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INFINITE SEMIGROUPS WHOSE NONTRIVIAL HOMOMORPHS ARE ALL ISOMORPHIC

BRUCE A. JENSEN

An infinite semigroup S such that every nontrivial homomorph of it is isomorphic to S is called an HI semigroup. Every commutative HI semigroup is a group and thus it is isomorphic to the group $Z(p)^{\infty}$, for some prime P. An infinite Brandt semigroup is HI if and only if it has a trivial structure group. An inverse HI semigroup containing a primitive idempotent is either Brandt or else it is isomorphic to a trasfinite chain of extensions of a Brandt semigroup K by isomorphic copies of K (where K has the trivial group as its structure group). Necessary and sufficient conditions are given for a semigroup of the latter type to yield an HI semigroup and an example is constructed.

In his monograph *Infinite Abelian Groups*, I. Kaplansky includes as exercises the following results concerning an infinite abelian group G:

(1) If every subgroup of G is isomorphic to G, G is cyclic.

(2) If every subgroup of G is finite, G is isomorphic to the group $Z(p^{\infty})$ for some prime p.

(3) If every proper homomorph of G is finite, G is cyclic.

(4) If every nontrivial homomorph of G is isomorphic to G, G is isomorphic to the group $Z(p^{\infty})$ for some prime p.

In generalizing these results to semigroups, (1) can easily be disposed of. Suppose S is a semigroup such that every subsemigroup of S is isomorphic to S. It is clear that S must be cyclic, say $S = \{a, a^2, \cdots\}$. However, $T = \{a^2, a^3, \cdots\}$ is a noncyclic subsemigroup of S and thus T is not isomorphic to S, a contradiction.

In [3], Jensen and Miller prove that any infinite semigroup S such that every subsemigroup of S is finite is a group. Thus in particular, if S is commutative, S is isomorphic to $Z(p^{\infty})$ for some prime p.

Defining an HF semigroup to be an infinite semigroup with the property that every proper homomorph is finite, it is shown in [3] that a commutative semigroup S containing at least three elements is an HF semigroup if and only if S can be (isomorphically) imbedded in an infinite cyclic group with zero adjoined. In [2] the structure of HF inverse semigroups is investigated. The structure of all HF inverse semigroups that contain a primitive idempotent is determined up to the determination of all HF groups. The author is unaware of any general results concerning nonabelian HF groups although obviously some exist, e.g., any infinite simple group.

Throughout this paper, E_S denotes the set of idempotents of the semigroup S and A^* denotes the set of nonzero elements of any $A \subseteqq S$. If S is a Brandt semigroup with structure group G and index set Λ, we write $S = B(G; \Lambda)$ and we denote the elements of S^* by $\{(i, g, j) \mid i, j \in \Lambda, g \in G\}$. Then

$$(i, g, j)(i', g', j') = \begin{cases} (i, gg', g') & \text{if } j = i' , \\ 0 & \text{otherwise} . \end{cases}$$

If I is an ideal of the semigroup S, we identify $(S/I)^*$ with $S \backslash I = \{x \in S \mid x \notin I\}$. Thus $S/I = (S \backslash I) \cup \{\overline{0}\}$, where $\overline{0}$ is the zero of S/I. Except when variations are noted in this paper, the terminology and notation is the same as that used in [1].

1. Commutative *HI* semigroups.

THEOREM 1. *If S is a commutative HI semigroup, then S is a group.*

Proof. Assume that S is not a group. Thus there is an $x \in S$ such that $xS \neq S$. Then $S \cong S/xS$ so S contains a zero. We first show that under this assumption S is nil.

Let $Z = \{x \in S \mid xy = 0 \text{ for some } y \in S^*\}$. Since Z is an ideal of S either $S \cong S/Z$ or $S = Z$. If $S \cong S/Z$, it follows that $Z = \{0\}$, so S^* is a proper subsemigroup of S and the map $\theta: S \rightarrow \{0, 1\}$ which sends each element of S^* onto 1 and sends 0 onto 0 is a homomorphism of S onto the multiplicative semigroup $\{0, 1\}$, a contradiction of the *HI* property of S. Thus $S = Z$.

For a fixed element $a \in S^*$ define the set $A = \{x \in S \mid xa^n = 0 \text{ for}$ some postive integer $n\}$. Clearly A is an ideal of S. If $a \notin A$, $a \in S/A$ and since $Z = S \cong S/A$, there is an element $b \in S \backslash A$ such that $ab \in A$, say $(ab)a^n = 0$ or $ba^{n+1} = 0$, so $b \in A$, a contradiction. Thus $a \in A$ and hence a is nilpotent. Since a was arbitrary it follows that S is nil.

Let $x \in S$ such that $x \in xS$, say $x = xe$. Then $x = xe^n$ for each positive integer n. Since S is nil, this implies that $x = 0$. Thus, if $x \in S^*$, $x \in S/xS$ and $S/xS \cong S$. But $x(S/xS) = \overline{0}$ so there exists $y \in S^*$ such that $yS = 0$. Let $J = \{x \in S \mid xS = 0\}$. J is an ideal of S so either $J = S$ or $S \cong S/J$. If $S = J$, every nonempty subset of S is an ideal of S and thus $S \cong S/A$ for each $A \subseteq S$, $A \neq \varnothing$. Clearly this is not the case. By a similar argument, it follows that $S = S^2$. We now have $S \cong S/J$, so there is an $a \in S/J$ such that $a(S/J) = \overline{0}$, i.e., $a(S \backslash J) \subset J$, and hence $aS \subset J$. Therefore, $aS = aS^2 \subseteq JS = 0$, so $aS = 0$. But this contradicts the choice of a, and our proof is complete.

COROLLARY. *If S is a commutative HI semigroup, then $S = Z(p^\infty)$, for some prime p.*

2. Inverse *HI* semigroups containing a primitive idempotent. Following the notation of [1], p. 72, let $J(x) = S^1xS^1$ and $I(x) = \{y \in J(x) \mid J(y) \neq J(x)\}$. It easily follows (see [1], p. 73) that if $J(x) \neq \varnothing$, $I(x)$ is an ideal of S such that $J(x)/I(x)$ is either 0-simple or the null semigroup of order 2.

LEMMA 1. *If S is an inverse HI semigroup, either S is simple or else $S = S^\circ$ and S contains a unique 0-simple ideal K contained in every nonzero ideal of S.*

Proof. Suppose S is not simple, say $a \in S$ such that $S \neq SaS = S^1aS^1 = J(a)$. Then $S \cong S/J(a)$, so S contains a zero. It follows from the remark above that $J(a)/I(a)$ is 0-simple. $S \cong S/I(a)$ implies S contains a 0-simple ideal $K \cong J(a)/I(a)$.

Let U denote the union of all ideals B of S such that $K \cap B = 0$. U is a proper ideal of $S(K \neq 0)$ so $S \cong \bar{S} = S/U$. Since $K \cap U = 0$, we can consider K as an ideal of $\bar{S}$. It easily follows that every nonzero ideal of $\bar{S}$ has nonzero intersection with the 0-simple ideal K. $S \cong \bar{S}$ implies the desired result.

We call this unique 0-simple ideal the kernel of S.

LEMMA 2. *Let $S = S^\circ$ be an inverse semigroup and let B be an ideal of S. Then B is a Brandt subsemigroup of S if and only if $B = SeS$ for some primitive idempotent $e \in S$.*

Proof. Let e be primitive in S and let I be an ideal of S such that $I \subseteqq SeS$. Suppose $e \notin I$ and let $f \in E_I \subseteqq I \subseteqq SeS$, say $f = aeb$. Then $a^{-1}fb^{-1} = he$, where $h = a^{-1}abb^{-1} \in E_S$. If $he = e$, $e \in SfS \subseteqq I$, contrary to our assumption. Therefore, by the primitivity of e, $he = 0$, so $f = aheb = 0$ and thus $E_I = \{0\}$. It follows that $I = \{0\}$. Clearly $(SeS)^2 = SeS$, so SeS is 0-simple and hence Brandt.

Conversely, suppose B is Brandt ideal of S and let $e \in E_{B^*}$. Clearly $B = SeS$. If $f \in E_S$, since $ef \in B$ and $(ef)e = ef$, either $ef = 0$ or $ef = e$, so e is primitive in S.

THEOREM 2. *The Brandt semigroup $B = B(G; \Lambda)$ is HI if and only if $|G| = 1$ and Λ is infinite (i.e., if and only if B is homomorphically simple and infinite).*

Proof. By [5], $B(1; \Lambda)$ is a homomorph of B and $|B(1; \Lambda)| \geqq 2$.

Hence B is HI if and only if $B = B(1; \Lambda)$ where Λ is infinite.

LEMMA 3. *Let S be an inverse HI semigroup with Brandt kernel K. Then $K = B(1; \Lambda)$ for some index set Λ.*

Proof. Let ρ be a congruence relation on K such that $|K/\rho| > 1$. If ρ' denotes the identity extension of ρ to all of S ($x\rho' y$ if and only if $x = y$ or $x\rho y$), it follows from [2] that ρ' is a congruence relation on S. Since $|S/\rho'| > 1$, $S \cong S/\rho'$. Thus K, the unique Brandt kernel of S, is isomorphic to $K/\rho' = K/\rho$, the unique Brandt kernel of S/ρ'. It follows from Preston [5], that $B(G; \Lambda) \cong B(G\theta; \Lambda)$ and thus $G \cong G\theta$ for every homomorphism θ on G. Therefore $G = 1$.

THEOREM 3. *Let S be an inverse HI semigroup containing a primitive idempotent e. Then S satisfies one of the following:*

(1) *S is an HI group,*

(2) *S is an HI Brandt semigroup,*

(3) *S has a transfinite composition series such that every factor is isomorphic to a (fixed) Brandt semigroup $B(1; \Lambda)$ for some index set Λ.*

Proof. If $0 \notin S$, S is simple so $SeS = S$. Therefore $E_S = E_{SeS} = \{e\}$, so S is a group.

Next assume $0 \in S$. If $SeS = S$, it follows from Lemma 2 that S is Brandt, so suppose $SeS \neq S$. By Lemma 3, the kernel $K = SeS \cong B(1; \Lambda)$ for some index set Λ. If $x \in S^*$, then $S \cong S/I(x)$ so $S/I(x)$ contains a unique kernel $\bar{K} \cong K$. By the remark at the beginning of this section, $J(x)/I(x)$ is Brandt and hence it is the Brandt kernel of $S/I(x)$; that is, the factor $J(x)/I(x)$ in the composition series of S is isomorphic to K. Moreover, S cannot contain a maximal proper ideal A since this would imply $S \cong S/A$ is 0-simple. Thus the composition series is infinite.

THEOREM 4. *The ideals of an inverse HI semigroup S containing a primitive idempotent are well ordered by inclusion such that for each proper ideal A of S there is a unique ideal A' of S with the properties* (1) $A \subset A'$ and (2) $A \subset B$ *implies $A' \subseteqq B$ for any ideal B of S. We call A' the successor of A.*

Proof. If $0 \notin S$, S is simple and the theorem holds trivially, so assume $0 \in S$. Suppose S has ideals A and B such that $A \not\subseteqq B$ and $B \not\subseteqq A$. Then $S \cong \bar{S} = S/(A \cap B)$, and $\bar{A} = A/(A \cap B)$ and $\bar{B} = B/(A \cap B)$ are ideals of $\bar{S}$ such that $\bar{A} \cap \bar{B} = \bar{0}$, a contradiction of Lemma 1. Thus the ideals are linearly ordered by the inclusion relation.

If A is a proper ideal of S, $S \cong S/A$ so S/A contains an ideal $\bar{K} \cong K$ (the Brandt kernel of S). Then $\bar{K}$ is of the form A'/A for some ideal A' of S. Since inclusion linearly orders the ideals, it follows that A' is unique. Clearly A' satisfies conditions (1) and (2) of the theorem.

Next let $\mathscr{A}$ denote a nonempty collection of ideals of S. Let $B = \cap \{A \mid A \in \mathscr{A}\}$, so either $B = S$ (and hence $B \in \mathscr{A}$ and B is the least element of $\mathscr{A}$) or else $S \cong S/B$. Let B' denote the successor of B and let $x \in B' \backslash B$. It follows from the definition of B that there is an ideal $A_x \in \mathscr{A}$ such that $x \notin A_x$. By Lemma 1 applied to $S/B \cong S$ we have $B \subseteqq A_x \subset B'$. Therefore, $B = A_x \in \mathscr{A}$ and B is the least element of $\mathscr{A}$.

THEOREM 5. *Let S be an inverse semigroup containing a primitive idempotent e such that S is the union of the chain of ideals*

$$\{0\} = S_0 \subset S_1 \subset S_2 \subset \cdots$$

and such that for each $i \geqq 1$,

$$S_i/S_{i-1} \cong B(1; \Lambda) ,$$

where Λ is some (fixed) index set. Then S is HI if and only if for each $i \geqq 2$, there exist distinct idempotents f, g_1 and g_2 with $f \in S_i \backslash S_{i-1}$ and g_1, $g_2 \in S_{i-1} \backslash S_{i-2}$ such that $g_1 < f$ and $g_2 < f$. Furthermore, if this is the case, then every idempotent of $S_i \backslash S_{i-1}$ has at least two nonzero idempotents under it.

Before proving the theorem, we introduce the following notation:

$$B_n^* = S_n \backslash S_{n-1}, n \geqq 1 .$$

Thus S_n can be considered as the extension of S_{n-1} by B_n. Note that $B_1 = S_1$ is the kernel of S.

$$B_n = B(1_n; \Lambda) = \{(i, n, j) \mid i, j \in \Lambda\} \cup \{0_n\}$$

where

$$(i, n, j)(i', n, j') = \begin{cases} (i, n, j') & \text{if } j = i' \\ 0_n & \text{otherwise} . \end{cases}$$

Therefore, under the multiplication of S, if $j = i'$, the above product remains the same, while if $j \neq i'$, the above product lies in S_{n-1}.

For simplicity, we write

$$e_{n,i} = (i, n, i) .$$

Thus, the theorem asserts that S is *HI* if and only if for each $n \geqq 2$

there exists $i, r, s \in \Lambda$ such that

(1) $$e_{n,i} > e_{n-1,r} \text{ and } e_{n,i} > e_{n-1,s}, \text{ where } e_{n-1,r} \neq e_{n-1,s}.$$

Proof of Theorem 5. First assume that S is *HI*. It follows from Lemma 2 that for each $i \in \Lambda$, $e_{2,i}$ is not primitive in S so there exists $v_1 = v_1(i) \in \Lambda$ such that $e_{1,v_1} < e_{2,i}$. Assume inductively that for each $i \in \Lambda$, there exists $v_{n-1} = v_{n-1}(i) \in \Lambda$ such that $e_{n-1,v_{n-1}} < e_{n,1}$. Since $S \cong S/S_{n-1}$, it follows that for each $i \in \Lambda$ there exists $v_n \in \Lambda$ such that $e_{n,v_n} < e_{n+1,i}$.

Suppose that for each $i \in \Lambda$, $e_{2,i}$ has exactly one nonzero idempotent under it. Let ρ be the congruence relation on S generated by the relation $\rho_0 = \{e_{2,1}, e_{1,v}\}$, where $e_{1,v} < e_{2,1}$. If ρ is not one-to-one on S_1 it follows that $S_2/\rho = 0$ so there exist $x, y \in S$ such that $xe_{2,1}y \neq 0$ and $xe_{1,v}y = 0$. Therefore the idempotent $e = x^{-1}xe_{2,1}yy^{-1} \neq 0$. Since $e \leqq e_{2,1}$, it follows from our assumption that either $e = e_{2,1}$ or $e = e_{1,v}$. Thus in either case, we have

$$e_{1,v} = e_{1,v} \cdot e = e_{1,v}x^{-1}xe_{2,1}yy^{-1} = x^{-1}xe_{1,v}yy^{-1} = 0 ,$$

a contradiction. Therefore, ρ merely identifies corresponding terms of B_1 and B_2. Relabeling if necessary, we have $e_{1,j} < e_{2,j}$ for each j in Λ, and by induction $e_{n,j} < e_{n+1,j}$, $n \geqq 1, j \in \Lambda$. Define σ to be the congruence relation on S generated by the relation

$$\sigma_0 = \{(e_{n,i}, e_{m,i}) \mid n, m \geqq 1, i \in \Lambda\} .$$

Clearly σ is one-to-one on S_1, and since S/σ has no proper nonzero ideals, we cannot have $S \cong S/\sigma$, a contradiction.

To prove the sufficiency of the condition let S be an inverse semigroup of the type described in the theorem and suppose that for each $n \geqq 1$ there exist distinct $a(n)$ and $b(n)$ in Λ such that

$$e_{n,a(n)} < e_{n+1,1} \text{ and } e_{n,b(n)} < e_{n+1,1} ,$$

and let τ be a congruence relation on S that is not one-to-one. Since $x\tau y$ implies $xx^{-1}\tau yy^{-1}$ and $x^{-1}x\tau y^{-1}y$, it follows from the structure of S that τ is not one-to-one on E_S.

If $e_{n,u}\tau e_{n,v}$, $u \neq v$, then $e_{n,u}\tau e_{n,u}e_{n,v}$ so that we may assume without loss of generality that there exist integers n and m with $n > m$ and $r, s \in \Lambda$ such that

$$e_{n,r}\tau e_{m,s} .$$

Then

(2) $$(1, n, r)e_{n,r}(r, n, 1)\tau(1, n, r)e_{m,s}(r, n, 1) .$$

Upon multiplying both sides of (2) by $e_{n-1,a}$ we obtain the relation

$e_{n-1,a}\tau e_{r,s}$ for some $r < n-1$ and some $s \in \Lambda$. As above, this implies

$$e_{n-1,1}\tau e_{u_2,v_2}, \text{ where } u_2 < n-1 .$$

(If $e_{u_1,v_1} = e_{n-1,a}$ multiply both sides of (2) by $e_{n-1,b}$.)

Continuing in this manner we conclude that $e_{1,1}\tau 0$ and thus by the transitivity of τ we conclude that $|S_n/\tau| = 1$. If there exists an integer N such that τ is not one-to-one on S_N but is one-to-one on S/S_N, then $S/\tau = S/S_N \cong S$. If no such integer N exists, $|S/\tau| = 1$. Hence S is *HI*.

The final assertion will follow if, when S is *HI*, for each $i \in \Lambda$, there exist $r, s \in \Lambda, r \neq s$, such that

$$e_{2,i} > e_{1,r} \text{ and } e_{2,i} > e_{1,s} .$$

Without loss of generality assume $e_{2,1} > e_{1,1}$ and $e_{2,1} > e_{1,2}$. For each $i \in \Lambda$,

$$(i, 2, 1)(1, 1, 1) = (a_i, 1, 1) \text{ for some } a_i \in \Lambda ,$$

and

$$(i, 2, 1)(2, 1, 2) = (b_i, 1, 2) \text{ for some } b_i \in \Lambda .$$

Therefore

$$(i, 2, 1)(1, 1, 1)(1, 2, i) = e_{1,a_i}$$

and

$$(i, 2, 1)(2, 1, 2)(1, 2, i) = e_{1,b_i} .$$

Clearly $e_{1,a_i} < e_{2,i}$ and $e_{1,b_i} < e_{2,i}$. Furthermore,

$$\begin{aligned} e_{1,a_i}e_{1,b_i} &= (i, 2, 1)e_{1,1}(1, 2, i)(i, 2, 1)e_{1,2}(i, 2, i) \\ &= (1, 2, 1)e_{1,1}e_{2,1}e_{1,2}(1, 2, i) = (i, 2, 1)e_{1,2}e_{1,1}(1, 2, i) = 0 . \end{aligned}$$

Therefore, $a_i \neq b_i$, and the proof is complete.

We conclude with an example of an *HI* inverse semigroup of the type described in Theorem 5. Let N denote the set of positive integers, and let $\{B_n \mid n \in N\}$ be a collection of pairwise disjoint isomorphic copies of the Brandt semigroup $B(1, N)$. As in Theorem 5, write the nonzero elements of B_n in the form (i, n, j), for $i, j \in N$, let 0_n denote the zero of B_n, and write 0 for 0_1.

Let $S_1 = B_1$ and let S_{n+1} be the extension of S_n by B_{n+1} where multiplication is defined as follows:

$$\text{If } \alpha, \beta \in S_n, \ \alpha \circ \beta = \alpha\beta .$$

$$\text{If } \alpha, \beta \in B^*_{n+1}, \alpha \circ \beta = \begin{cases} \alpha\beta & \text{if } \alpha\beta \neq 0_{n+1} \\ 0 & \text{if } \alpha\beta = 0_{n+1} \end{cases}.$$

Products between B^*_{n+1} and S_n are defined recursively as follows:

$$(i, n, j) \circ (r, n-1, s) = \begin{cases} (2i-1, n-1, s) & \text{if } r = 2j-1\,, \\ (2i, n-1, s) & \text{if } r = 2j\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

$$(s, n-1, r) \circ (j, n, i) = [(i, n, j) \circ (r, n-1, s)]^{-1}$$

$$(i, n, j) \circ (r, n-k-1, s) = [(i, n, j) \circ (f(r), n-k, f(r))] \circ (r, n-k-1, s)\,,$$

where $f(r)$ is the greatest integer less than or equal to $\dfrac{r+1}{2}$.

$$(s, n-k-1, r) \circ (j, n, i) = [(i, n, j) \circ (r, n-k-1, s)]^{-1}$$

$$(i, n, j) \circ 0 = 0 \circ (i, n, j) = 0\,.$$

Defining $S = \bigcup_{n \geqq 1} S_n$, it can be shown that S is a semigroup as follows:

Let $\phi_1 = (i, n, j)$, $\phi_2 = (r, m, s)$ and $\phi_3 = (u, p, v)$. First observe that $(\phi_1\phi_2)\phi_3 = \phi_1(\phi_2\phi_3)$ if $|n-m| \leqq 1$ and $|m-p| \leqq 1$. Because of the way multiplication is defined it is sufficient to consider the following cases to show this: (i) $m = n$, $p = n-1$; (ii) $m = p = n-1$; (iii) $m = n-1$, $p = n$; (iv) $m = n+1$, $p = n$; (v) $m = n-1$, $p = n-2$. Associativity can be shown in each of the above cases by direct computation. Clearly this can be generalized to show that any product where consecutive factors come from $B^*_i \cup B^*_j$, with $|i-j| \leqq 1$, can be associated in any manner.

Next, observe that $e_{n,r} < e_{n+1,f(r)} < e_{n+2,f^2(r)} < \cdots$, where $f^{k+1}(r) = f(f^k(r))$. Thus every product in S can be written as a product where consecutive factors are of the form $(i, n, j) \circ (r, m, s)$ such that $|n-m| \leqq 1$. Therefore, applying the observation made above, we see that S is a semigroup. Furthermore,

$$e_{n,1} < e_{n+1,1} \text{ and } e_{n,2} < e_{n+1,1} \text{ for each } n \geqq 1\,,$$

so by Theorem 4, S is *HI*.

The following example illustrates the associativity of S:

Let $\phi_1 = (3, n, 2)$, $\phi_2 = (3, n-1, 2)$ and $\phi_3 = (3, n-2, 2)$. Then

$$(\phi_1\phi_2)\phi_3 = (5, n-1, 2)\phi_3 = (9, n-2, 2)$$

and

$$\begin{aligned} \phi_1(\phi_2\phi_3) &= \phi_1(5, n-2, 2) = [\phi_1(3, n-1, 3)](5, n-2, 2) \\ &= (5, n-1, 3)(5, n-2, 2) = (9, n-2, 2)\,. \end{aligned}$$

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PORTLAND STATE UNIVERSITY