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"THE δ-POINCARÉ ESTIMATE"

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The main application of the theorem proved here is to establish the local solvability of a system of linear partial differential equations, in the analytic case, by a homological procedure based on the associated Spencer resolution and δ-cohomology. The theorem states that the δ-cohomology associated with an involutive system of partial differential equations vanishes in a normed sense. From this one can show that the Spencer resolution associated with an involutive system is exact for analytic data, and thus by a result of D. G. Quillen the corresponding inhomogeneous system has local solutions, provided the inhomogeneous term is analytic and satisfies the appropriate compatibility conditions in the overdetermined case. It is well known that if an arbitrary system is prolonged a sufficient number of times, the resulting system will have vanishing δ-cohomology. According to a result of J. P. Serre this is equivalent to the resulting system being involutive. Thus the question of local solvability reduces to the involutive case, and we obtain the classical existence theorem of Cartan-Kähler.

We prove the theorem only in the case of first order systems. However, there is no loss of generality here because any linear system of partial differential equations can be changed into an equivalent first order system by viewing the lower order derivatives as new variables (see Quillen [3], Prop. 8.2). The theorem proved here was first stated by Spencer [4], who subsequently gave a proof in his paper [5] which, however, is incomplete. Later Ehrenpreis, Guillemin, and Sternberg [1] obtained estimates, by a method different from ours, which are equivalent as far as the above application is concerned. The result of Quillen mentioned above is contained in [3], the result of Serre can be found in [2].

1. The δ-sequence. Let M be a C^∞ manifold of dimension n, and let E and F be vector bundles over M with fiber dimensions m and l. Denote by $\underline{E}$ and $\underline{F}$ the sheaves of germs of C^∞ sections of E and F.

We introduce the jet bundles $J_\mu(E)$ for nonnegative integers μ. The fiber of $J_\mu(E)$ over $q \in M$ is obtained from the stalk $\underline{E}_q$ by indentifying germs which agree up to order μ (in any local coordinate). Coordinates in $J_\mu(E)$ are introduced as follows. Choose a coordinate neighborhood $U \subset M$, with coordinate $x = (x_1, \cdots, x_n)$, over which E

is trivial, and choose a coordinate in the fibers of $E\,|\,U$. Then for $q \in U$, $\sigma \in J_\mu(E)_q$ will have components

$$\sigma_p^j = \partial_p f^j(q)\ ; \qquad |\,p\,| \leqq \mu,\ j = 1, \cdots, m$$

where $f = (f^1, \cdots, f^m)$ is a local section representing σ, $p = (p_1, \cdots, p_n)$ is an n-tuple of nonnegative integers, $|\,p\,| = p_1 + \cdots + p_n$, and

$$\partial_p = \partial^{|p|}/\partial x_1^{p_1} \cdot \cdots \cdot \partial x_n^{p_n}\ .$$

There is a map $j_\mu\colon \underline{E} \to \underline{J_\mu(E)}$ which is given in local coordinates by

$$f \longrightarrow \{\partial_p f^j \,|\, |\,p\,| \leqq \mu,\ j = 1, \cdots, m\}\ .$$

There is also a projection $\pi\colon J_\mu(E) \to J_{\mu-1}(E)$ given in local coordinates by

$$\{\sigma_p^j \,|\, |\,p\,| \leqq \mu,\ j = 1, \cdots, m\} \longrightarrow \{\sigma_p^j \,|\, |\,p\,| \leqq \mu - 1,\ j = 1, \cdots, m\}\ .$$

The same definitions apply to $J_\mu(F)$. We define $S_{d,\mu}$ to be the kernel of $\pi\colon J_\mu(E) \to J_{\mu-1}(E)$, and $T_{d,\mu}$ to be the kernel of $\pi\colon J_\mu(F) \to J_{\mu-1}(F)$.

Now let $\mathscr{D}\colon \underline{E} \to \underline{F}$ be a first-order linear partial differential operator. $\mathscr{D}$ is represented by maps $\rho_\mu\colon J_\mu(E) \to J_{\mu-1}(F)$, $\mu \geqq 1$, which make the diagrams

$$\begin{array}{ccc} \underline{J_\mu(E)} & \xrightarrow{\rho_\mu} & \underline{J_{\mu-1}(F)} \\ \uparrow{\scriptstyle j_\mu} & & \uparrow{\scriptstyle j_{\mu-1}} \\ \underline{E} & \xrightarrow{\mathscr{D}} & \underline{E} \end{array} \tag{1.1}$$

commute. It follows that $\rho_\mu(S_{d,\mu}) \subset T_{d,\mu-1}$, and we define S_μ to be the kernel of $\rho_\mu\colon S_{d,\mu} \to T_{d,\mu-1}$. We shall assume that S_μ is a subbundle of $S_{d,\mu}$ for each $\mu \geqq 1$.

Let $U \subset M$ be a coordinate neighborhood, with coordinate $x = (x_1, \cdots, x_n)$, over which E and F are trivial. In terms of this coordinate there are maps $\delta_\nu\colon J_\mu(E)\,|\,U \to J_{\mu-1}(E)\,|\,U\,(\nu = 1, \cdots, n)$ given by

$$(\delta_\nu \sigma)_p^j = \sigma_{p+1_\nu}^j\ ; \qquad |\,p\,| \leqq \mu - 1,\ j = 1, \cdots, m$$

where $p + 1_\nu = (p_1, \cdots, p_{\nu-1}, p_\nu + 1, p_{\nu+1}, \cdots, p_n)$. The maps δ_ν represent the operator

$$\frac{\partial}{\partial x_\nu}\colon \underline{E}\,|\,U \longrightarrow \underline{E}\,|\,U$$

in the sense of (1.1). It follows that $\delta_\nu(S_{d,\mu}\,|\,U) \subset S_{d,\mu-1}\,|\,U$ and that $\delta_\nu(S_\mu\,|\,U) \subset S_{\mu-1}\,|\,U$. Similarly, we have maps $\delta_\nu\colon J_\mu(F)\,|\,U \to J_{\mu-1}(F)\,|\,U$ with $\delta_\nu(T_{d,\mu}\,|\,U) \subset T_{d,\mu-1}\,|\,U$. We shall use the following notation for various objects G (e.g., for one of the bundles $S_\mu\,|\,U$ or for a fiber

$(S_\mu)_q$ or $(S_{d,\mu})_q$ over $q \in U$):

$$
\begin{aligned}
&G^n = G \\
(1.2) \qquad &G^\nu = \{\sigma \in G \mid \delta_n(\sigma) = \cdots = \delta_{\nu+1}(\sigma) = 0\} , \qquad 1 \leqq \nu < n \\
&G^0 = 0 .
\end{aligned}
$$

DEFINITION. The local coordinate x is called regular at $q \in U$ if the maps

$$(1.3) \qquad \delta_\nu \colon (S_{\mu+1})^\nu_q \longrightarrow (S_\mu)^\nu_q$$

are surjective for $1 \leqq \nu \leqq n$ and all $\mu \geqq 1$. The operator $\mathscr{D}$ is called involutive if there is a regular coordinate at each $q \in M$.

We denote by T^* the cotangent bundle of M and form the δ-sequence

$$0 \longrightarrow S_{\mu+n} \xrightarrow{\delta} S_{\mu+n-1} \otimes \Lambda^1 T^* \xrightarrow{\delta} \cdots \xrightarrow{\delta} S_\mu \otimes \Lambda^n T^* \longrightarrow 0 ,$$

where δ is formal exterior differentiation. In terms of a local coordinate x, δ is given by

$$\delta\zeta = \sum_1^n dx_\nu \wedge \delta_\nu \zeta .$$

In terms of a local coordinate x on a neighborhood $U \subset M$ and a coordinate in the fibers of $E \mid U$ we can define a norm $\|\cdot\|$ in the fibers of $S^\mu \mid U$ by

$$\|\sigma\| = \sup\{|\sigma^j_p| \mid |p| = \mu,\ j = 1, \cdots, m\} .$$

This norm can be extended to $(S_\mu \otimes \Lambda^r T^*) \mid U$ by

$$\|\zeta\| = \sup\{\|\zeta_{i_1,\dots,i_r}\| \mid 1 \leqq i_1 < i_2 < \cdots < i_r \leqq n\}$$

if $\zeta = \Sigma\zeta_{i_1,\dots,i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}$, $\zeta_{i_1,\dots,i_r} \in S_\mu$. The main theorem of this paper can now be stated.

THEOREM. *Let $\mathscr{D}$ be involutive. Then for every $q \in M$ there exists a local coordinate $x = (x_1, \cdots, x_n)$ on a neighborhood U of q, a coordinate in $E \mid U$, and a constant $K > 0$ such that whenever $\xi \in (S_\mu \otimes \Lambda^r T^*) \mid U$, $r \geqq 1$ and $\delta\xi = 0$, then $\xi = \delta\zeta$ for some $\zeta \in (S_{\mu+1} \otimes \Lambda^{r-1} T^*) \mid U$ with $\|\zeta\| \leqq K\|\xi\|$.*

2. The constant coefficient case. In this section we shall assume that $M \subset \boldsymbol{R}^n$ is the open unit disk and that $E = M \times \boldsymbol{R}^m$ and $F = M \times \boldsymbol{R}^l$ are trivial bundles. We assume that $\mathscr{D} \colon \underline{E} \to \underline{F}$ has the form

(2.1) $$\mathscr{D}f = \sum_1^n A_i \frac{\partial f}{\partial x_i}$$

in the given coordinates, where each A_i is a constant $l \times m$ matrix, $f = (f^1, \cdots, f^m)$, and $\partial f/\partial x_i = (\partial f^1/\partial x_i, \cdots, \partial f^m/\partial x_i)$.

Denote by E' and F' the restrictions of the bundles E and F to the unit disk in $\boldsymbol{R}^{n-1} = \boldsymbol{R}^{n-1} \times 0 \subset \boldsymbol{R}^n$, and define the operator $\mathscr{D}'\colon \underline{E}' \to \underline{F}'$ by

$$\mathscr{D}'f = \sum_1^{n-1} A_i \frac{\partial f}{\partial x_i} .$$

The data already defined for $\mathscr{D}$ are defined for $\mathscr{D}'$ in a corresponding manner. The maps $\rho'_\mu\colon J_\mu(E') \to J_{\mu-1}(F')$ represent $\mathscr{D}'$, $S'_{d,\mu}$ is the kernel of $\pi\colon J_\mu(E') \to J_{\mu-1}(E')$, $T'_{d,\mu}$ is the kernel of $\pi\colon J_\mu(F') \to J_{\mu-1}(F')$, and S'_μ is the kernel of $\rho'_\mu\colon S'_{d,\mu} \to T'_{d,\mu-1}$.

Because of the assumption of constant coefficients, it will suffice to work in the fibers over a single point in M, say 0. To simplify notation we will abbreviate $(S_{d,\mu})_0$, $(S_\mu)_0$, $(S'_\mu)_0$, etc., by $S_{d,\mu}$, S_μ, S'_μ, etc. According to this convention we can write $S'_\mu = S_\mu^{n-1}$ (see (1.2)).

The element $\sigma \in S_{d,\mu}$ can be identified with the polynomial

$$\sigma = \sum_{|p|=\mu} \frac{x^p}{p!} \sigma_p ,$$

where $p! = p_1! \cdot \cdots \cdot p_n!$, $x^p = x_1^{p_1} \cdot \cdots \cdot x_n^{p_n}$, and $\sigma_p = (\sigma_p^1, \cdots, \sigma_p^m) \in \boldsymbol{R}^m$. Similarly, $\tau \in T_{d,\mu}$ can be identified with

$$\tau = \sum_{|p|=\mu} \frac{x^p}{p!} \tau_p .$$

Under these identifications ρ_μ goes over into $\mathscr{D}$ (see (1.1)).

We now identify $S_{d,\mu}$ with $S'_{d,0} \oplus S'_{d,1} \oplus \cdots \oplus S'_{d,\mu}$ by writing $\sigma \in S_{d,\mu}$ as a polynomial

$$\sigma = \sum_{\nu=0}^{\mu} \frac{x_n^{\mu-\nu}}{(\mu-\nu)!} \sigma^{(\nu)} ,$$

where $\sigma^{(\nu)}$ is a homogeneous polynomial of degree ν in $x_1, \cdots, x_{n-1}$ with coefficients in $\boldsymbol{R}^m$, i.e., $\sigma^{(\nu)} \in S'_{d,\nu}$. We write $\sigma = (\sigma^{(0)}, \cdots, \sigma^{(\mu)})$. Similarly, we identify $T_{d,\mu}$ with $T'_{d,0} \oplus \cdots \oplus T'_{d,\mu}$.

Now let $\sigma = (\sigma^{(0)}, \cdots, \sigma^{(\mu)}) \in S_{d,\mu}$ and let $\rho_\mu \sigma = \tau = (\tau^{(0)}, \cdots, \tau^{(\mu-1)})$. We have

$$\begin{aligned}\tau = \rho_\mu \sigma &= \mathscr{D}\Big(\sum_0^\mu \frac{x_n^{\mu-\nu}}{(\mu-\nu)!}\sigma^{(\nu)}\Big)\\ &= \sum_0^\mu \Big\{\mathscr{D}'\Big(\frac{x_n^{\mu-\nu}}{(\mu-\nu)!}\sigma^{(\nu)}\Big) + A_n \frac{\partial}{\partial x_n}\Big(\frac{x_n^{\mu-\nu}}{(\mu-\nu)!}\sigma^{(\nu)}\Big)\Big\}\\ &= \sum_1^\mu \frac{x_n^{\mu-\nu}}{(\mu-\nu)!}\mathscr{D}'(\sigma^{(\nu)}) + \sum_0^{\mu-1} \frac{x_n^{\mu-\nu-1}}{(\mu-\nu-1)!}A_n\sigma^{(\nu)}\\ &= \sum_1^\mu \frac{x_n^{\mu-\nu}}{(\mu-\nu)!}\{\rho'_\nu\sigma^{(\nu)} + A_n\sigma^{(\nu-1)}\}\,.\end{aligned}$$

Thus

$$\tau^{(\nu-1)} = \rho'_\nu\sigma^{(\nu)} + A_n\sigma^{(\nu-1)} \tag{2.2}$$

for $\nu = 1, \cdots, \mu$.

PROPOSITION 1. Let $\mathscr{D}$ be given by (2.1) and assume that the coordinate x is regular at 0. Then there exist nonzero constants $t_1, \cdots, t_n$, a constant $c > 0$, and maps

$$r_\mu\colon \rho_\mu(S_{d,\mu}) \longrightarrow S_{d,\mu}\,, \qquad \mu \geqq 1\,,$$

with $\rho_\mu r_\mu = 1$, $\|r_\mu\| \leqq c$, and $\delta_n r_{\mu+1} = r_\mu \delta_n$ for $\mu \geqq 1$, where the norm $\|\cdot\|$ is defined by the coordinate $(t_1x_1, \cdots, t_nx_n)$ in M.

Proof (*by induction on* n). The case $n = 1$ is trivial; assume the proposition true when n is replaced by $n-1$. Since the coordinate $(x_1, \cdots, x_{n-1})$ is regular for $\mathscr{D}'$, the inductive hypothesis yields $t_1, \cdots, t_{n-1}$, $c' > 0$, and maps

$$r'_\mu\colon \rho'_\mu(S'_{d,\mu}) \longrightarrow S'_{d,\mu}\,, \qquad \mu \geqq 1$$

with $\rho'_\mu r'_\mu = 1$ and $\|r'_\mu\| \leqq c'$ in the norm defined by $(t_1x_1, \cdots, t_{n-1}x_{n-1})$.

The map δ_ν defined by the coordinate $(t_1x_1, \cdots, t_{n-1}x_{n-1})$ is a nonzero multiple of the map δ_ν defined by $(x_1, \cdots, x_n)$. Thus if we change to the coordinate $(t_1x_1, \cdots, t_{n-1}x_{n-1}, x_n)$ the kernels and images in (1.3) do not change; and the new coordinate is regular at 0 for $\mathscr{D}$. We shall assume that this change has already been made so that $t_1 = \cdots = t_{n-1} = 1$.

Let $r_1\colon \rho_1(S_{d,1}) \to S_{d,1}$ be any splitting of the sequence

$$0 \longrightarrow S_1 \longrightarrow S_{d,1} \xrightarrow{\rho_1} \rho_1(S_{d,1}) \longrightarrow 0\,;$$

we can assume that $\|r_1\| \leqq c'$. Let $\mu > 1$, $\sigma \in S_{d,\mu}$, and $\tau = \rho_\mu\sigma$. Let

$$\begin{aligned}\sigma &= (\sigma^{(0)}, \cdots, \sigma^{(\mu)})\,,\\ \tau &= (\tau^{(0)}, \cdots, \tau^{(\mu-1)})\,,\end{aligned}$$

so that by (2.2)

$$\tau^{(\nu-1)} = \rho'_\nu \sigma^{(\nu)} + A_n \sigma^{(\nu-1)} , \qquad 1 \leqq \nu \leqq \mu .$$

We define $r_\mu \tau = \eta = (\eta^{(0)}, \cdots, \eta^{(\mu)})$ as follows. Define $\eta^{(0)}$ and $\eta^{(1)}$ by

$$(\eta^{(0)}, \eta^{(1)}) = r_1(\tau^{(0)}) .$$

This is possible since $(\tau^{(0)}) = \rho_1(\sigma^{(0)}, \sigma^{(1)}) \in \rho_1(S_{d,1})$. Now $\rho_1(\eta^{(0)} - \sigma^{(0)}, \eta^{(1)} - \sigma^{(1)}) = 0$, and thus $(\eta^{(0)} - \sigma^{(0)}, \eta^{(1)} - \sigma^{(1)}) \in S_1$. Since $\delta_n : S_2 \rightarrow S_1$ is surjective, there is a $\kappa^{(2)} \in S'_{d,2}$ such that $(\eta^{(0)} - \sigma^{(0)}, \eta^{(1)} - \sigma^{(1)}, \kappa^{(2)}) \in S_2$. By (2.2) we have $\rho'_2 \kappa^{(2)} + A_n(\eta^{(1)} - \sigma^{(1)}) = 0$. Since $\tau^{(1)} = \rho'_2 \sigma^{(2)} + A_n \sigma^{(1)}$, we have $\tau^{(1)} - A_n \eta^{(1)} = \rho'_2(\sigma^{(2)} + \kappa^{(2)}) \in \rho'_2(S'_{d,2})$. Thus we may define

$$\eta^{(2)} = r'_2(\tau^{(1)} - A_n \eta^{(1)}) .$$

It follows from (2.2) that $\rho_2(\eta^{(0)}, \eta^{(1)}, \eta^{(2)}) = (\tau^{(0)}, \tau^{(1)})$. Repeating this procedure several times we obtain $r_\mu \tau = \eta = (\eta^{(0)}, \cdots, \eta^{(\mu)})$ such that

$$\begin{aligned} &\rho_\mu \eta = \tau \\ &\eta^{(\nu)} = r'_\nu(\tau^{(\nu-1)} - A_n \eta^{(\nu-1)}) , \qquad 2 \leqq \nu \leqq \mu . \end{aligned}$$

The map r_μ is clearly linear and the commutativity $\delta_n r_{\mu+1} = r_\mu \delta_n$ is an immediate consequence of the construction. We turn to the statement about norms.

We have $\| \eta \| = \sup_\nu \| \eta^{(\nu)} \|$, $\| \tau \| = \sup_\nu \| \tau^{(\nu)} \|$, and $\| \eta^{(\nu)} \| \leqq c' \| \tau \|$ for $\nu = 1, 2$. For $2 \leqq \nu \leqq \mu$ we have

$$\| \eta^{(\nu)} \| \leqq c' \| \tau \| + c' \| A_n \| \| \eta^{(\nu-1)} \| .$$

Suppose we make the coordinate change $y_n = t_n x_n$. Then (2.1) becomes

$$\mathscr{D} f = \sum_1^{n-1} A_i \frac{\partial f}{\partial x_i} + (t_n A_n) \frac{\partial f}{\partial y_n} .$$

We shall assume that this coordinate change has already been made with $t_n \neq 0$ small enough to insure $\| A_n \| \leqq 1/2c'$. Then for $\nu \geqq 2$,

$$\begin{aligned} \| \eta^{(\nu)} \| &\leqq c' \| \tau \| + \frac{1}{2} \| \eta^{(\nu-1)} \| \\ &\leqq c'(1 + 2^{-1} + \cdots + 2^{2-\nu}) \| \tau \| + 2^{1-\nu} \| \eta^{(1)} \| \\ &\leqq 3c' \| \tau \| . \end{aligned}$$

We may take $c = 3c'$.

PROPOSITION 2. Assume that x is regular at 0 and that the conclusion of Proposition 1 holds with $t_1 = \cdots = t_n = 1$. Then there exists a constant $C > 0$ and maps

$$\eta_\mu\colon S_{\mu-1} \longrightarrow S_\mu\,, \qquad \mu \geqq 2\,,$$

such that $\delta_n\eta_\mu = 1$ and $\|\eta_\mu\| \leqq C$ for $\mu \geqq 2$.

Proof. For $\mu \geqq 2$ we have the exact commutative diagram (see (1.2)):

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S^{n-1}_{d,\mu} & \xrightarrow{\rho_\mu} & \rho_\mu(S_{d,\mu})^{n-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_\mu & \longrightarrow & S_{d,\mu} & \xrightarrow{\rho_\mu} & \rho_\mu(S_{d,\mu}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \delta_n} & & \\
0 & \longrightarrow & S_{\mu-1} & \longrightarrow & S_{d,\mu-1} & \xrightarrow{\rho_{\mu-1}} & \rho_{\mu-1}(S_{d,\mu-1}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{2.3}
$$

By hypothesis $\delta_n\colon S_\mu \to S_{\mu-1}$ is surjective, and thus by diagram chasing $\rho_\mu(S^{n-1}_{d,\mu}) = \rho_\mu(S_{d,\mu})^{n-1}$. Let c, r_μ be the data from Proposition 1. Since $\delta_n r_\mu = r_{\mu-1}\delta_n$, we have $r_\mu(\rho_\mu(S_{d,\mu})^{n-1}) \subset S^{n-1}_{d,\mu}$. Define $\delta_{-n}\colon S_{d,\mu-1} \to S_{d,\mu}$ by

$$(\delta_{-n}\sigma)^j_p = \begin{cases} \sigma^j_{p-1_n}\,, & \text{if } p_n > 0 \\ 0\,, & \text{if } p_n = 0 \end{cases}$$

where $p - 1_n = (p_1, \cdots, p_{n-1}, p_n - 1)$. Then $\delta_n\delta_{-n} = 1$. Diagram chasing now shows that the map $\eta_\mu\colon S_{\mu-1} \to S_\mu$ defined by

$$\eta_\mu\sigma = (\delta_{-n} - r_\mu\rho_\mu\delta_{-n})\sigma\,, \qquad \sigma \in S_{\mu-1}\,,$$

satisfies $\delta_n\eta_\mu = 1$. If we take $M \geqq \|A_1\| + \cdots + \|A_n\|$, then $\|\rho_\mu\| \leqq M$ for all $\mu \geqq 1$. Thus

$$\|\eta_\mu\| \leqq \|\delta_{-n}\| + \|r_\mu\|\,\|\rho_\mu\|\,\|\delta_{-n}\| \leqq 1 + cM\,.$$

We may take $C = 1 + cM$.

PROPOSITION 3. Let $\mathscr{D}$ be given by (2.1) and assume that the coordinate x is regular at 0. Then there exist constants $c, K > 0$ and nonzero constants $t_1, \cdots, t_n$ such that:

(i) there exist maps $r_\mu\colon \rho_\mu(S_{d,\mu}) \to S_{d,\mu} (\mu \geqq 1)$ satisfying $\rho_\mu r_\mu = 1$, $\delta_n r_{\mu+1} = r_\mu\delta_n$, $\|r_\mu\| \leqq c$;

(ii) for any $\xi \in S_\mu \otimes \Lambda^r T^* (\mu \geqq 1, r \geqq 1)$ satisfying $\delta\xi = 0$ there exists $\zeta \in S_{\mu+1} \otimes \Lambda^{r-1} T^*$ with $\delta\zeta = \xi$ and $\|\zeta\| \leqq K\|\xi\|$; where the norm $\|\cdot\|$ is defined by the coordinate $(t_1x_1, \cdots, t_nx_n)$.

Proof (*by induction on* n). The case $n = 1$ is trivial; assume the proposition true when n is replaced by $n - 1$.

Applying the inductive hypothesis to $\mathscr{D}'$, we obtain nonzero constants $t_1, \cdots, t_{n-1}$ and constants $c', K' > 0$ such that (i) and (ii) hold for the primed data. By the inductive step of Proposition 1, there exist $t_n \neq 0$ and $c > 0$ such that (i) holds for $\mathscr{D}$. We can assume that a coordinate change has already been made so that $t_1 = \cdots = t_n = 1$. Proposition 2 yields a constant $C > 0$ and maps $\eta_\mu \colon S_{\mu-1} \to S_\mu$ such that $\delta_n \eta_\mu = 1$ and $\| \eta_\mu \| \leqq C$ for $\mu \geqq 2$.

Now let $\xi \in S_\mu \otimes \Lambda^r T^*$ and assume $\delta\xi = 0$. Write $\xi = dx_n \wedge \xi^1 + \xi^2$, where ξ^1 and ξ^2 have no terms involving dx_n. If we set $\tau = \eta_{\mu+1}\xi^1$ (more precisely, $\tau = (\eta_{\mu+1} \otimes 1)(\xi^1)$), then $\delta\tau = dx_n \wedge \xi^1 +$ (terms not involving dx_n) so that $\xi - \delta\tau$ has no terms involving dx_n. Since $\delta(\xi - \delta\tau) = 0$, we must have $dx_n \wedge \delta_n(\xi - \delta\tau) = 0$ and thus $\delta_n(\xi - \delta\tau) = 0$. This means that $\xi - \delta\tau \in S'_\mu \otimes \Lambda^r T^{*\prime}$ and thus by the inductive hypothesis $\xi - \delta\tau = \delta\sigma$ for some $\sigma \in S'_{\mu+1} \otimes \Lambda^{r-1} T^{*\prime}$ with $\| \sigma \| \leqq K' \| \xi - \delta\tau \|$.

If we let $\zeta = \sigma + \tau$, then $\delta\zeta = \xi$ and $\| \zeta \| \leqq \| \sigma \| + \| \tau \| \leqq K' \| \xi \| + K' \| \delta\tau \| + \| \tau \|$. Since $\| \delta \| \leqq n$ and $\| \tau \| \leqq C \| \xi \|$, we have $\| \zeta \| \leqq (K' + nK'C + C) \| \xi \|$, and we may take $K = K' + nK'C + C$.

With part (ii) of Proposition 3 the theorem is proved in the constant coefficient case. In treating the general case, we shall use some additional results.

PROPOSITION 4. *Let $\mathscr{D}$ be given by* (2.1) *and let $\mu \geqq 1$. Then the maps*

$$\delta_\nu \colon S^\nu_{\mu+1} \longrightarrow S^\nu_\mu \qquad (\nu = 1, \cdots, n) \tag{2.4}$$

are surjective if and only if the maps

$$\rho_{\mu+1} \colon S^\nu_{d,\mu+1} \longrightarrow \rho_{\mu+1}(S_{d,\mu+1})^\nu \qquad (\nu = 1, \cdots, n-1) \tag{2.5}$$

are surjective (see (1.2)).

Proof. For $0 \leqq \nu \leqq n-1$ we have the following exact commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S^\nu_{\mu+1} \arrow[r] \arrow[d] & S^\nu_{d,\mu+1} \arrow[r] \arrow[d] & \rho_{\mu+1}(S^{\nu+1}_{d,\mu+1})^\nu \arrow[d] & \\
0 \arrow[r] & S^{\nu+1}_{\mu+1} \arrow[r] \arrow[d, "\delta_{\nu+1}"] & S^{\nu+1}_{d,\mu+1} \arrow[r] \arrow[d, "\delta_{\nu+1}"] & \rho_{\mu+1}(S^{\nu+1}_{d,\mu+1}) \arrow[r] \arrow[d, "\delta_{\nu+1}"] & 0 \\
0 \arrow[r] & S^{\nu+1}_{\mu} \arrow[r] & S^{\nu+1}_{d,\mu} \arrow[r] \arrow[d] & \rho_{\mu}(S^{\nu+1}_{d,\mu}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

By diagram chasing we find that the surjectivity of the maps (2.4) is equivalent to

$$\rho_{\mu+1}(S^{\nu}_{d,\mu+1}) = \rho_{\mu+1}(S^{\nu+1}_{d,\mu+1})^{\nu}$$

for $1 \leqq \nu \leqq n-1$. Thus if the maps (2.4) are surjective, then

$$\begin{aligned}\rho_{\mu+1}(S^{\nu}_{d,\mu+1}) &= \rho_{\mu+1}(S^{\nu+1}_{d,\mu+1})^{\nu} = [\rho_{\mu+1}(S^{\nu+2}_{d,\mu+1})^{\nu+1}]^{\nu} \\ &= \rho_{\mu+1}(S^{\nu+2}_{d,\mu+1})^{\nu} = \cdots = \rho_{\mu+1}(S^{n}_{d,\mu+1})^{\nu} = \rho_{\mu+1}(S_{d,\mu+1})^{\nu}\end{aligned}$$

and the maps (2.5) are surjective. Conversely, if the maps (2.5) are surjective, then for $1 \leqq \nu \leqq n-1$ we have $\rho_{\mu+1}(S^{\nu}_{d,\mu+1}) \subset \rho_{\mu+1}(S^{\nu+1}_{d,\mu+1})^{\nu} \subset \rho_{\mu+1}(S_{d,\mu+1})^{\nu} = \rho_{\mu+1}(S^{\nu}_{d,\mu+1})$, and the maps (2.4) are surjective.

PROPOSITION 5. Suppose the maps (2.4) are surjective for $\mu = k \geqq 1$. Then they are surjective for $\mu = k+1$ (and thus for all $\mu \geqq k$).

Proof. By Proposition 4 the maps (2.5) are surjective for $\mu = k$. We will show by induction on ν that the maps (2.5) are surjective for $\mu = k+1$. The inductive step follows from the five lemma and the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & S^{\nu-1}_{d,k+2} \arrow[r] \arrow[d, "\rho_{k+2}"] & S^{\nu}_{d,k+2} \arrow[r, "\delta_\nu"] \arrow[d, "\rho_{k+2}"] & S^{\nu}_{d,k+1} \arrow[r] \arrow[d, "\rho_{k+1}"] & 0 \\ 0 \arrow[r] & \rho_{k+2}(S_{d,k+2})^{\nu-1} \arrow[r] & \rho_{k+2}(S_{d,k+2})^{\nu} \arrow[r, "\delta_\nu"] & \rho_{k+1}(S_{d,k+1})^{\nu} \arrow[r] \arrow[d] & 0 \\ & & & 0 & \end{tikzcd}$$

Note that when $\nu = 1$, the left column consists of zeros so that the induction has a beginning.

PROPOSITION 6. Let $\mathscr{D}$ be given by (2.1) and assume that $\mathscr{D}$ is involutive. Assume that the maps (2.4) are surjective for $\mu \geqq k+1$, where $k \geqq 1$. Then these maps are surjective for $\mu \geqq k$ (and thus for all $\mu \geqq 1$).

Proof. Since $\mathscr{D}$ is involutive, there is a regular coordinate at 0, and by Proposition 3 the δ-sequences

$$\begin{aligned}0 \longrightarrow S_{k+r} \longrightarrow S_{k+r-1} \otimes T^* \longrightarrow \cdots \\ \longrightarrow S_k \otimes \Lambda^r T^* \longrightarrow \delta(S_k \otimes \Lambda^r T^*) \longrightarrow 0\end{aligned}$$

are exact. In particular, we obtain the exact diagram

$$\begin{tikzcd}
S_{k+2} \otimes \Lambda^{n-1} T^* \arrow[r] \arrow[d, "\delta_n"] & S_{k+1} \otimes \Lambda^n T^* \arrow[r] \arrow[d, "\delta_n"] & 0 \\
S_{k+1} \otimes \Lambda^{n-1} T^* \arrow[r] \arrow[d] & S_k \otimes \Lambda^n T^* \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

and we conclude that $\delta_n\colon S_{k+1} \to S_k$ is surjective. We now claim that the

$$\begin{aligned}
0 \longrightarrow S_{k+r}^{n-1} \longrightarrow S_{k+r-1}^{n-1} \otimes T^* \longrightarrow \cdots \\
\longrightarrow S_k^{n-1} \otimes \Lambda^r T^* \longrightarrow \delta(S_k^{n-1} \otimes \Lambda^r T^*) \longrightarrow 0
\end{aligned}$$

are exact. Indeed, this follows by applying homology to the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & S_{k+r}^{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & S_{k+1}^{n-1} \otimes \Lambda^{r-1} T^* \arrow[r] \arrow[d] & S_k^{n-1} \otimes \Lambda^r T^* \arrow[r] \arrow[d] & \delta(S_k^{n-1} \otimes \Lambda^r T^*) \arrow[d] \\
0 \arrow[r] & S_{k+r} \arrow[r] \arrow[d, "\delta_n"] & \cdots \arrow[r] & S_{k+1} \otimes \Lambda^{r-1} T^* \arrow[r] \arrow[d, "\delta_n"] & S_k \otimes \Lambda^r T^* \arrow[r] \arrow[d, "\delta_n"] & \delta(S_k \otimes \Lambda^r T^*) \\
0 \arrow[r] & S_{k+r-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & S_k \otimes \Lambda^{r-1} T^* \arrow[r] \arrow[d] & \delta_n(S_k \otimes \Lambda^r T^*) \arrow[d] & \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

As before, we conclude that $\delta_{n-1}\colon S_{k+1}^{n-1} \to S_k^{n-1}$ is surjective. Several repetitions of the above argument complete the proof.

3. The general case. We now leave the constant coefficient case and return to the general situation described in § 1. Accordingly, we no longer use the abbreviation for bundles introduced in § 2; from now on S_μ will denote a bundle and not the fiber over a particular point.

The results of § 2 carry over to the general case in a pointwise fashion. After introducing coordinates about $q \in M$, we observe that the fibers $(S_\mu)_q$ depend only on the coefficients of the principal part of $\mathscr{D}$ at q. For the purposes of studying these fibers we may therefore assume that $\mathscr{D}$ has the form (2.1). Thus the arguments of § 2 apply to the fibers over each $q \in M$ separately. The following proposition, which is implicitly contained in Quillen [3] and in the work of Serre (see [2]), will allow us to make these arguments uniformly for q in small compact sets.

PROPOSITION 7. Let $\mathscr{D}$ be involutive and assume each S_μ is a bundle. Let x be a local coordinate on a neighborhood $V \subset M$ which is regular at $q \in V$. Then there is a neighborhood U of q, contained in V, such that:

(i) x is regular at each $p \in U$

(ii) each $(S_\mu \mid U)^\nu$, $1 \leqq \nu \leqq n$, $\mu \geqq 1$, is a bundle.

Proof. Since x is regular at q, the maps

$$\delta_n \colon S_{\mu+1} \mid V \longrightarrow S_\mu \mid V\,, \qquad 1 \leqq \mu \leqq 2n$$

have maximal rank at q. It follows that they have maximal rank in a smaller neighborhood U of q, and thus the maps

$$\delta_n \colon S_{\mu+1} \mid U \longrightarrow S_\mu \mid U\,, \qquad 1 \leqq \mu \leqq 2n$$

are surjective. Moreover, $(S_\mu \mid U)^{n-1}$ is a bundle for $2 \leqq \mu \leqq 2n+1$, and this permits us to repeat the argument for the maps

$$\delta_{n-1} \colon (S_{\mu+1} \mid U)^{n-1} \longrightarrow (S_\mu \mid U)^{n-1}\,, \qquad 2 \leqq \mu \leqq 2n\,.$$

Eventually, we obtain a neighborhood U such that

$$\delta_\nu \colon (S_{\mu+1} \mid U)^\nu \longrightarrow (S_\mu \mid U)^\nu\,, \qquad n \leqq \mu \leqq 2n, 1 \leqq \nu \leqq n$$

are surjective and $(S_\mu \mid U)^\nu$ is a bundle for $n \leqq \mu \leqq 2n, 1 \leqq \nu \leqq n$. From Propositions 5 and 6 we conclude that x is regular at each $p \in U$.

(ii) now follows by a remark of D. G. Quillen. For each $p \in U$ and $1 \leqq \nu \leqq n$ we have the following exact sequence (see the proof of Proposition 6).

$$0 \longrightarrow (S^\nu_{k+n})_p \longrightarrow (S^\nu_{k+n-1})_p \otimes T^*_p \longrightarrow \cdots \longrightarrow (S^\nu_k)_p \otimes \Lambda^n T^*_p \longrightarrow 0\,.$$

The exactness expresses a relation among the dimensions of the spaces involved. Accordingly, if $\dim (S^\nu_\mu)_p$ is a constant function of $p \in V$ for each $k < \mu \leqq k+n$ (resp. $k \leqq \mu < k+n$), then the same is true for $\mu = k$ (resp. $\mu = k+n$). This provides the inductive step which yields the proof of (ii).

The following proposition contains the theorem stated in § 1.

PROPOSITION 8. Let $\mathscr{D}$ be involutive and let the coordinate $x = (x_1, \cdots, x_n)$ be regular at $q \in M$. Then there exists a neighborhood U of q, on which x is defined, a coordinate in the fibers of $E \mid U$, nonzero constants $t_1, \cdots, t_n$, and a constant $K > 0$ such that whenever $\xi \in (S_\mu \otimes \Lambda^r T^*) \mid U$ and $\delta\xi = 0$, then $\xi = \delta\zeta$ for some $\zeta \in (S_{\mu+1} \otimes \Lambda^{r-1} T^*) \mid U$

with $\|\zeta\| \leqq K\|\xi\|$, where the norm $\|\cdot\|$ is defined by the coordinate $(t_1x_1, \cdots, t_nx_n)$.

Proof. Choose a compact neighborhood U of q which satisfies the conclusion of Proposition 7 and which is contained in the domain of definition of the coordinate x. We can assume that $E \mid U$ and $F \mid U$ are trivial, and we fix a coordinate in the fibers of each. We apply Proposition 3 to the fibers over each $p \in U$ and claim that the constants $t_1, \cdots, t_n, K$ can be chosen uniformly for $p \in U$. Indeed, from the proofs of Propositions 1, 2, and 3, we see that the choice of these constants depends on upper bounds for each of the coefficient matrices for the principal part of $\mathscr{D}$ and upper bounds for the chosen splittings of the sequences

$$0 \longrightarrow (S_1 \mid U)^\nu \longrightarrow (S_{d,1} \mid U)^\nu \xrightarrow{\rho_1} \rho_1((S_{d,1} \mid U)^\nu) \longrightarrow 0 ,$$

$1 \leqq \nu \leqq n$. Since each $(S_1 \mid U)^\nu$ is a bundle, these splittings can be chosen as bundle maps, which are thus bounded uniformly for $p \in U$. Since the coefficient matrices can be uniformly bounded, the proof is complete.

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