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TRANSLATION KERNELS ON DISCRETE ABELIAN GROUPS

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Let G be a compact Abelian group with discrete countable dual group $\Gamma = \hat{G}$ and let $f \in L^1(G)$ with Fourier transform $F = \hat{f}$. If V is a finite subset of Γ we consider the operator F_V on $L^2(V)$:

$$(F_V\varphi)(\gamma) = \sum_{\tau \in V} F(\gamma - \tau)\varphi(\tau) \qquad \varphi \in L^2(V),\ \gamma \in V .$$

Then if $\{V_n\}$ is any suitably restricted sequence of finite subsets of Γ we show that

$$\lim_{n\to\infty} | F_{V_n} | = \lim_{n\to\infty} \{\max_{||\varphi||_2=1} | (F_{V_n}\varphi, \varphi) |\} = || f ||_\infty$$

where $| F_V |$ is the operator norm of F_V on $L^2(V)$ and $(F_V\varphi, \varphi)$ denotes the inner product of $F_V\varphi$ and φ (over V).

This result is then translated into a statement concerning a special class of infinite matrices which generalize the classical Toeplitz matrices. We then apply these results in evaluating the norm of a special type of linear operator.

In [1] the author considered the asymptotic distribution of eigenvalues and characteristic numbers of certain sequences of operators $\{F_{V_n}\}$ over a locally compact group Γ associated with sequences $\{V_n\}$ of Borel sets of Γ of finite nonzero measure satisfying

$$(*) \qquad \lim_{n\to\infty} | \gamma V_n \triangle V_n |/| V_n | = 0 \qquad \text{for all } \gamma \in \Gamma ,$$

where $|\ |$ is left Haar measure on Γ. We write $\{V_n\} \in W_\Gamma$, and say $\{V_n\}$ has the weak ratio property in case $(*)$ is satisfied (see [2]). In this paper we are considering countable Abelian Γ and a more general family T_Γ of sequences $\{V_n\}$ than those in W_Γ (and hence in general the asymptotic distribution of the characteristic numbers of $\{F_{V_n}\}$ does not exist, [2]) but still restricted enough to guarantee an asymptotic formula for the *maximal* characteristic number of F_{V_n} as $n \to \infty$.

1. The basic theorem.

Γ denotes an arbitrary countably infinite discrete Abelian group equipped with the counting measure.

DEFINITION 1. A sequence $\{V_n\}$ of finite nonempty subsets of Γ has *the translation property*, written $\{V_n\} \in T_\Gamma$, if and only if to every finite subset $\Gamma_0 \subseteqq \Gamma$ there corresponds an $n_0 = n_0(\Gamma_0)$ such that for $n \geqq n_0$ there exists a $\tau_n = \tau_n(\Gamma_0) \in \Gamma$ with the property that $\tau_n + \Gamma_0 \subseteqq V_n$.

PROPOSITION 1. (i) $\{V_n\} \in T_\Gamma$ *and* $V_n \subseteqq V_n^*$ *for* $n \in N^+$ *implies* $\{V_n^*\} \in T_\Gamma$. (ii) W_Γ *is properly contained in* T_Γ.

Proof. (i) is immediate from the definition. To prove (ii), first assume $\{V_n\} \in W_\Gamma$ and fix any finite nonempty subset $\Gamma_0 \subseteqq \Gamma$. Set $C = \Gamma_0 \cup \{0\}$. We then readily conclude (see [2] where many properties of W_Γ are established)

$$\lim_{n\to\infty} \frac{|C + V_n|}{|V_n|} = 1\,. \tag{1}$$

But $C + V_n = \bigcup_{\tau \in V_n}(\tau + \Gamma_0) \cup V_n$. Hence if $\tau + \Gamma_0 \not\subseteqq V_n$ for all $\tau \in V_n$, we have $|(\tau + \Gamma_0) \sim V_n| \geqq 1$ for all $\tau \in V_n$ and consequently

$$|C + V_n| \geqq |V_n| + \frac{|V_n|}{|\Gamma_0|} \tag{2}$$

since we may choose $|V_n|$ elements $\tau + \gamma_\tau \in (\tau + \Gamma_0) \sim V_n$, where $\tau \in V_n$ and $\gamma_\tau \in \Gamma_0$, and no element is duplicated more than $|\Gamma_0|$ times. But for sufficiently large n (2) violates (1) and therefore there is a $\tau_n \in V_n$ for which $\tau_n + \Gamma_0 \subseteqq V_n$. Hence $W_\Gamma \subseteqq T_\Gamma$.

We now show inclusion is proper. For let $\{V_n\} \in W_\Gamma$ ($\neq \varnothing$, by [2]); we shall construct a sequence $V_n^* \supseteqq V_n$ such that $\{V_n^*\} \notin W_\Gamma$, which completes the proof of (ii) upon appealing to (i). Fix any $\gamma \in \Gamma \sim \{0\}$. We inductively construct a sequence $\nu_1^{(n)}, \cdots, \nu_{|V_n|}^{(n)}$ as follows: $\nu_1^{(n)} \notin V_n + \{0, \pm\gamma\}$, and

$$\nu_k^{(n)} \notin (V_n \cup \{\nu_1^{(n)}, \cdots, \nu_{k-1}^{(n)}\}) + \{0, \pm\gamma\} \quad (2 \leqq k \leqq |V_n|)\,. \tag{I}$$

We set $V_n^* = V_n \cup \{\nu_1^{(n)}, \cdots, \nu_{|V|}^{(n)}\}$ and verify that

$$\frac{|(\gamma + V_n^*) \cap V_n^*|}{|V_n^*|} \leqq \frac{1}{2} \qquad (n \in N^+)$$

implying $\{V_n^*\} \notin W_\Gamma$. For

$$\begin{aligned}
&(\gamma + V_n^*) \cap V_n^* \\
&= ((\gamma + V_n) \cap V_n^*) \cup (\{\gamma + \nu_1^{(n)}, \cdots, \gamma + \nu_{|V_n|}^{(n)}\} \cap (V_n \cup \{\nu_1^{(n)}, \cdots, \nu_{|V_n|}^{(n)}\})) \\
&= ((\gamma + V_n) \cap V_n^*)
\end{aligned}$$

since the second term in the union is empty by (I). Hence $|(\gamma + V_n^*) \cap V_n^*| \leqq |\gamma + V_n| = |V_n|$, and therefore for $n \in N^+$

$$\frac{|(\gamma + V_n^*) \cap V_n^*|}{|V_n^*|} = \frac{|(\gamma + V_n^*) \cap V_n^*|}{2|V_n|} \leqq \frac{|V_n|}{2|V_n|} = \frac{1}{2}\,.$$

We now prove a result, of independent interest, which is critical in the proof of Theorem 1.

PROPOSITION 2. *Let G be a compact Abelian group (with measure normalized to one), let $f \in L^1(G)$, and let ρ be any positive number. Then*

$$\|f\|_\infty = \sup_\omega \left| \int_G |\omega(x)|^\rho f(x)dx \right|$$

where ω ranges over all trigonometric polynomials on G satisfying $\|\omega\|_\rho \leqq 1$.

Proof. Recall that a trigonometric polynomial is a finite linear combination of characters on G. Clearly

$$\left| \int_G |\omega(x)|^\rho f(x)dx \right| \leqq \|f\|_\infty \int_G |\omega(x)|^\rho dx = \|f\|_\infty \|\omega\|_\rho^\rho \leqq \|f\|_\infty .$$

We divide the proof of the converse inequality into two cases:

To prove the converse inequality, we first consider the case $\|f\|_\infty < +\infty$. Fix any $\delta > 0$ (until the conclusion of the argument). Let $S = S(\delta)$ be a measurable subset of the complex plane of diameter less than δ and such that

$$E = f^{-1}(S),\ \|\chi_E f\|_\infty = \|f\|_\infty ,$$

where χ_E denotes the characteristic function of E. Hence for $s \in S$ and $x \in E$ we have

$$\big| |s| - |(\chi_E f)(x)| \big| \leqq |s - (\chi_E f)(x)| < \delta ,$$

and consequently also

$$\big| |s| - \|f\|_\infty \big| = \big| |s| - \|\chi_E f\|_\infty \big| \leqq \delta .$$

Therefore, if $g = \chi_E/|E|$ then

$$\begin{aligned} 0 &\leqq \|f\|_\infty - \left| \int_G fgdx \right| \leqq \big| \|f\|_\infty - |s| \big| + \left| |s| - \left| \int_G fgdx \right| \right| \\ &\leqq \big| \|f\|_\infty - |s| \big| + \left| s - \int_G fgdx \right| \\ &= \big| \|f\|_\infty - |s| \big| + \left| \frac{1}{|E|} \int_E (s - \chi_E f)dx \right| \leqq 2\delta . \end{aligned} \tag{3}$$

We next wish to approximate g by a *continuous* function h, and at this point the estimate is rather delicate because this is also needed later in the case $\|f\|_\infty = +\infty$ and consequently we must avoid $\|f\|_\infty$ as a factor in the error of estimation. Now since $f \in L^1(G)$, to every $\varepsilon > 0$ there corresponds an $\eta = \eta(\varepsilon)$ such that for all measurable subsets T of G of measure at most η

$$\int_T |f|\, dx < \varepsilon\ .$$

We now choose $\gamma = \gamma(\delta)$ satisfying

$$(4)\quad \text{(i)}\quad \gamma < \delta\,|E|,\quad \text{(ii)}\quad \int_T |f|\, dx < \delta\,|E| \text{ if } |T| < \gamma\ .$$

Furthermore, since Haar measure is regular, we may find an open set E^+ and a closed set E^- such that

$$(4')\qquad E^- \subseteqq E \subseteqq E^+\ ,\qquad |E^+ \sim E^-| < \gamma\ .$$

Finally (by Urysohn's Lemma, since G is a normal topological space) there exists a continuous $h_0: G \to [0, 1]$ such that $h_0 \mid E^- \equiv 1$ and $h_0 \mid G \sim E^+ \equiv 0$. Our candidate for h is then defined to be the nonnegative function $h = h_0/|E|$. Let us now estimate $\int_G fgdx - \int_G fhdx$:

$$\begin{aligned}
&\left| \int_G fgdx - \int_G fhdx \right| \leqq \int_G |f|\,|g - h|\, dx \\
(5)\quad &= \left(\int_{E^-} + \int_{E^+\sim E^-} + \int_{G\sim E^+}\right)|f|\,|g - h|\, dx = \int_{E^+\sim E^-} |f|\,|g - h|\, dx \\
&\leqq \max |g - h| \int_{E^+\sim E^-} |f|\, dx \leqq \frac{1}{|E|}\cdot \delta\,|E| = \delta
\end{aligned}$$

by (4), (4′) and the definitions of g and h. Also, we have

$$\int_{E^-} hdx \leqq \int_G hdx = \int_{E^+} hdx \leqq \|h\|_\infty\,|E^+|\ ,$$

implying the estimate

$$(6)\qquad \frac{|E^-|}{|E|} \leqq \|h\|_1 \leqq \frac{|E^+|}{|E|} \leqq 1 + \delta \text{ by virtue of (4) and (4')}\ .$$

Lastly, to any $\alpha > 0$ we may correspond a trigonometric polynomial ω_α satisfying $\|h^{1/\rho} - \omega_\alpha\|_\infty < \alpha$, and consequently $\|h^{1/\rho} - |\omega_\alpha|\,\|_\infty < \alpha$ since $h^{1/\rho} \geqq 0$. Thus by choosing $\alpha_0 = \alpha_0(\delta)$ sufficiently small we may conclude

$$(7)\qquad \|h - |\omega_{\alpha_0}|^\rho\|_1 \leqq \|h - |\omega_{\alpha_0}|^\rho\|_\infty < \delta\ .$$

Also,

$$\|\omega_{\alpha_0}^\rho\|_1 \leqq \|h - |\omega_{\alpha_0}|^\rho\|_1 + \|h\|_1 \leqq \delta + (1 + \delta) = 1 + 2\delta\ .$$

We now let

$$\omega = \omega_{\alpha_0}/(1 + 2\delta)^{1/\rho},\ \text{implying } \|\omega\|_\rho \leqq 1\ .$$

Finally,

$$\left|\int_G |\omega|^\rho f dx\right| = \frac{1}{1+2\delta}\left|\int_G |\omega_{\alpha_0}|^\rho f dx\right|$$

$$(8)\qquad \geqq \frac{1}{1+2\delta}\left(\left|\int_G fg dx\right| - \left|\int_G fg dx - \int_G fh dx\right| - \left|\int_G hf dx - \int_G |\omega_{\alpha_0}|^\rho f dx\right|\right)$$

$$\geqq \frac{1}{1+2\delta}((\|f\|_\infty - 2\delta) - \delta - \delta\|f\|_1)\ .$$

By (3), (5), and (7). Our assertion follows upon letting $\delta \to 0$.

In case $\|f\|_\infty = +\infty$, we let S_n be a measurable subset of the complex plane of diameter less than δ and such that $E_n = f^{-1}(S_n)$, $\|\chi_{E_n} f\|_\infty > n$. Equations (3) – (8) still hold with $\|f\|_\infty$ replaced by $\|\chi_{E_n} f\|_\infty > n$ wherever it occurs, and we readily construct trigonometric polynomials ω_n with $\|\omega_n\|_\rho \leqq 1$ and such that $\int_G |\omega_n|^\rho f dx$ is unbounded as $n \to +\infty$.

We now are ready to prove the basic theorem.

THEOREM 1. *Let G be a compact group (with measure normalized to one), let $f \in L^1(G)$, and let $F = \hat{f} \in L^\infty(\Gamma)$, the Fourier Transform of f. Furthermore, let $\{V_n\} \in T_\Gamma$ and let F_{V_n} be the Hilbert-Schmidt operator on $L^2(V_n)$:*

$$(F_{V_n}\psi)(\gamma) = \int_{V_n} F(\gamma - \tau)\psi(\tau)d\tau = \sum_{\tau \in V_n} F(\gamma - \tau)\psi(\tau)$$
$$(\psi \in L^2(V_n),\ \gamma \in V_n)\ .$$

Let $(F_{V_n}\psi, \psi)_{V_n}$ denote the inner product of $F_{V_n}\psi$ and ψ over V_n, *and let $|F_{V_n}|$ denote the maximal characteristic number of F_{V_n} as an operator on the Hilbert space $L^2(V_n)$. Then*

(i) $$\lim_{n\to\infty} \max_{\|\psi\|_2 = 1} |(F_{V_n}\psi, \psi)_{V_n}| = \|f\|_\infty\ .$$

(ii) $$\lim_{n\to\infty} |F_{V_n}| = \|f\|_\infty\ .$$

Proof. (i) By definition,

$$(F_{V_n}\psi, \psi)_{V_n} = \sum_{\gamma,\tau \in V_n} F(\gamma - \tau)\psi(\tau)\overline{\psi(\gamma)}$$
$$= \sum_{\gamma,\tau \in V_n}\left[\int_G \overline{(\gamma - \tau, x)} f(x)dx\right]\psi(\tau)\overline{\psi(\gamma)}$$
$$= \int_G \left[\sum_{\gamma,\tau \in V_n} (\tau, x)\psi(\tau)\overline{(\gamma, x)\psi(\gamma)}\right] f(x)dx$$
$$= \int_G \left|\sum_{\tau \in V_n} (\tau, x)\psi(\tau)\right|^2 f(x)dx\ .$$

Note that

$$\omega_\psi(x) = \sum_{\tau \in V_n} (\tau, x)\psi(\tau)$$

is a trigonometric polynomial on G, and $\psi \to \omega_\psi$ is an isometry of $L^2(V_n)$ into $L^2(G)$ since $\|\omega_\psi\|_2^2 = \sum_{\tau \in V_n} |\psi(\tau)|^2 = \|\psi\|_2^2$. Therefore

$$(\dagger) \qquad \max_{\|\psi\|_2=1} |(F_{V_n}\psi, \psi)_{V_n}| = \max_{\|\omega\|_2=1} \left| \int_G \left|\omega(x)\right|^2 f(x)dx \right|$$

where ω ranges over linear combinations of characters on G generated by elements in V_n. Hence by Proposition 2 ($\rho = 2$),

$$\lim_{n\to\infty} \max_{\|\psi\|_2=1} |(F_{V_n}\psi, \psi)_{V_n}| \leqq \|f\|_\infty .$$

On the other hand, let ω be *any* trigonometric polynomial on G, say

$$\omega(x) = \sum_{1 \leqq i \leqq k} (\gamma_i, x)c_i \qquad (c_i \in \mathbb{C},\ \gamma_i \in \Gamma) .$$

Let $\Gamma_0 = \{\gamma_1, \cdots, \gamma_k\}$, a finite subset of Γ. Now since $\{V_n\} \in T_\Gamma$ there exists an n_0 such that for $n \geqq n_0$ there exists $\tau_n \in \Gamma$ such that $\tau_n + \Gamma_0 \subseteqq V_n$. Hence for $n \geqq n_0$,

$$\omega_n(x) = (\tau_n, x)\omega(x) = \sum_{1 \leqq i \leqq k} (\tau_n + \gamma_i, x)c_i$$

is a linear combination of characters on G generated by elements of V_n. Since $|\omega(x)| = |\omega_n(x)|$ for all $x \in G$, the proof of (i) is completed by again applying Proposition 2 with $\rho = 2$.

(ii) Recall that $|F_{V_n}|$ is the norm of F_{V_n} considered as an operator on $L^2(V_n)$, i.e.,

$$|F_{V_n}| = \max_{\|\psi\|_2=1} \|F_{V_n}\psi\|_2 .$$

but by the Cauchy-Schwarz Inequality, for $\|\psi\|_2 = 1$

$$|(F_{V_n}\psi, \psi)_{V_n}| \leqq \|F_{V_n}\psi\|_2 \|\psi\|_2 = \|F_{V_n}\psi\|_2 \leqq |F_{V_n}|$$

and therefore by (i),

$$\varliminf_{n\to\infty} |F_{V_n}| \geqq \varliminf_{n\to\infty} \max_{\|\psi\|_2=1} |(F_{V_n}\psi, \psi)_{V_n}| = \|f\|_\infty .$$

Thus, if $\|f\|_\infty = +\infty$ nothing remains to be proved. If $\|f\|_\infty < +\infty$ we have $f \in L^1(G) \cap L^\infty(G)$, and therefore by [3], p. 445, $|F_{V_n}| \leqq \|f\|_\infty$ for all $n \in N^+$. Hence $\varlimsup_{n\to\infty} |F_{V_n}| \leqq \|f\|_\infty$, and consequently $\lim_{n\to\infty} |F_{V_n}| = \|f\|_\infty$ in this case as well.

We now conversely prove that the hypothesis $\{V_n\} \in T_\Gamma$ is in fact necessary for the conclusion of Theorem 1. More precisely,

THEOREM 1′. *Using the notation of Theorem* 1, *if* $\{V_n\}$ *is any sequence of finite subsets of* Γ *for which conclusion* (i) *holds for all trigonometric polynomials* f *on* G, *then* $\{V_n\} \in T_\Gamma$.

Proof. Assume $\{V_n\} \notin T_\Gamma$, i.e., there exists a finite subset Γ_0 of Γ such that no translate of Γ_0 lies in V_m for an appropriate subsequence $m \to \infty$. We then assert that

$$f(x) = \frac{1}{|\Gamma_0|} \sum_{\tau \in \Gamma_0} (\tau, x) \qquad (\|f\|_\infty = f(0) = 1)$$

is a trigonometric polynomial for which (i) fails. More precisely we show for all these m:

$$\operatorname*{Max}_{\|\psi\|_2=1} |(F_{V_m}\psi, \psi)_{Vm}| \leqq \left(1 - \frac{1}{2|\Gamma_0|}\right)\|f\|_\infty .$$

Recalling relation (†) of the proof of Theorem 1. We have:

$$(\dagger) \qquad \operatorname*{Max}_{\|\psi\|_2=1} |(F_{Vm}\psi, \psi)_{Vm}| = \operatorname*{Max}_{\|\omega\|_2=1} \left| \int_G |\omega(x)|^2 f(x)dx \right|$$

where ω ranges over all linear combinations of characters on G generated by elements in V_m.

However, any such ω is of the form

$$\omega(x) = \sum_{\tau \in Vm} (\tau, x)a_\tau$$

where

$$\sum_{\tau \in V_m} |a_\tau|^2 = \|\omega\|_2^2 \leqq 1 ,$$

implying

$$|\omega(x)|^2 = \sum_{\tau_1, \tau_2 \in V_m} (\tau_1 - \tau_2, x)a_{\tau_1}\bar{a}_{\tau_2} ,$$

and finally

$$\int_G |\omega(x)|^2 f(x)dx = \frac{1}{|\Gamma_0|} \sum_{\substack{\tau_1, \tau_2 \in V_m \\ \tau_2 - \tau_1 \in \Gamma_0}} a_{\tau_1}\bar{a}_{\tau_2} .$$

Consequently,

$$\begin{aligned}\left|\int_G |\omega(x)|^2 f(x)dx\right| &\leqq \frac{1}{|\Gamma_0|}\sum_{\substack{\tau_1,\tau_2\in V_m\\ \tau_2-\tau_1\in\Gamma_0}} |a_{\tau_1}||a_{\tau_2}| \leqq \frac{1}{2|\Gamma_0|}\sum_{\substack{\tau_1,\tau_2\in V_m\\ \tau_2-\tau_1\in\Gamma_0}} (|a_{\tau_1}|^2+|a_{\tau_2}|^2)\\ &= \frac{1}{2|\Gamma_0|}\left(\sum_{\tau_1\in V_m}|a_{\tau_1}|^2\left(\sum_{\substack{\tau_2\in V_m\\ \tau_2-\tau_1\in\Gamma_0}}1\right)+\sum_{\tau_2\in V_m}|a_{\tau_2}|^2\left(\sum_{\substack{\tau_1\in V_m\\ \tau_2-\tau_1\in\Gamma_0}}1\right)\right)\\ &= \frac{1}{2|\Gamma_0|}\left(\sum_{\tau_1\in V_m}|a_{\tau_1}|^2|(V_m-\tau_1)\cap\Gamma_0|\right.\\ &\qquad\left.+\sum_{\tau_2\in V_m}|a_{\tau_2}|^2|(\tau_2-V_m)\cap\Gamma_0|\right)\\ &\leqq \frac{1}{2|\Gamma_0|}\left\{(|\Gamma_0|-1)\sum_{\tau_1\in V_m}|a_{\tau_1}|^2+|\Gamma_0|\sum_{\tau_2\in V_m}|a_{\tau_2}|^2\right\}\\ &= \left(1-\frac{1}{2|\Gamma_0|}\right)\sum_{\tau\in V_m}|a_\tau|^2 \leqq \left(1-\frac{1}{2|\Gamma_0|}\right)\\ &= \left(1-\frac{1}{2|\Gamma_0|}\right)||f||_\infty\end{aligned}$$

since no translate $V_m - \tau_1$ contains Γ_0 by hypothesis. Our assertion now readily follows.

2. A class of doubly-infinite matrices.

We now translate the theorem of the preceding section into a statement concerning a class of doubly-infinite complex matrices $M = (\alpha_{i,j})_{i,j=1}^{\infty}$ whose entries $\alpha_{i,j}$ are determined by a "group law".

DEFINITION 2. Let $M = (\alpha_{i,j})_{i,j=1}^{\infty}$ be a matrix with complex entries. We then write

$$M \sim (\Gamma, \Lambda, F)$$

if and only if

(i) Γ is a countable Abelian group.

(ii) Λ is a subset of Γ.

(iii) $F: \Gamma \to \mathbb{C}$.

(iv) There exists an ordering of $\Lambda = \{\lambda_1, \cdots, \lambda_n, \cdots\}$ such that for all $i, j \in N^+$,

$$\alpha_{i,j} = F(\lambda_i - \lambda_j) .$$

REMARK. For *any* $M = (\alpha_{i,j})_{i,j=1}^{\infty}$ with complex entries we may take Γ to be $Q^\times$, the multiplicative group of rational numbers, and Λ to be $P = \{p_n : n \in N^+\}$, the set of all positive integral primes, upon defining F by

$$F(r) = \begin{cases}\alpha_{i,j} & \text{if } r = p_i/p_j\\ 0 & \text{otherwise} .\end{cases}$$

We then have $M \sim (Q^{\times}, P, F)$.

Under suitable restrictions on (Γ, Λ, F) we shall be able to compute the norm and quadratic norm of the matrix M, which are defined as follows:

DEFINITION 3. The norm of M, $|M|$, and the quadratic norm of M, $|M|_I$, are defined by

$$|M| = \sup_{||X||_2 \leqq 1} ||MX||_2 , \qquad |M|_I = \sup_{||X||_2 \leqq 1} |(MX, X)| ,$$

where $X = (\{x_i\})$ ranges over elements of the complex Hilbert space l^2 with only *finitely* many $x_i \neq 0$, and $MX = (\{\sum_j \alpha_{i,j} x_j\})$.

LEMMA 1. *If M induces a bounded operator on l^2, then*

$$\text{(i)} \quad |M| = \sup_{||X||_2 \leqq 1} ||MX|| , \qquad \text{(ii)} \quad |M|_I = \sup_{||X||_2 \leqq 1} |(MX, X)| ,$$

where $X = (\{x_i\})$ ranges over all elements of l^2 (with $||X||_2 \leqq 1$). Hence in this case $|M|$ is the standard norm of M considered as a bounded linear operator on l^2.

Proof. For $x \in l^2$, let X_n be the projection of X on its first n components (0 elsewhere). Since M is bounded and consequently closed, $\lim_{n \to \infty} MX_n = MX$ and (i) follows since X_n has at most n nonzero components. Also

$$(MX, X) = (MX_n, X_n) + (M(X - X_n), X_n) + (MX, X - X_n) ,$$

and therefore

$$\begin{aligned} &|(MX, X) - (MX_n, X_n)| \\ &\leqq ||M(X - X_n)||_2 \, ||X_n||_2 + ||MX||_2 \, ||X - X_n||_2 \to 0 \end{aligned}$$

as $n \to \infty$, and (ii) clearly follows.

THEOREM 2. *Let $M \sim (\Gamma, \Lambda, F)$ where*

(i) *$F \in A(\Gamma)$, i.e., $F = \hat{f}$ for some $f \in L^1(G)$.*

(ii) *To each finite subset $\Gamma_0 \subseteqq \Gamma$ there corresponds a $\gamma = \gamma(\Gamma_0)$ such that $\gamma + \Gamma_0 \subseteqq \Lambda$.*

Then $|M| = |M|_I = ||f||_\infty$.

Proof. Assume $\Lambda = \{\lambda_1, \cdots, \lambda_n, \cdots\}$ as in Definition 2, and set $V_n = \{\lambda_1, \cdots, \lambda_n\}$. Then hypothesis (ii) clearly implies $\{V_n\} \in T_\Gamma$. The theorem will follow from the two inequalities

(i) $|M| \leqq ||f||_\infty$

(ii) $||f||_\infty \leqq |M|_I$,

since $|M|_I \leqq |M|$ by the Cauchy-Schwarz inequality.

(i): If $||f||_\infty = +\infty$ there is nothing to prove. Otherwise $f \in L^1(G) \cap L^\infty(G)$, and therefore the operator $M': L^2(\Gamma) \to L^2(\Gamma)$ defined by

$$(M'\varphi)(\gamma) = \sum_{\tau \in \Gamma} F(\gamma - \tau)\varphi(\tau) \qquad (\varphi \in L^2(\Gamma)\ ,\ \gamma \in \Gamma)$$

has norm $|M'| = ||f||_\infty$ by [3], § 3.2., p. 441. Hence if we only consider φ with support in Λ and restrict γ to Λ, M' restricts to an operator $M'': L^2(\Lambda) \to L^2(\Lambda)$ with $|M''| \leqq |M'|$. Now consider the isometry of l^2 onto $L^2(\Lambda)$ given by $X = (\{x_n\}) \rightsquigarrow \varphi_X$ where $\varphi(\lambda_n) = x_n$ for $n \in N^+$. Then for this $\varphi = \varphi_X$ and $\lambda_i \in \Lambda$,

$$(M''\varphi)(\lambda_i) = \sum_{\lambda \in \Lambda} F(\lambda_i - \lambda)\varphi(\lambda) = \sum_j F(\lambda_i - \lambda_j)\varphi(\lambda_j) = \sum_j \alpha_{i,j}x_j$$

which is the i^{th} component of MX, and therefore $|M| = |M''| \leqq |M'| = ||f||_\infty$ (and thus M induces a bounded linear operator if $||f||_\infty < +\infty$).

(ii): For $n \in N^+$, consider the isometry of $L^2(V_n)$ (which is none other than n-dimensional Euclidean space) into l^2 given by $\varphi \to X_\varphi$ where $X_\varphi = (\{x_j^\varphi\})$ and $x_j^\varphi = \varphi(\lambda_j)$ for $1 \leqq j \leqq n$ and 0 otherwise. Hence X_φ has only finitely many nonzero components, and each $X \in l^2$ with only finitely many nonzero components is in the image of $L^2(V_n)$ under the above isometry for $n = n(X)$ sufficiently large. Now consider F_{V_n} on $L^2(V_n)$:

$$\begin{aligned}(F_{V_n}\varphi, \varphi)_{V_n} &= \sum_{1 \leqq i \leqq n} \sum_{1 \leqq j \leqq n} F(\lambda_i - \lambda_j)\varphi(\lambda_j)\overline{\varphi(\lambda_i)} \\ &= \sum_{1 \leqq i, j \leqq n} \alpha_{i,j}x_j\bar{x}_i = (MX_\varphi, X_\varphi) \qquad \text{(in } l^2\text{)}\ .\end{aligned}$$

But by Theorem 1 (i) $\lim_{n \to \infty} \max_{||\varphi||_2 = 1} |(F_{V_n}\varphi, \varphi)_{V_n}| = ||f||_\infty$ and therefore $|M|_I = ||f||_\infty$ since

$$|M|_I = \sup_{\substack{||x||_2 \leqq 1 \\ x_i \neq 0 \text{ finitely}}} |(MX, X)| = \lim_{n \to \infty} \max_{||\varphi||_2 \leqq 1} |(F_{V_n}\varphi, \varphi)_{V_n}| = ||f||_\infty\ .$$

COROLLARY 1. (1) *Hypothesis* (i) *of Theorem* 2 *is satisfied if*

$$\text{(i)}' \qquad \sum_{\gamma \in \Gamma} |F(\gamma)|^2 < +\infty\ .$$

(2) *Hypothesis* (ii) *of Theorem* 2 *is satisfied if*

$$\text{(ii)}' \qquad \Lambda + \Lambda \subseteqq \Lambda \quad \textit{and} \quad \text{(ii)}'' \ \Lambda \textit{ generates } \Gamma\ .$$

Proof. (1), (i)$'$ implies $f(x) = \sum_{\gamma \in \Gamma}(\gamma, x)F(\gamma) \in L^2(G)$ and therefore also $f \in L^1(G)$ since G is compact and hence of finite measure. Clearly $F = \hat{f} \in A(\Gamma)$.

(2) First note that any element of Γ is the difference of two elements in Λ, i.e., $\Gamma = \Lambda - \Lambda$. For by (ii)$''$, if $\gamma \in \Gamma$ we have $\gamma = \lambda_{i_1} + \cdots + \lambda_{i_k} - \cdots - \lambda_{i_n}$ for some suitable finite sequence of integers $i_1, \cdots, i_n$ (if no terms with a plus sign occur we may take $k = 0$, if none with a minus sign occur take $k = n$). But by (ii)$'$, $\gamma^+ = \lambda_{i_1} + \cdots + \lambda_{i_k} \in \Lambda$ if $k > 0$, $\gamma^- = \lambda_{i_{k+1}} + \cdots + \lambda_{i_n} \in \Lambda$ if $k < n$. If $k = 0$ we may write $\gamma = \lambda_1 - (\lambda_1 + \cdots + \lambda_n + \lambda_1)$ and similarly $\gamma = (\lambda_1 + \cdots + \lambda_n + \lambda_1) - \lambda_1$ if $k = n$.

Now let $\Gamma_0 = \{\gamma_1, \cdots, \lambda_k\}$ be a nonempty finite subset of Γ. Then for appropriate $a_i, b_i \in N^+$ we have

$$\gamma_i = \lambda_{a_i} - \lambda_{b_i} \qquad (1 \leqq i \leqq k) .$$

Consequently, for $1 \leqq i \leqq k$,

$$\gamma_i = \lambda_{a_i} + \lambda_{b_1} + \cdots + \hat{\lambda}_{b_i} + \cdots + \lambda_{b_k} - (\lambda_{b_1} + \cdots + \lambda_{b_k})$$

where $(\hat{\ })$ denotes deletion of a term. Hence if we set $\gamma = \lambda_{b_1} + \cdots + \lambda_{b_k}$ we have $\gamma + \Gamma_0 \subseteqq \Lambda$ since

$$\lambda_{a_i} + \lambda_{b_1} + \cdots + \hat{\lambda}_{b_i} + \cdots + \lambda_{b_k} \in \Lambda$$

by (ii)$'$.

We now apply Theorem 2 to completely solve the norm evaluation in the case $M \sim (\Gamma, \Lambda, F)$ where $F \geqq 0$ and Λ satisfies (ii). We make use of the following simple lemma:

LEMMA 2. *If* $M = (\alpha_{i,j})_{i,j=1}^{\infty}$ *and* $M' = (\alpha'_{i,j})_{i,j=1}^{\infty}$ *where*

$$\alpha_{i,j} \geqq \alpha'_{i,j} \geqq 0 \qquad \text{for all } i, j \in N^+$$

then

(i) $$|M| = \sup_{||x||_2 \leqq 1} ||MX|| , \quad |M|_I = \sup_{||x||_2 \leqq 1} (MX, X)$$

where $X = (\{x_i\})$ *has only finitely many nonzero coordinates, all positive.*

(ii) $$|M'| \leqq |M| , \quad |M'|_I \leqq |M|_I .$$

Proof. For $X = (\{x_i\}) \in l^2$ we define $X^+ = (\{|x_i|\})$. Note $||X||_2 = ||X^+||_2$ and X^+ and X have the same cardinality of nonzero coordinates. Also, $\alpha_{i,j} \geqq 0$ clearly implies $||MX||_2 \leqq ||MX^+||_2$ and $|(MX, X)| \leqq (MX^+, X^+)$ and (i) readily follows. But $\alpha_{i,j} \geqq \alpha'_{i,j} \geqq 0$ also implies each component of $M'X^+$ is dominated by the corresponding component of MX^+ and hence (ii) follows from (i).

THEOREM 3. *Let* $M \sim (\Gamma, \Lambda, F)$ *where*

(i) *$F(\gamma) \geqq 0$ for all $\gamma \in \Gamma$.*

(ii) *If Γ_0 is any finite subset of Γ there exists a $\gamma = \gamma(\Gamma_0)$ such that $\gamma + \Gamma_0 \subseteqq \Lambda$. Then*

$$|M| = |M|_I = \sum_{\gamma \in \Gamma} F(\gamma) \qquad (\textit{possibly } +\infty) .$$

Proof. Since $|M|_I \leqq |M|$ it suffices to show that

$$(1) \quad |M| \leqq \sum_{\gamma \in \Gamma} F(\gamma) , \qquad (2) \quad |M|_I = \sum_{\gamma \in \Gamma} F(\gamma) .$$

If $\sum_{\gamma \in \Gamma} F(\gamma) < +\infty$ there is nothing to prove since in this case the result is included in Theorem 2 because $f(x) = \sum_{\gamma \in \Gamma}(\gamma, x)F(\gamma)$ is a continuous function on G, $F = \hat{f}$, and $\|f\|_\infty = f(0) = \sum_{\gamma \in \Gamma} F(\gamma)$.

On the other hand, if $\sum_{\gamma \in \Gamma} F(\gamma) = +\infty$ then F may not be in $A(\Gamma)$ and hence we cannot apply Theorem 2 directly. Clearly (1) is true in this case and we need only verify (2). Let $\Gamma' = \{\gamma_1, \cdots, \gamma_n\}$ be any finite subset of Γ and define

$$M_{\Gamma'} = (\alpha_{i,j}^{\Gamma'})_{i,j=1}^{\infty}$$

where

$$\alpha_{i,j}^{\Gamma'} = \begin{cases} F(\gamma_\nu) & \text{if } \lambda_i - \lambda_j = \gamma_\nu \in \Gamma' \\ 0 & \text{otherwise ,} \end{cases}$$

i.e., $M_{\Gamma'} \sim (\Gamma, \Lambda, F_{\Gamma'})$ where $F_{\Gamma'}(\gamma) = F(\gamma)I_{\Gamma'}(\gamma)$. Since $F \geqq 0$, $\alpha_{i,j} \geqq \alpha_{i,j}^{\Gamma'} \geqq 0$ for $i, j \in N^+$, and Lemma 2 implies $|M|_I \geqq |M_{\Gamma'}|_I$. But by Theorem 2

$$|M_{\Gamma'}|_I = \operatorname*{ess\,sup}_{x \in G} \left| \sum_{\gamma \in \Gamma'} (\gamma, x) F_{\Gamma'}(\gamma) \right| = \sum_{\gamma \in \Gamma'} F(\gamma)$$

since $\sum_{\gamma \in \Gamma}(\gamma, x)F_{\Gamma'}(\gamma)$ is continuous and $F_{\Gamma'} \geqq 0$. This in turn implies

$$|M|_I \geqq \sup_{|\Gamma'| < +\infty} \sum_{\gamma \in \Gamma'} F(\gamma) = +\infty .$$

COROLLARY 2. *Under the hypothesis of Theorem* 3.

$$|M| = |M|_I = \sup_{i \in N^+} \left(\sum_{j \in N^+} \alpha_{i,j} \right) = \sup_{j \in N^+} \left(\sum_{i \in N^+} \alpha_{i,j} \right) .$$

Proof. We prove only $|M| = |M|_I = \sup_{i \in N^+}(\sum_{j \in N^+} \alpha_{i,j})$ the proof of the other equality being similar. By Theorem 3. we need only verify $\sup_{i \in N^+}(\sum_{j \in N^+} \alpha_{i,j}) = \sum_{\gamma \in \Gamma} F(\gamma)$. First

$$\sum_{j \in N^+} \alpha_{i,j} = \sum_{j \in N^+} F(\lambda_i - \lambda_j) = \sum_{\gamma \in \lambda_i - \Lambda} F(\gamma) \leqq \sum_{\gamma \in \Gamma} F(\gamma) .$$

Let $\Gamma' = \{\gamma_1, \cdots, \gamma_n\}$ be any finite subset of Γ and let $\Gamma_0 = \{0, -\gamma_1, \cdots, -\gamma_n\}$. Condition (ii) insures the existence of an $\alpha \in \Gamma$ such that $\alpha + \Gamma_0 \subseteqq \Lambda$. In particular $\alpha \in \Lambda$, say $\alpha = \lambda_{k(\alpha)}$. But

$$\Gamma' \subseteqq -\Gamma_0 = \alpha - (\alpha + \Gamma_0) \subseteqq \lambda_{k(\alpha)} - \Lambda ,$$

and thus for $i = k(\alpha)$ we have

$$\sum_{j \in N^+} \alpha_{i,j} = \sum_{\gamma \in \lambda_{k(\alpha)} - \Lambda} F(\gamma) \geqq \sum_{\gamma \in \Gamma'} F(\gamma)$$

since $F \geqq 0$, and our assertion follows.

3. An application.

In this section we apply the results of § 2 to evaluate the norm of a special type of linear operator.

DEFINITION 4. Let T be the circle group, considered as the real numbers $R^+ \bmod 2\pi$, and let $L^2 = L^2(T, dt)$ be the associated Hilbert function space with respect to normalized Lebesgue measure. Let $\mathscr{M} \subseteqq L^2$ be the submanifold

$$\mathscr{M} = \left\{ f \in L^2 \colon \int_T f(t)dt = 0 \right\} .$$

Furthermore, let $Z' = Z \sim \{0\}$ and for $a = \{a_n\}_{n \in Z'} \in L^1(Z')$ define $H_a \colon \mathscr{M} \to \mathscr{M}$ by

$$(H_a f)(t) = \sum_{n \in Z'} a_n f(nt)$$

(where equality of functions is to be taken in the L^2 sense).

We now show that the mapping $a \rightsquigarrow H_a$ is a one-to-one bounded linear transformation from $L^1(Z')$ into $\mathscr{M}^*$, the dual space of $\mathscr{M}$. For

$$\begin{aligned} \| H_a f \|_2^2 &= \left\| \sum_{n \in Z'} a_n f(nt) \sum_{m \in Z'} \overline{a_m f(mt)} \right\|_1 \\ &= \left\| \sum_{m,n \in Z'} a_n \overline{a_m} f(nt) \overline{f(mt)} \right\|_1 \leqq \sum_{m,n \in Z'} | a_n | \, | a_m | \, \| f(nt) f(mt) \|_1 \\ &\leqq \| \sum_{m,n \in Z'} | a_n | \, | a_m | \, \| f(nt) \|_2 \, \| f(mt) \|_2 = \Big(\sum_{n \in Z'} | a_n | \Big)^2 \| f \|_2^2 = \| a \|_1^2 \| f \|_2^2 \end{aligned}$$

since $\| f(nt) \|_2 = \| f(t) \|_2$ for all $n \in Z'$. Therefore $\| H_a \|_{op} \leqq \| a \|_1$. Also, $f \in \mathscr{M}$ implies $H_a f \in \mathscr{M}$ since

$$\int_T (H_a f)(t)dt = \sum_{n \in Z'} a_n \int_T f(nt)dt = 0 .$$

Therefore, since H_a is clearly linear, $H_a \in \mathscr{M}^*$ and the mapping

$a \rightsquigarrow H_a$ is bounded and linear from $L^1(\boldsymbol{Z}')$ to $\mathscr{M}^*$. Finally, the mapping is one-to-one since

$$H_a(e^{it}) = \sum_{n \in \boldsymbol{Z}'} a_n e^{int} = 0 \Leftrightarrow a = 0 .$$

We now apply Corollary 1 to evaluate the norm of H_a.

THEOREM 4. *Let $a = \{a_n\}_{n \in \boldsymbol{Z}'} \in L^1(\boldsymbol{Z}')$, and for $r \in Q^\times$ let*

$$F(r) = \sum_{\substack{m/n=r \\ m,n \in \boldsymbol{Z}'}} a_m \bar{a}_n .$$

Then

$$\| H_a \|_{op} = \max_{x \in \hat{Q}^\times} \left| \sum_{r \in Q^\times} (r, x) F(r) \right|^{\frac{1}{2}}$$

where $\hat{Q}^\times$ is the compact dual of the discrete group $Q^\times$.

Proof. Let $f \in \mathscr{M}$, and let the Fourier expansion of f be

$$f(t) = \sum_{m \in \boldsymbol{Z}'} b_m e^{imt} .$$

Then

$$(H_a f)(t) = \sum_{n \in \boldsymbol{Z}'} a_n f(nt) = \sum_{n \in \boldsymbol{Z}'} a_n \Big[\sum_{m \in \boldsymbol{Z}'} b_m e^{imnt} \Big]$$
$$= \sum_{m,n \in \boldsymbol{Z}'} a_n b_m e^{imnt} = \sum_{p \in \boldsymbol{Z}'} c_p e^{ipt} ,$$

where $c_p = \sum_{mn=p \; m,n \in \boldsymbol{Z}'} a_n b_m$, and L^2 convergence is the justification for the rearrangement of summation. Therefore

$$\| H_a f \|_2^2 = \sum_{p \in \boldsymbol{Z}'} | c_p |^2 = \sum_{p \in \boldsymbol{Z}'} \Big(\sum_{\substack{mn=p \\ m,n \in \boldsymbol{Z}'}} \overline{a_n b_m} \sum_{\substack{m'n'=p \\ m',n' \in \boldsymbol{Z}'}} a_{n'} b_{m'} \Big)$$
$$= \sum_{\substack{m,n,m',n' \in \boldsymbol{Z}' \\ mn=m'n'}} \bar{a}_n \bar{b}_m a_{n'} b_{m'} = \sum_{m,m' \in \boldsymbol{Z}'} \Big\{ \Big(\sum_{\substack{n,n' \in \boldsymbol{Z}' \\ n'/n=m/m'}} a_{n'} \overline{a_n} \Big) b_{m'} \overline{b_{m'}} \Big\}$$

where the manipulation of the quadruple sum is justified by absolute convergence:

$$\sum_{\substack{m,n,m',n' \in \boldsymbol{Z}' \\ mn=m'n'}} | \bar{a}_n \bar{b}_m a_{n'} b_{m'} | = \sum_{n,n' \in \boldsymbol{Z}'} | a_n | \, | a_{n'} | \Big(\sum_{\substack{m,m' \in \boldsymbol{Z}' \\ n'/n=m/m'}} | b_m | \, (b_{m'}) \Big)$$
$$\leqq \sum_{n,n' \in \boldsymbol{Z}'} | a_n | \, | a_{n'} | \Big(\sum_{m \in \boldsymbol{Z}'} | b_m |^2 \Big) = \| a \|_1^2 \, \| f \|_2^2 < +\infty$$

by Cauchy-Schwarz. Upon setting

$$\alpha_{i,j} = \sum_{m/n=i/j} a_m \bar{a}_n = F\left(\frac{i}{j}\right) \qquad \text{for } i, j \in \boldsymbol{Z}'$$

and

$$M = (\alpha_{ij}) \quad (\text{order } \boldsymbol{Z}' = (1, -1, 2, -2, \cdots)) ,$$

we obtain

$$M \sim (Q^\times, \boldsymbol{Z}', F) .$$

Also, upon identifying $\mathscr{M}$ with $\mathscr{L}^2$ by $f \leftrightarrow X_f = (b_1, b_{-1}, b_2, b_{-2}, \cdots)$ we have

$$\| H_a f \|_2^2 = (MX_f, X_f) .$$

But

$$\sum_{r \in Q^\times} | F(r) | = \sum_{\substack{i,j \in \boldsymbol{Z}' \\ (i,j)=1 \\ i>0}} \left| F\left(\frac{i}{j}\right) \right| \leqq \sum_{\substack{i,j \in \boldsymbol{Z}' \\ (i,j)=1 \\ i>0}} \sum_{m/n=i/j} | a_n | \, | a_m |$$
$$= \left(\sum_{n \in \boldsymbol{Z}'} | a_n | \right)^2 = \| a \|_1^2 < +\infty ,$$

and hence

$$f(x) = \sum_{r \in Q^\times} (r, x) F(r)$$

is a continuous function on $\hat{Q}^\times$ with Fourier transform F. The theorem follows upon applying Theorem 2 (Corollary 1 (i)) to $M \sim (Q, \boldsymbol{Z}', F)$.

COROLLARY 3. *If* $a = \{a_n\}_{n \in \boldsymbol{Z}'} \in L^1(\boldsymbol{Z}')$ *and* $a_n \geqq 0$ *for all* $n \in \boldsymbol{Z}'$, *then*

$$\| H_a \|_{op} = \| a \|_1 .$$

Proof. By Theorem 4,

$$\| H_{ap} \|_{op} = \max_{x \in \hat{Q}^\times} \left| \sum_{r \in Q^\times} (r, x) F(r) \right|^{\frac{1}{2}} \leqq \left(\sum_{r \in Q^\times} | F(r) | \right)^{\frac{1}{2}} = \| a \|_1$$

since

$$a_n \geqq 0, \; |F(r)| = \sum_{\substack{m/n=r \\ m,n \in \boldsymbol{Z}'}} a_n a_m , \text{ and } \sum_{r \in Q^\times} | F(r) | = \| a \|_1^2 .$$

But upon setting $x = 0$ we obtain

$$\left| \sum_{r \in Q^\times} (r, 0) F(r) \right|^{\frac{1}{2}} = \left| \sum_{r \in Q^\times} F(r) \right|^{\frac{1}{2}} = \Bigl(\sum_{r \in Q^\times} | F(r) | \Bigr)^{\frac{1}{2}} = \| a \|_1 ,$$

and thus the proof is complete.

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