a point which is isolated in $f^{-1}y$. Then Y is a countable sum of subspaces, each of which is homeomorphic to a subspace of X.*

Proof. Without loss of generality, we may suppose that all $S_i \in \gamma$ are closed in X. For each i, let $f_i^* = f \mid S_i$, and let

$$X_i = \{x \in S_i \mid f_i^{*-1} f_i^* x = x\} .$$

Let $Y_i = fX_i$, and let us show that X_i is homeomorphic to Y_i and $Y = \bigcup_{i=1}^{\infty} Y_i$

Since f is closed, so is its restriction f_i^* to the closed set S_i. Now $X_i = f_i^{*-1} Y_i$, so if f_i: $X_i \to Y_i$ is defined by $f_i = f_i^* \mid X_i$, then f_i is also closed. Since f_i is clearly continuous and one-to-one, it is a homeomorphism.

It remains to show that $Y = \bigcup_{i=1}^{\infty} Y_i$. Let $y \in Y$, and let x be an isolated point of $f^{-1}y$. Then $0x = X \backslash (f^{-1}y \backslash x)$ is a neighborhood of x. Since γ is a net in X, there exists an $S_k \in \gamma$ such that $x \in S_k \subseteqq 0x$. Then $x = S_k \cap f^{-1}y$, and thus $x \in X_k$. It follows that

$$y = fx \in f_k X_k = Y_k \subseteqq \bigcup_{i=1}^{\infty} Y_i .$$

This completes the proof of (3.2).

LEMMA (3.3). *If $f: X \to Y$ is a closed mapping of a space X with a countable net γ, such that card $(f^{-1}y) < \mathfrak{c}$ for all $y \in Y$, then Y is a countable sum of subspaces each of which is homeomorphic to a subspace of X.*

Proof. By (2.1), $Y = Y_0 \cup Y_1$, where Y_0 is countable and $f^{-1}y$ is bicompact for each $y \in Y_1$. Consider $f_1 = f \mid X_1$, where $X_1 = f^{-1} Y_1$. If $y \in Y_1$, then card $f_1^{-1}y < \mathfrak{c}$, so $f_1^{-1}y$ has an isolated point. Thus X_1,

f_1 satisfy the condition of (3.2), and hence Y_1 is a countable sum of subspaces which are homeomorphic to subspaces of $X_1 \subseteqq X$. But Y_0 is a countable sum of points, and hence the conclusion follows.

Proof of (3.1). Let $Y_0 = \{y \in Y \mid \text{card } f^{-1}y < \mathfrak{c}\}$. Consider $f_0 = f \mid X_0$, where $X_0 = f^{-1}Y_0$. The map $f_0: X_0 \to Y_0$ satisfies all the conditions of (3.3). Since X is a space with a countable net, X is hereditarily Lindelöf. Hence every subspace of X is countable-dimensional [17]. By (3.3), Y_0 is thus also countable-dimensional. Therefore $Y \setminus Y_0$ is uncountable-dimensional, and the proof of (3.1) is complete.

Theorem (3.1) is new even for compact metric X, where it can be rephrased as follows:

COROLLARY (3.4). *If $f: X \to Y$ is a mapping of a countable-dimensional compact metric space X onto an uncountable-dimensional space Y, then $\{y \in Y \mid \text{card } f^{-1}y = \mathfrak{c}\}$ is uncountable-dimensional.*

REMARK 1. Another application of (2.2): A space is called *dyadic* [14] if there exists a dyadic bicompact extension of this space. We call a dyadic space *nowhere countable* if there are no nonempty countable open sets in it.

THEOREM (3.5). *If a nowhere countable dyadic space Y is a closed image of a separable metric space, then there exists a countable base in Y.*

Proof. By (3.1), there exists a countable set $Y_0 \subseteqq Y$ such that, if $Y_1 = Y \setminus Y_0$, $X_1 = f^{-1}Y_1$, and $f_1: X_1 \to Y_1$ is defined by $f_1 = f \mid X_1$, then f_1 is a perfect map. Hence Y_1 is a space with countable base. Since Y is nowhere countable, Y_1 is dense in Y. Hence Y is metrizable, by a theorem of B. Efimov [7]. This completes the proof.[3)]

REMARK 2. The following generalization of a theorem of J. Nagata [13] could also be proved.

THEOREM (3.6). *Let X be a compact metric space, and $f: X \to Y$ a map such that $f^{-1}y$ is finite for all $y \in Y$. Then, for each countable-dimensional $X' \subseteqq X$, the space $Y' = fX'$ is countable-dimensional.*

[3] This theorem shows a simple way for constructing nondyadic spaces.

BIBLIOGRAPHY

1. P. S. Aleksandrov, *Some results in the theory of topological spaces obtained within the last twenty-five years*, Uspehi Mat. Nauk (92) **15** (1960), 25-95. (Russian Math. Surveys **15** (1960), 23-83).
2. ———, *On some basic trends in general topology*. Uspehi Mat. Nauk **19** (1964), 3-46.
3. A. Arhangel'skiĭ, *An addition theorem for the weight of sets lying in bicompacts*, Dokl. Akad. Nauk SSSR **126** (1959), 239-241.
4. ———, *Some type of factor mappings and the relations between classes of topological space*, Dokl. Akad. Nauk SSSR **153** (1963), 743-746. (Soviet Math. Dokl. **4** (1963), 1726-1729).
5. ———, *On a class of spaces containing all metric and all bicompact spaces*, Dokl. Akad. Nauk SSSR **151** (1963), 751-754, (Soviet Math. Dokl. **4** (1963), 1051-1055).
6. A. Archangielski and W. Holsztynski, *Sur les reseaux dans les espaces topologiques*, Bull. Acad. Polon. Sci. Ser. Math. **11** (1963), 493-797.
7. B. Efimov, *On dyadic spaces*, Dokl. Akad. Nauk SSSR **151** (1963), 1021-1024. (Soviet Math. Dokl. **4** (1963), 1131-1134).
8. J. L. Kelly, *General Topology*, van Nostrand, New York, 1955.
9. B. T. Levšenko, *On the concept of compactness and point-finite coverings*, Mat. Sb. N. S. **42** (84) (1957), 478-484.
10. E. Michael, *A note on closed maps and compact sets*, Israel J. Math. 2 (1964), 173-176.
11. K. Morita, *On closed mappings*, Proc. Japan Acad. **32** (1956), 539-543.
12. K. Morita and S. Hanai, *Closed mappings and metric spaces*, Proc. Japan Acad. **32** (1956), 10-14.
13. J. Nagata, *On countable dimensional spaces*, Proc. Japan Acad. **34** (1958), 146-149.
14. V. I. Ponomarev, *On dyadic spaces*, Fund. Math. **52** (1963), 1013-1016.
15. E. G. Skljarenko, *Some remarks on spaces having an infinite number of dimensions*. Dokl. Akad. Nauk SSSR **126** (1959), 1203-1206.
16. ———, *Two theorems on infinite-dimensional spaces*, Dokl. Akad. Nauk SSSR **143** (1962), 1053-1056. (Soviet Math. Dokl. **3** (1962), 547-550).
17. Yu. M. Smirnov, *On the theory of finally compact spaces*, Ukrain. Mat. Zurnal **3** (1951), 52-60.
18. A. H. Stone, *Metrizability of decomposition spaces*, Proc. Amer, Math. Soc. **7** (1956), 690-700.
19. E. A. Vaĭnšteĭn, *On closed mappings*, Moskov. Gos. Univ. Uč. Zap. 155, Mat. **5** (1952), 3-53.

Received May 14, 1965.

UNIVERSITY OF MOSCOW
MOSCOW, U.S.S.R.