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ON THE LATTICE OF CLOSED SUBSPACES OF HILBERT SPACE

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The purpose of this note is to answer two questions which have arisen in connection with the lattice-theoretic characterization of the set of closed subspaces of a Hilbert space of countably infinite dimension which appears in "Axioms for nonrelativistic quantum mechanics," Pacific Journal of Mathematics, Vol. 11, No. 3, 1961, pages 1151–1169.

The material in this section is to replace [2, p. 1165, lines 10–32]. Up to that point it has been shown that the lattice under each finite element of P is isomorphic to the lattice of subspaces of a Hilbert space over a field D which is either real or complex. The orthocomplementation induced in a Hilbert space by such an isomorphism gives rise to an involution of D (vide infra). In this section we show that such an involution is continuous, thereby closing a gap brought to our attention by a comment of M. D. Maclaren.

Let $a \in P_f$ with $n = \dim a > 0$. Choose pairwise orthogonal points $A_0, \cdots, A_n$ in (a) and in each line $l_i = A_0 \vee A_i$, $i = 1, \cdots, n$, choose, a point E_i different from A_0 and A_i. Clearly the points $A_0, E_1, \cdots, E_n$ are independent and the choice of $A_1 \vee \cdots \vee A_n$ as improper hyperplane, A_0 as origin and $E_1, \cdots, E_n$ as unit points leads to the unique introduction of homogeneous coordinates in (a) in standard fashion. In particular, the proper points of l_1 are precisely those with homogeneous coordinates $(1, \lambda, 0, \cdots, 0)$ which we abbreviate as $(1, \lambda)$ — λ, of course, being any member of the field D that has been constructed. The topology for D is obtained as follows: The subset N of D is a neighborhood of 0 if $\{(1, \nu): \nu \in N\}$ is a neighborhood of A_0 in l_1. Under this topology, D is either the real or complex field (cf. [2, Lemma 2.11 et seq., p. 1164]).

It is shown in [1] that there then exist an involution σ of D and numbers (= members of D) $\eta_0, \cdots, \eta_n$ such that

(1) $\eta_i^\sigma = \eta_i$,

(2) $\sum x_i \eta_i x_i^\sigma = 0$ if and only if all $x_i = 0$,

(3) If $(x_0, \cdots, x_n) \in (a)_0$, then $a(x_0, \cdots, x_n)'$ (the complement of $(x_0, \cdots, x_n)$ in $(a)) = \vee \{(y_0, \cdots, y_n) \in (a)_0: \sum y_i \eta_i x_i^\sigma = 0\}$

Note that by (2), no η_i is 0 and that $1, \eta_1/\eta_0, \cdots, \eta_n/\eta_0$ defines the same orthomorplementation as $\eta_0, \cdots, \eta_n$; i.e., we may assume that $\eta_0 = 1$.

Again confining our attention to l_1, observe that if $\lambda \neq 0$ and $l_1(1, \lambda)'$ (the point of l_1 orthogonal to the point $(1, \lambda)$) has coordinates

$(1, \mu)$, then $\mu = -1/\eta_1\lambda^\sigma$. Hence if $\lambda_m \to 1$ and is never 0, $(1, \lambda_m) \to (1, 1)$ by definition (of the topology for D) so $(1, \lambda_m)' \to (1, 1)' = (1, -1/\eta_1)$ by [2, Lemma 2.8]. But $(1, \lambda_m)' = (1, \mu_m)$ with $\mu_m = -1/\eta_1\lambda_m^\sigma$. Then $(1, \mu_m) \to (1, -1/\eta_1)$ which implies $\mu_m \to -1/\eta_1$; i.e., $-1/\eta_1\lambda_m^\sigma \to -1/\eta_1$ so $\lambda_m^\sigma \to 1$. Thus, σ is continuous at 1 and hence is continuous (if $\lambda_m \to 0$ then $\lambda_m + 1 \to 1$ so $(\lambda_m + 1)^\sigma = \lambda_m^\sigma + 1 \to 1$ so $\lambda_m^\sigma \to 0$). Of course, this result was automatic in the real case. It follows that σ is either the identity or, in the complex case, conjugation. It follows now from (2) that $\eta_1, \cdots, \eta_n$ are positive real numbers. If D is the complex numbers, σ is conjugation, for otherwise $(1, i\eta_1^{-1/2}, 0, \cdots, 0)$ would be self-orthogonal.

Taking the Hilbert space of $n + 1$ tuples of D as H_a, the mapping $(x_0, \cdots, x_n) \to \{\lambda(x_0, \cdots, x_n): \lambda \in D\}$ clearly induces a continuous isomorphism φ_a of (a) on the lattice L_a of subspaces of H_a such that the orthocomplementation induced by φ_a in L_a is obtained from the inner product $(x, y) = \sum x_i\eta_i\bar{y}_i$ for H_a.

2. The following is a replacement for [2, p. 1165, lines 33 to 41]. Its purpose is to insure that all the isometries $\psi_{b,a}$ are linear rather than conjugate linear. I am indebted to V. S. Varadarajan for calling my attention to this omission.

Let $a \leqq b$ be finite and suppose that, in accordance with what has preceded, we have selected a Hilbert space H_a over D of dimension $1 + \dim a$ and a continuous isomorphism φ_a of (a) on the lattice L_a of subspaces of H_a which is orthogonality-preserving in the sense that

$$\varphi_a(c) \perp \varphi_a(d) \text{ if and only if } c \perp d . \tag{13}$$

Suppose that H_b, φ_b, have been similarly chosen for b.

Now $\varphi_b\varphi_a^{-1}$ is a continuous, orthogonality-preserving isomorphism of L_a in L_b. Hence, as is well-known and not difficult to show, there exists a continuous automorphism σ of D and a σ-isometry $\psi_{b,a}$, unique up to multiplication by a number of modulus one, providing $\dim a > 0$ (see below), such that $\psi_{b,a}$ induces $\varphi_b\varphi_a^{-1}$ in the sense that $\varphi_b\varphi_a^{-1}[v] = [\psi_{b,a}v]$ for all $v \in H_a$, where $[v]$ denotes the linear subspace generated by v. A σ-isometry ψ of H is a mapping of H in itself with the following three properties:

$$\begin{aligned} &\text{Additivity:} && \psi(u + v) = \psi(u) + \psi(v) \\ &\sigma\text{-linearty:} && \psi(\lambda u) = \lambda^\sigma\psi(u) \\ &\sigma\text{-isometry:} && (\psi(u), \psi(v)) = (u, v)^\sigma . \end{aligned} \tag{14}$$

A σ-isometry is said to be *linear* or *conjugate-linear* when σ is the identity or conjugation respectively.

If D is the real field, the automorphism σ is the identity, while in the complex case, in view of its continuity, σ may be either the identity or conjugation. Observe that if dim $a = 0$ and u, v are unit vectors in H_a, $\varphi_b(a)$ respectively, then $\lambda u \to \lambda v$ and $\lambda u \to \bar{\lambda} v$ both induce the mapping $\varphi_b\varphi_a^{-1}$ of L_a in L_b. In other words, $\psi_{b,a}$ may be chosen both linear and conjugate-linear when dim $a = 0$, independent of the choice of H_a, φ_a and H_b, φ_b. In general, the linearity of $\psi_{b,a}$ may be achieved through the proper choice of H_b, φ_b as follows. Suppose that $\psi_{b,a}$ inducing $\varphi_b\varphi_a^{-1}$ is conjugate-linear. Let $\{v_i\}$ be a complete orthonormal set for H_b and define $\gamma: H_b \to H_b$ by: $\gamma(\sum \lambda_i v_i) = \sum \bar{\lambda}_i v_i$. Let φ denote the automorphism of L_b induced by γ and let $\bar{\varphi}_b = \varphi \circ \varphi_b$. Then $\bar{\varphi}_b$ is a continuous, orthogonality-preserving isomorphism of (b) on L_b which is induced by the linear isometry $\bar{\psi}_{b,a} = \gamma \circ \psi_{b,a}$.

Suppose now that dim $a > 0$, that H_a, φ_a have been chosen arbitrarily and that for every finite $b > a$, H_b, φ_b has been chosen as above so that $\varphi_b\varphi_a^{-1}$ is "linear" in the sense that every isometry of H_a in H_b which induces it is linear. For each finite $c \not> a$ let $H_c = \varphi_{a\vee c}(c)$ and let $\varphi_c = \varphi_{a\vee c} \mid (c)$. Then $\varphi_{a\vee c}\varphi_c^{-1}$ is linear, for it is induced by the projection in $H_{a\vee c}$ of its subspace H_c.

Now that H_c, φ_c have been assigned to every finite c, it remains to show that $\varphi_{c_1}\varphi_{c_2}^{-1}$ is in fact linear whenever $c_2 < c_1$. The type of argument we shall use involves the introduction of $c_3 < c_2$ for which both $\varphi_{c_1}\varphi_{c_3}^{-1}$ and $\varphi_{c_2}\varphi_{c_3}^{-1}$ are known to be linear. The linearity of $\varphi_{c_1}\varphi_{c_2}^{-1}$ then follows from the equation $\varphi_{c_1}\varphi_{c_3}^{-1} = (\varphi_{c_1}\varphi_{c_2}^{-1})(\varphi_{c_2}\varphi_{c_3}^{-1})$.

Given finite $c_2 < c_1$, let $b_i = c_i \vee a$, $i = 1, 2$. Now $b_2 \leqq b_1$ and $\varphi_{b_1}\varphi_{b_2}^{-1}$ is linear, for $\varphi_{b_i}\varphi_a^{-1}$, $i = 1, 2$ are linear by construction and $\varphi_{b_1}\varphi_a^{-1} = (\varphi_{b_1}\varphi_{b_2}^{-1})(\varphi_{b_2}\varphi_a^{-1})$. Since $\varphi_{b_1}\varphi_{b_2}^{-1}$ is linear and $\varphi_{b_2}\varphi_{c_2}^{-1}$ is linear by construction, $\varphi_{b_1}\varphi_{c_2}^{-1} = (\varphi_{b_1}\varphi_{b_2}^{-1})(\varphi_{b_2}\varphi_{c_2}^{-1})$ is linear. Finally, since $\varphi_{b_1}\varphi_{c_2}^{-1}$ is linear and $\varphi_{b_1}\varphi_{c_i}^{-1}$ is linearby construction, the linearity of $\varphi_{c_1}\varphi_{c_2}^{-1}$ follows from the equation $\varphi_{b_1}\varphi_{c_2}^{-I} = (\varphi_{b_1}\varphi_{c_1}^{-1})(\varphi_{c_1}\varphi_{c_2}^{-1})$.

Thus, each finite c has been provided with H_c, φ_c in such a way that $c < d$ implies $\varphi_d\varphi_c^{-1}$ may be induced by a linear isometry $\psi_{d,c}$ of H_c in H_d which is unique up to multiplication by a number of modulus one. Our next task is to show that these arbitrary multipliers may be chosen consistently; i.e., so that

$$a < b < c \text{ implies } \psi_{c,a} = \psi_{c,b}\psi_{b,a}\,. \tag{15}$$

3. Erratum, page 1167, line 4 from bottom.
For "$\sum_{i=1}^{n} \lambda_i(u)\psi_{b_n,a_i}$" read "$\sum_{i=1}^{n} \lambda_i(u)\psi_{b_n,a_i}u_i$".

REFERENCES

1. G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*, Ann. of Math. **37** (1936), 823-843.
2. N. Zierler, *Axioms for nonrelativistic quantum mechanics*, Pacific J. Math. **11** (1961), 1151-1169.

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