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ON THE STRUCTURE TOPOLOGY OF SIMPLEX SPACES

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This paper studies the hull-kernel topology of the maximal ideal space of separable simplex spaces. We show that the properties of local compactness, first countability, and second countability are equivalent.

A simplex space is an ordered Banach space V with closed positive cone whose dual is an L-space [4, 5, 6]. Let $P_1(V)$ be the positive linear functionals on V with norm less than or equal to one. The pure states are the extreme points of $P_1(V)$ and are denoted by $EP_1(V)$. We shall write E^+ for the nonzero extreme points of $P_1(V)$, i.e.,

$$E^+ = EP_1(V) - \{0\} .$$

We let Z be the closure, in the weak* topology, of E^+.

We let max V be the set of closed maximal ideals of V equipped with the hull-kernel topology. The closed ideals of V are in a one-to-one order inverting correspondence with the closed faces of $P_1(V)$ containing zero. Thus, max V may be identified with E^+ as point sets. By this identification we can transfer the hull-kernel topology to E^+ and define a new topology, called the structure topology, on E^+. Its closed sets are the nonzero extreme points of a closed face containing zero. Hence, the structure topology is weaker than the weak* topology.

In this paper we shall consider various topological properties of max V. In particular, we consider compactness, local compactness, first countability, second countability, and standard Borel structure. It was conjectured in [5] that for separable simplex spaces the latter four properties are equivalent. This is very nearly correct, as we see in Theorem 3.3 and Proposition 3.6.

In section 1, we study the structure topology and introduce several new maps. We give several criteria for determining whether a set in E^+ is structure closed. In section 2 we consider the property of first countability for max V.

Finally, in section 3, we state and prove the main theorems. We show that if V is separable, then max V is compact if and only if 0 does not belong to Z. We also show, for V separable, that the properties of first countability, second countability, and local compactness are equivalent for max V.

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0. Conventions. The notation and definitions are those used in [4, 5, 6]. We shall use results freely from these papers. Throughout, we assume that vector spaces have nonzero elements. In normed spaces, the subscript λ on a subset indicates the subset intersected with the closed ball of radius λ.

For any set $A \subseteqq P_1(V)$, $\bar{A}$ will denote the weak* closure of A and A^+ will denote $A - \{0\}$.

For any net, Greek subscripts, e.g., α, β, denote arbitrary index sets while Latin subscripts, e.g., i, j, n, denote the natural numbers as an index set, i.e., $\{x_n\}$ is a sequence.

For the entire paper, V *will always denote a separable simplex space.*

Throughout, propositions are stated in terms of the hull-kernel topology of $\max V$ and proven for the structure topology of E^+. Hopefully, this will cause no confusion.

1. The structure topology and several maps. In this section, we prove some preliminary results concerning the structure topology.

As V is separable, V_1^* is weak* metrizable and, thus, so is $P_1(V)$. As $P_1(V)$ is a simplex, Choquet's Theorem asserts that for each $q \in P_1(V)$ there is a unique maximal probability measure π_q which represents q and for which

$$\pi_q(P_1(V) - EP_1(V)) = 0 .$$

[10, p. 70]. We shall always denote this measure by π_q.

Let $S[q]$ be the smallest weak* closed face of $P_1(V)$ containing 0 and q, for $q \in P_1(V)$. As $S[q]$ is compact, metrizable, and convex, the Choquet Theorem applies equally as well to it. Hence, there is a maximal probability measure μ which represents q and for which

$$\mu(S[q] - \text{extreme points of } S[q]) = 0 .$$

As $S[q]$ is a face, its extreme points are extreme in $P_1(V)$ and so

$$\mu(S[q] - EP_1(V)) = 0 .$$

Since $P_1(V)$ is a simplex, $\mu = \pi_q$. Hence

$$\operatorname{supp} \pi_q \subseteqq S[q] .$$

For $q \in P_1(V)$, we let $\operatorname{supp}^+ \pi_q = \{y \in P_1(V)^+ \mid$ each neighborhood $N \subseteqq P_1(V)$ of y satisfies $\pi_q(N) > 0\}$. Then, $\operatorname{supp}^+ \pi_q = \operatorname{supp} \pi_q - \{0\}$, consistent with our convention. The first proposition gives some relations between the structure and weak* topologies. Recall that for $A \subseteqq P_1(V)$, the weak* closure of A is denoted by $\bar{A}$.

PROPOSITION 1.1. (A) *Let* $q \in P_1(V)^+$. *Then* $\operatorname{supp}^+ \pi_q$ *is the closure in* $P_1(V)^+$ *of* $\operatorname{supp}^+ \pi_q \cap E^+$. *Further,*

$$E^+ \cap S[q] = \textit{structure closure } (\operatorname{supp}^+ \pi_q \cap E^+) .$$

(B) *Let* $D \subseteqq P_1(V)$. *Suppose for each* $q \in D$ *we have* $\operatorname{supp} \pi_q \subseteqq D$. *If* D *is weak* closed, then the weak* closed convex hull of* D *is a face of* $P_1(V)$ *and* $D \cap E^+$ *is structurally closed.*

(C) *Let* $D \subseteqq E^+$. *Then the following are equivalent:*

(1) D *is structure-closed.*

(2) *For each nonzero* $q \in \bar{D}$, $S[q] \cap E^+ \subseteqq D$.

(3) *For each nonzero* $q \in \bar{D}$, $\operatorname{supp}^+ \pi_q \cap E^+ \subseteqq D$.

(4) (a) D *is weak* closed relative to* E^+.

(b) *For each* $q \in \bar{D} - EP_1(V)$, $\operatorname{supp}^+ \pi_q \cap E^+ \subseteqq D$.

Proof. (A) Since $P_1(V)^+$ is a locally compact metric space,

$$\pi_q(P_1(V)^+ - \operatorname{supp}^+ \pi_q) = 0 .$$

Since π_q is a maximal measure,

$$\pi_q(P_1(V)^+ - E^+) = 0 .$$

Suppose y is not in the closure in $P_1(V)^+$ of $\operatorname{supp}^+ \pi_q \cap E^+$. Then there is a relatively open set $N \subseteqq P_1(V)^+$ about y such that

$$N \cap (\operatorname{supp}^+ \pi_q \cap E^+) = \varnothing .$$

But then N is open in $P_1(V)$ and obviously

$$\pi_q(N) \leqq \pi_q(P_1(V)^+ - (\operatorname{supp}^+ \pi_q \cap E^+)) = 0 .$$

Thus $y \notin \operatorname{supp}^+ \pi_q$. The other inclusion is trivial.

For the second conclusion, from the discussion preceeding the proposition,

$$\operatorname{supp}^+ \pi_q \cap E^+ \subseteqq S[q] \cap E^+ .$$

As the latter is structurally closed,

$$\text{structure closure } (\operatorname{supp}^+ \pi_q \cap E^+) \subseteqq S[q] \cap E^+ .$$

For the other inclusion, let K be the closed face containing zero which satisfies

$$K \cap E^+ = \text{structure closure } (\operatorname{supp}^+ \pi_q \cap E^+) ,$$

which exists by the definition of the structure topology. As K is closed and contains zero, the first part implies that $\operatorname{supp} \pi_q \subseteqq K$. As K is convex, $q \in K$. But then $S[q] \subseteqq K$ and so $S[q] \cap E^+ \subseteqq K \cap E^+$.

(B) The first conclusion is [5, Th. 3.3] while the second follows easily from the definition of the structure topology and the Milman Theorem [10, p. 9].

(C) (1) $\Rightarrow$ (2). Let q be a nonzero element of $\bar{D}$. Then there is a sequence $\{p_n\} \subseteqq D$ such that $p_n \rightarrow q$. Then $\{p_n\}$ tends structurally to each element of $S[q] \cap E^+$ [6, Lemma 2.3] and so $S[q] \cap E^+ \subseteqq D$.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). Let $\{q_n\} \subseteqq D$ and $q \in E^+$ satisfy $q_n \rightarrow q$. Then $q \in \bar{D}$ and so $\operatorname{supp}^+ \pi_q \cap E^+ = \{q\} \subseteqq D$. Therefore D is weak* closed relative to E^+.

(4) $\Rightarrow$ (1). Let $F = \bar{D} \cup \{0\}$. If $q \in F - \{0\}$, then $\operatorname{supp}^+ \pi_q \cap E^+ \subseteqq D$ and so $\operatorname{supp}^+ \pi_q \subseteqq \bar{D}$ by part (A). Hence $\operatorname{supp} \pi_q \subseteqq F$ and so (B) implies that $F \cap E^+$ is structurally closed. Since D is weak* closed in E^+, $F \cap E^+ = \bar{D} \cap E^+ = D$.

We must now define certain maps. Let $\tilde{\Phi}: Z \rightarrow$ structure closed subsets of E^+ by

$$\tilde{\Phi}(q) = S[q] \cap E^+ .$$

If $0 \in Z$, then

$$\tilde{\Phi}(0) = \varnothing .$$

We let $\tilde{\psi}: E^+ \rightarrow$ subsets of Z by

$$\tilde{\psi}(p) = \{q \in Z \mid p \in \tilde{\Phi}(q)\} .$$

Hence, for each $p \in E^+$,

$$0 \notin \tilde{\psi}(p) .$$

We extend these maps to set functions by letting

$$\Phi(A) = \bigcup_{q \in A} \tilde{\Phi}(q)$$

for every set $A \subseteqq Z$, and, for any $B \subseteqq E^+$,

$$\psi(B) = \bigcup_{p \in B} \tilde{\psi}(p) = \{q \in Z \mid \tilde{\Phi}(q) \cap B \neq \varnothing\} .$$

For each $q \in Z$, we shall write $\Phi(q)$ to mean $\Phi(\{q\})$. Similarly, for each $p \in E^+$, we write $\psi(p)$ to mean $\psi(\{p\})$. Then, obviously, for $q \in Z$,

$$\Phi(q) = \tilde{\Phi}(q)$$

and for $p \in E^+$,

$$\psi(p) = \tilde{\psi}(p) .$$

We may, alternately, describe the maps Φ and ψ in terms of the relation R in $E^+ \times Z$ defined by

$$pRq \text{ if and only if } p \in S[q] \cap E^+ .$$

We have, for any $A \subseteqq Z$,

$$\Phi(A) = \{p \mid pRq, \text{ some } q \in A\}$$

and for any $B \subseteqq E^+$,

$$\psi(B) = \{q \mid pRq, \text{ some } p \in B\} .$$

Elementary relations for these maps are contained in the next lemma.

LEMMA 1.2. (1) *For* $A \subseteqq Z^+$, $\psi\Phi(A) \supseteqq A$.

(2) *For* $B \subseteqq E^+$, $\Phi\psi(B) \supseteqq B$.

(3) *Let* $p \in E^+$ *and* $q \in Z$. *Then* $q \in \psi(p)$, $p \in \Phi(q)$, *and* pRq *are equivalent.*

(4) *For* $B \subseteqq E^+$, $E^+ \cap \psi(B) = B$.

(5) *Let* $q \in Z$. *Then* $\Phi(q) =$ *structure closure* $(\mathrm{supp}^+ \pi_q \cap E^+)$.

Proof. They are all obvious.

If A and B are any topological spaces, a map $\Gamma: A \to$ subsets of B is called *lower semi-continuous* if whenever $U \subseteqq B$ is open then

$$\{x \in A \mid \Gamma(x) \cap U \neq \varnothing\}$$

is open in A [2, Th. 1, p. 115].

PROPOSITION 1.3. *Φ is lower semi-continuous when E^+ is given the structure topology. In fact, if $U \subseteqq E^+$, then U is structurally open if and only if $\psi(U)$ is weak* open in Z.*

Proof. Let $U \subseteqq E^+$. Then

$$\begin{aligned} Z - \psi(U) &= \{q \in Z \mid \Phi(q) \cap U = \varnothing\} \\ &= \{q \in Z \mid \Phi(q) \subseteqq E^+ - U\} . \end{aligned}$$

Let us first suppose that U is structurally open. Then $E^+ - U$ is structurally closed. Hence, there is a closed face K containing zero so that $K \cap E^+ = E^+ - U$. Let $q \in K \cap Z$. Then $S[q] \subseteqq K$. Thus $\Phi(q) \subseteqq E^+ - U$ and so $q \in Z - \psi(U)$. If $q \in Z - \psi(U)$, then

$\operatorname{supp} \pi_q \subseteqq \{0\} \cup \Phi(q) \subseteqq \{0\} \cup (K \cap E^+) \subseteqq K$ and so $q \in K$. Therefore, $Z - \psi(U) = K \cap Z$ and so $Z - \psi(U)$ is weak* closed.

Second, let us suppose that $\psi(U)$ is open. Then $(Z - \psi(U)) \cap E^+ = E^+ - U$ is weak* closed in E^+. In order to show that $E^+ - U$ is structure closed, we let q be any nonzero element of $\overline{E^+ - U}$. Since $\overline{E^+ - U} \subseteqq \overline{Z - \psi(U)} \cap \bar{E}^+ = Z - \psi(U)$, we have that $S[q] \cap E^+ = \Phi(q) \subseteqq E^+ - U$. Hence, $E^+ - U$ is structurally closed [Proposition 1.1 (C)].

The main reason we introduced the map Φ is the following.

PROPOSITION 1.4. *Suppose $\{p_n\} \subseteqq E^+$ and $q \in Z$. If $p_n \to q$, then*

(1) *$F = \{p_n\} \cup \Phi(q)$ is structurally closed.*

(2) *Let $p_{n_0} \in F - \Phi(q)$. Then $\{p_{n_0}\}$ is structurally open relative to F.*

(3) *$\{p_n\}$ converges structurally to each element of $\Phi(q)$.*

Proof. The conclusions are all trivial if $q = p_N = p_{N+1} = \cdots$ so we may assume that $\{p_n\}$ is not eventually the constant sequence $\{q\}$. For (1), we let $D = \{p_n\} \cup S[q]$. We claim that for each $z \in D$ we have $\operatorname{supp} \pi_z \subseteqq D$. Indeed, if $z = p_{n_0}$, $\operatorname{supp} \pi_{p_{n_0}} = \{p_{n_0}\} \subseteqq D$. If $z \in S[q]$, then $\operatorname{supp} \pi_z \subseteqq S[z] \subseteqq S[q]$. Since $S[q]$ is closed and $\{p_n\} \cup \{q\}$ is closed, $\bar{D} = D$. Therefore, $D \cap E^+ = \{p_n\} \cup (S[q] \cap E^+) = \{p_n\} \cup \Phi(q) = F$ is structurally closed [Proposition 1.1 (B)]. To show (2), we let $I = \{i \mid p_i \neq p_{n_0}\}$. As k runs over I, $p_k \to q$. By part (1), $\{p_k \mid k \in I\} \cup \Phi(q)$ is structurally closed, i.e., $\{p_{n_0}\}$ is structurally open relative to F. Part (3) is contained in [6, Lemma 2.3].

COROLLARY 1.5. *Suppose $\{p_n\} \subseteqq E^+$ and $q \in Z$. If $p_n \to q$, then the set of structure convergence points of $\{p_n\}$, the set of structure cluster points of $\{p_n\}$, and $\Phi(q)$ are the same set.*

Proof. We need only show that if $x \in E^+$ is a structure cluster point of $\{p_n\}$ then $x \in \Phi(q)$. Let $F = \{p_n\} \cup \Phi(q)$. As F is structure-closed, x a cluster point implies $x \in F$. Part (2) of the above shows that such an x is not in $F - \Phi(q)$. Hence, $x \in \Phi(q)$.

2. Preliminary results. We study in this section the property of first countability for $\max V$. We shall derive several equivalent properties. Given that $\max V$ is first countable, we can find structure open sets by the following.

LEMMA 2.1. *Let $x \in E^+$ and W be any w^*-open set containing*

$\psi(x)$. *If* x *has a countable structure base, then* $E^+ \cap W$ *has structure interior and, further,* x *is in the structure interior of* $(E^+ \cap W)$.

Proof. The proof will be by contradiction. Let $U_1 \supset U_2 \supset \cdots$ be a structure base at x. Suppose that there is an $x_n \in U_n - (E^+ \cap W)$. Then $\{x_n\}$ converges structurally to x. Since Z is compact metric, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $y \in Z$ such that $x_{n_k} \to y$. Therefore $y \in \psi(x)$ and so $y \in W$. But

$$x_{n_k} \in (U_{n_k} - (E^+ \cap W)) \cap E^+$$

and so $x_{n_k} \notin W$ for each n_k. This contradicts the assumption that W is a w^*-neighborhood of y.

LEMMA 2.2. *Let* $x \in E^+$. *Let* $\tilde{U}_\varepsilon = E^+ \cap \{z \mid \operatorname{dist}(z, \psi(x)) < \varepsilon\}$ *and let* U_ε *be the structure interior of* $\tilde{U}_\varepsilon$. *Assume* $\psi(x)$ *is compact and that* $x \in U_\varepsilon$ *for each* $\varepsilon > 0$.

Then, if U *is a structure neighborhood of* x, *there exists an* $\varepsilon > 0$ *such that* $x \in U_\varepsilon \subseteqq \tilde{U}_\varepsilon \subseteqq U$. *In particular,* $\{U_{1/n}\}$ *form a countable structure base at* x.

Proof. Let U be a structure neighborhood of x. First note that $\psi(U)$ is a neighborhood of $\psi(x)$ [Proposition 1.3]. As $\psi(x)$ is compact, there is an $\varepsilon > 0$ such that

$$\psi(x) \subseteqq \{z \mid \operatorname{dist}(z, \psi(x)) < \varepsilon\} \subseteqq \psi(U) .$$

Intersecting with E^+, we have $x \in \tilde{U}_\varepsilon \subseteqq \psi(U) \cap E^+ = U$.

LEMMA 2.3. *Let* $x \in E^+$ *and suppose that* x *has a countable structure base. Then* $\psi(x)$ *is compact.*

Proof. Since Z is compact, it suffices to show that $\psi(x)$ is closed. Let $\{q_n\} \subset \psi(x)$ and $q \in Z$ satisfy $q_n \to q$. Let $O_1 \supset O_2 \supset \cdots$ be a w^*-base at q. We may assume that $q_n \in O_n$. Let $G_1 \supset G_2 \supset \cdots$ be a structure base at x. As $\psi(x) \subset \psi(G_j)$ and $\psi(G_j)$ is open $j = 1, 2, \cdots$, $\psi(G_n) \cap O_n$ is a w^*-open neighborhood of q_n, $n = 1, 2, \cdots$. Hence, there is a $p_n \in E^+ \cap \psi(G_n) \cap O_n$. Consequently, $p_n \in G_n$ and so $\{p_n\}$ converges structurally to x. As $p_n \in O_n$, $p_n \to q$. But then $x \in \Phi(q)$, i.e., $q \in \psi(x)$ and so $\psi(x)$ is indeed compact.

Putting these three lemmas together yields the following.

COROLLARY 2.4. *Let* $x \in E^+$ *and suppose that* x *has a countable structure base. Let* $\tilde{U}_\varepsilon = E^+ \cap \{z \mid \operatorname{dist}(z, \psi(x)) < \varepsilon\}$ *and let* U_ε *be the* *structure interior of* $\tilde{U}_\varepsilon$. *Let* U *be any structure neighborhood of* x. *Then there exists an* $\varepsilon > 0$ *such that* $x \in U_\varepsilon \subseteqq \tilde{U}_\varepsilon \subseteqq U$. *In particular,* $\{U_{1/n}\}$ *form a countable structure base at* x.

If C and D are topological spaces, a map $\Gamma: C \to$ subsets of D has a *closed graph* (at c) if whenever $\{c_\alpha\} \subseteqq C$ converges to $c \in C$, $\{d_\alpha\} \subseteqq D$ converges to $d \in D$, and $d_\alpha \in \Gamma(c_\alpha)$, then $d \in \Gamma(c)$.

We can now state and prove the main theorem in this section.

THEOREM 2.5. *The following statements are equivalent:*

(1) Φ *has a closed graph.*

(2) (a) $\psi(x)$ *is compact for each* $x \in E^+$.

(b) *Let* $x \in E^+$ *and* $\tilde{U}(\varepsilon) = \{z \mid \operatorname{dist}(z, \psi(x)) < \varepsilon\} \cap E^+$. *Then* x *is in the structure interior of* $E^+ \cap \tilde{U}(\varepsilon)$.

(3) $\max V$ *is first countable.*

(4) $\max V$ *is second countable.*

Proof. We shall show $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ and $(3) \Rightarrow (4)$.

$(1) \Rightarrow (2)$. Let $x \in E^+$ and $\tilde{U}(\varepsilon) = \{z \mid \operatorname{dist}(z, \psi(x)) < \varepsilon\} \cap E^+$. We first show that x is in the structure interior of $\tilde{U}(\varepsilon)$. Indeed, suppose that there is a net $\{x_\alpha\} \subseteqq E^+ - \tilde{U}(\varepsilon)$ such that $\{x_\alpha\}$ converges structurally to x. Since Z is compact, there is a subnet $\{x_{\alpha_\beta}\}$ and a point $y \in Z$ such that $x_{\alpha_\beta} \to y$. Since $x_{\alpha_\beta} = \Phi(x_{\alpha_\beta})$, the closed graph condition implies that $x \in \Phi(y)$, i.e., $y \in \psi(x)$. Thus, $\operatorname{dist}(x_{\alpha_\beta}, \psi(x)) \to 0$ and so $x_{\alpha_\beta} \in \{z \mid \operatorname{dist}(z, \psi(x)) < \varepsilon\}$ eventually. It follows that $x_{\alpha_\beta} \in \tilde{U}(\varepsilon)$ eventually and so $x_\alpha \in \tilde{U}(\varepsilon)$ frequently. This contradicts the choice of $\{x_\alpha\}$ and so the claim is proven. Next, we must demonstrate that $\psi(x)$ is compact. In fact, we must only show that $\psi(x)$ is closed since Z is compact. Let $\{y_\alpha\}$ be a net in $\psi(x)$ and $y \in Z$ satisfy $y_\alpha \to y$. The closed graph condition then implies that $x \in \Phi(y)$, i.e., $y \in \psi(x)$.

$(2) \Rightarrow (3)$ is Lemma 2.2.

$(3) \Rightarrow (1)$. Suppose $\{x_\alpha\} \subseteqq E^+$, $x \in E^+$, $\{y_\alpha\} \subseteqq Z$ and $y \in Z$. Let $y_\alpha \to y$, $\{x_\alpha\}$ converges structurally to x, and $x_\alpha \in \Phi(y_\alpha)$. We must show that $x \in \Phi(y)$. Let $\tilde{U}_n = E^+ \cap \{z \mid \operatorname{dist}(z, \psi(x)) < 1/n\}$ and U_n be the structure interior of $\tilde{U}_n$. Then $\{U_n\}$ forms a structure base at x [Corollary 2.4]. Let $\{O_n\}$ be a w^*-base at y. Since $\{x_\alpha\}$ converges structurally to x, there is an α_n^{1} such that for each $\alpha \geqq \alpha_n^{1}$, $x_\alpha \in U_n$. Similarly, there is an α_n^{11} such that for each $\alpha \geqq \alpha_n^{11}$, $y_\alpha \in O_n$. Choose α_n larger than α_n^{1}, α_n^{11}, and α_{n-1}. Taking $y_n = y_{\alpha_n}$, $x_n = x_{\alpha_n}$, clearly $y_n \to y$, $\{x_n\}$ converges structurally to x, $x_n \in \Phi(y_n) \cap U_n$. Recall that we must prove that $x \in \Phi(y)$.

Let

$$\tilde{U}(\varepsilon, n) = E^+ \cap \{z \mid \operatorname{dist}(z, \psi(x_n)) < \varepsilon\} .$$

Since $x_n \in U_n$ and U_n is a structure neighborhood about x_n, there is an $\varepsilon(n) > 0$ such that $x_n \in \tilde{U}(\varepsilon(n), n) \subseteqq U_n$ [Corollary 2.4]. Let $\delta(n) = \min(\varepsilon(n), 1/n)$. Then $x_n \in \tilde{U}(\delta(n), n) \subseteqq U_n$. Since E^+ is dense in Z, we can find $z_n \in E^+$ such that $\operatorname{dist}(z_n, y_n) < \delta(n) \leqq 1/n$. Since $y_n \to y$, we have $z_n \to y$. Because $y_n \in \psi(x_n)$, we have $z_n \in E^+ \cap \{z \mid \operatorname{dist}(z, \psi(x_n)) < \delta(n)\} = \tilde{U}(\delta(n), n)$. As $\tilde{U}(\delta(n), n) \subseteqq U_n$, we have $z_n \in U_n$. Hence $\{z_n\}$ structurally converges to x. Thus, $x \in \Phi(y)$ [Corollary 1.5].

(3) $\Rightarrow$ (4). Let $\mathscr{S} = \{S_i\}$ be a countable base for Z. Let $x \in E^+$ and U be an arbitrary structure neighborhood of x. Then $\psi(U)$ is a w^*-open neighborhood of each $y \in \psi(x)$. For each $y \in \psi(x)$, choose $S_y \in \mathscr{S}$ such that $y \in S_y \subseteqq \psi(U)$. The neighborhoods $\{S_y \mid y \in \psi(x)\}$ cover $\psi(x)$. As $\psi(x)$ is compact [Lemma 2.3], a finite number of these neighborhoods cover $\psi(x)$. Thus,

$$\psi(x) \subseteqq \bigcup_{i=1}^{N} S_{y_i} \subseteqq \psi(U) .$$

But Lemma 2.1 implies that x is in the structure interior of

$$\left(E^+ \cap \bigcup_{i=1}^{N} S_{y_i}\right) .$$

Since $E^+ \cap \bigcup_{i=1}^{N} S_{y_i} \subseteqq E^+ \cap \psi(U) = U$, sets of the form

$$\{\text{structure interior of } E^+ \cap A \mid A \text{ is a finite union of sets from } \mathscr{S}\}$$

form a countable structure base.

The proofs of Lemma 2.1 and (3) $\Rightarrow$ (4) are adapted from [12, Lemma 1 and Lemma 3, § 4]

3. The main theorems. The preceeding section has thrown light on several of the topological properties which we wanted to discuss. We will now discuss the others.

Though usually compactness and sequential compactness are not comparable for nonfirst-countable spaces, we have the following result:

PROPOSITION 3.1 *Let* $K \subseteqq \max V$. *Then the following are equivalent*:

(1) K *is compact.*

(2) K *is sequentially compact.*

Proof. (1) $\Rightarrow$ (2). Let $K \subseteqq E^+$ be structurally compact and let

$\{x_n\}$ be a sequence in K. Since Z is compact metric, there is a subsequence $\{x_{n_k}\}$ and a point $y \in Z$ such that $x_{n_k} \to y$. As K is structure-compact, the net $\{x_{n_k}\} \subseteqq K$ has a cluster point $z \in K$. But then $z \in \Phi(y)$ [Corollary 1.5] and so $\{x_{n_k}\}$ converges structurally to $z \in K$ [Proposition 1.4]. Thus, K is sequentially compact.

$(2) \Rightarrow (1)$. Let $K \subseteqq E^+$ be structurally sequentially compact and let $\{x_\alpha\}$ be a net in K. Since Z is compact, there is a subnet $\{x_{\alpha_\beta}\}$ and a point $y \in Z$ such that $x_{\alpha_\beta} \to y$. As y has a countable neighborhood base, we may find a sequence $\{x_n\} \subseteqq E^+$ such that $x_n \to y$ and $\{x_n\} \subseteqq \{x_{\alpha_\beta}\}$. Then $\{x_n\}$ is a sequence in K and so there is a subsequence $\{x_{n_k}\}$ and a point $z \in K$ such that $\{x_{n_k}\}$ converges structurally to z. Then $z \in \Phi(y)$ and so $\{x_{\alpha_\beta}\}$ tends structurally to z [6, Lemma 2.3] and so K is compact.

We can now completely characterize those separable simplex spaces for which max V is compact.

THEOREM 3.2. *The following are equivalent:*

(1) max V *is compact.*

(2) max V *is sequentially compact.*

(3) $0 \notin Z$.

Proof. $(1) \Rightarrow (2)$ is a special case of Proposition 3.1.

$(2) \Rightarrow (3)$. Suppose $0 \in Z$. Then there is a sequence $\{p_n\} \subseteqq E^+$ such that $p_n \to 0$. Since max V is sequentially compact, there is a subsequence $\{p_{n_k}\}$ and a point $p \in E^+$ such that $\{p_{n_k}\}$ tends structurally to p. But then Corollary 1.5 implies that $p \in \Phi(0)$. However $\Phi(0) = \emptyset$ and we have a contradiction.

$(3) \Rightarrow (1)$. Let $\{U_\alpha\}$ be a structure-open cover of E^+. Then $\bigcup_\alpha U_\alpha = E^+$ and so $\psi(\bigcup_\alpha U_\alpha) = \bigcup_\alpha \psi(U_\alpha) = \psi(E^+) = Z^+$. Since $0 \notin Z$, $Z^+ = Z$ and so $\{\psi(U_\alpha)\}$ is an open cover of the compact set Z. Hence, there exists a finite subcover, i.e., there are sets $U_{\alpha_1}, \cdots, U_{\alpha_N}$ such that $\bigcup_{i=1}^N \psi(U_{\alpha_i}) = Z$. Then, intersecting both sides with E^+ yields

$$E^+ = \bigcup_{i=1}^{N} (E^+ \cap \psi(U_{\alpha_i})) = \bigcup_{i=1}^{N} U_{\alpha_i}$$

[Lemma 1.2(4)]. Hence, E^+ is structurally compact.

REMARK. The proof of $(3) \Rightarrow (1)$ establishes the fact that for a set $K \subseteqq \max V$, if $\psi(K)$ is compact, then K is compact.

We now come to the major result of this paper.

THEOREM 3.3. *Let V be a separable simplex space. Then the following are equivalent*:

(1) *Φ has a closed graph.*
(2) $\max V$ *is first countable.*
(3) $\max V$ *is second countable.*
(4) $\max V$ *is locally compact.*

Proof. Using Theorem 2.5, we must only show that (4) is equivalent to the others. We show first that first countability at one point implies local compactness at that point. Let $p \in E^+$ and suppose p has a countable structure neighborhood base. Let U be a structure open set containing p. We must find a structure-compact neighborhood K of p within U. Let $F = E^+ - U$. It is structurally closed and, hence, there is a closed face Q of $P_1(V)$, containing zero, such that $Q \cap E^+ = F$. Since $p \notin F$, $\psi(p) \cap Q = \emptyset$.

At this point, we shall specify the metric which we are using for $P_1(V)$. If $\{\xi_n\}$ is dense in V_1, then we take

$$\operatorname{dist}(x, y) = \sum_n 2^{-n} \,|\, x(\xi_n) - y(\xi_n) \,| \,.$$

Since Q and $\psi(p)$ are compact [Lemma. 2.3],

$$\operatorname{dist}(Q, \psi(p)) = \delta > 0 \,.$$

Let

$$D = \left\{ z \in Z \mid \operatorname{dist}(\psi(p), z) < \frac{\delta}{2} \right\}$$

and

$$T = \left\{ z \in Z \mid \operatorname{dist}(Q, z) \geqq \frac{\delta}{2} \right\} .$$

Then, clearly, T is compact, $T \cap Q = \emptyset$, and $D \subseteqq T$.

We first claim that $E^+ \cap T$ is a structure neighborhood of p within U. Indeed, $p \in E^+ \cap D \subseteqq E^+ \cap T$ and $E^+ \cap D$ is a structure neighborhood of p [Lemma 2.1]. Obviously, $E^+ \cap T \subseteqq U$.

We next claim that $E^+ \cap T$ is structure-compact. Let $\{p_\alpha\}$ be a net in $E^+ \cap T$. Since T is weak* compact, there is a subnet $\{p_{\alpha_\beta}\}$ and a point $q \in T$ such that $p_{\alpha_\beta} \to q$. Then $\{p_{\alpha_\beta}\}$ tends structurally to each point of $\Phi(q)$ [6, Lemma 2.3]. Hence it is sufficient to show $\Phi(q) \cap (E^+ \cap T) \neq \emptyset$. For $z \in P_1(V)$, let

$$f(z) = \text{dist}\,(z, Q)\ .$$

Obviously f is weak* continuous, $f(0) = 0$ and $f(q) \geqq \delta/2$. We next claim that f is convex. Indeed, let $x, y \in P_1(V)$ and $0 \leqq \lambda \leqq 1$. We must demonstrate that

$$f(\lambda x + (1 - \lambda)y) \leqq \lambda f(x) + (1 - \lambda)f(y)\ .$$

Choose q_x and q_y in Q such that

$$f(x) = \text{dist}\,(x, q_x)$$

and

$$f(y) = \text{dist}\,(y, q_y)\ .$$

Then

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &= \text{dist}\,(\lambda x + (1 - \lambda)y, Q) \\ &\leqq \text{dist}\,(\lambda x + (1 - \lambda)y, \lambda q_x + (1 - \lambda)q_y) \end{aligned}$$

since $\lambda q_x + (1 - \lambda)q_y \in Q$ by convexity

$$\begin{aligned} &= \sum_n 2^{-n} \,|\, \lambda x(\xi_n) + (1 - \lambda)y(\xi_n) - \lambda q_x(\xi_n) - (1 - \lambda)q_y(\xi_n) \,| \\ &\leqq \sum_n 2^{-n}(\lambda \,|\, x(\xi_n) - q_x(\xi_n) \,| + (1 - \lambda) \,|\, y(\xi_n) - q_y(\xi_n) \,|) \\ &= \lambda \sum_n 2^{-n} \,|\, x(\xi_n) - q_x(\xi_n) \,| + (1 - \lambda) \sum_n 2^{-n} \,|\, y(\xi_n) - q_y(\xi_n) \,| \\ &= \lambda\, \text{dist}\,(x, q_x) + (1 - \lambda)\, \text{dist}\,(y, q_y) \\ &= \lambda f(x) + (1 - \lambda)f(y)\ . \end{aligned}$$

Since $S[q]$ is convex and weak* compact, f restricted to $S[q]$ achieves its maximum at an extreme point of $S[q]$ [1, Satz 2]. As $S[q]$ is a face and $q \in S[q]$, there is a $p \in EP_1(V) \cap S[q]$ such that $f(p) \geqq f(q) \geqq \delta/2$. Since $f(0) = 0$, there is a $p \in E^+ \cap S[q] = \Phi(q)$ with $f(p) \geqq \delta/2$. But this means that there is a $p \in \Phi(q) \cap T$. Thus, $E^+ \cap T$ is structurally compact.

Next, we shall show that if max V is locally compact at a point, then that point has a countable neighborhood base. Let $x \in E^+$ and assume that E^+ is structurally locally compact at x. Let $\tilde{U}(\varepsilon) = \{z \in Z \mid \text{dist}\,(z, \psi(x)) < \varepsilon\} \cap E^+$ and $U(\varepsilon)$ be the structure-interior of $\tilde{U}(\varepsilon)$. We claim that $x \in U(\varepsilon)$. Indeed, suppose not. Then there is a net $\{x_\alpha\} \subseteqq E^+ - \tilde{U}(\varepsilon)$ such that $\{x_\alpha\}$ tends structurally to x. Since Z is compact, there is a subnet $\{x_{\alpha_\beta}\}$ and a point $y \in Z$ such that $x_{\alpha_\beta} \rightarrow y$. Suppose $x \notin \Phi(y)$. Let $U = E^+ - \Phi(y)$. It is a structure open set containing x. By local compactness, there is a structure-compact neighborhood K of x in U. Therefore, $x_{\alpha_\beta} \in$ structure-interior (K)

for all $\alpha_\beta \geqq \alpha_{\beta_0}$. Then $\{x_{\alpha_\beta} \mid \alpha_\beta \geqq \alpha_{\beta_0}\}$ still converges to y. Since Z is metrizable, we may select a sequence $\{x_n\} \subseteqq E^+$ such that $x_n \to y$ and $x_n \in \{x_{\alpha_\beta} \mid \alpha_\beta \geqq \alpha_{\beta_0}\}$. Then $\{x_n\} \subseteqq$ structure-interior (K) is a sequence in K. As K is sequentially compact [Proposition 3.1], there is a subsequence $\{x_{n_k}\}$ and a point $z \in K$ such that $\{x_{n_k}\}$ tends structurally to $z \in K$. As $x_{n_k} \to y$, we have $z \in \Phi(y)$ [Corollary 1.5]. Thus $z \in \Phi(y) \cap K$, which contradicts the choice of K. Hence, $x \in \Phi(y)$, i.e., $y \in \psi(x)$. As $x_{\alpha_\beta} \to y$, $\operatorname{dist}(x_{\alpha_\beta}, \psi(x)) \to 0$. Therefore $x_{\alpha_\beta} \in \tilde{U}(\varepsilon)$ eventually and so $x_\alpha \in \tilde{U}(\varepsilon)$ frequently. This contradicts the choice of the net $\{x_\alpha\}$ and so we have shown that $x \in U(\varepsilon)$.

In order to conclude that x has a countable structure neighborhood base, we need only show that $\psi(x)$ is compact [Lemma 2.2]. As Z is compact, we need only show that $\psi(x)$ is closed. Let $y \in \overline{\psi(x)}$. Suppose $y \notin \psi(x)$. Let $U = E^+ - \Phi(y)$. It is a structure neighborhood of x and so by hypothesis there is a compact neighborhood K of x in U. Let $G =$ structure interior (K). Then $\psi(G)$ is a weak* neighborhood of $\psi(x)$. Since $y \in \overline{\psi(G)}$, we may choose $\{x_k\} \subseteqq E^+ \cap \psi(G)$ such that $x_k \to y$. Since $\{x_k\} \subseteqq G \subseteqq K$, and K is sequentially structurally compact [Proposition 3.1], there is a subsequence $\{x_{k_j}\}$ and there is a point $z \in K$ such that $\{x_{k_j}\}$ tends structurally to z. As $x_{k_j} \to y$, we have $z \in \Phi(y)$ [Corollary 1.5]. Thus, $z \in \Phi(y) \cap K$, contradicting the choice of K. Hence $y \in \psi(x)$ and so $\psi(x)$ is compact.

REMARK. The proof of Theorem 3.3 established a stronger equivalence than that stated. Namely, we showed that first countability at a point is equivalent to local compactness at that point.

Specializing Theorem 3.3. to the case when cardinality $(Z-E^+)<\infty$, we get the following.

COROLLARY 3.4 *Let V be a separable simplex space. Suppose cardinality $(Z - E^+) < \infty$. Then* $\max V$ *is locally compact and second countable. Further, suppose $\{p_\alpha\}$ is a net in E^+ converging weak* to q and $p \in E^+$. Then the following are equivalent:*

(1) $p \in \Phi(q)$.

(2) $\{p_\alpha\}$ *converges structurally to p.*

Proof. We first note that $\psi(p)$, for each $p \in E^+$, is a finite set and so is trivially compact. Fix $p \in E^+$ and let U_ε be the structure-interior of $\{z \in E^+ \mid \operatorname{dist}(z, \psi(p)) < \varepsilon\}$. If we show that $p \in U_\varepsilon$, then Theorem 2.5 allows us to conclude the first statement. Let

$$A = \bigcup \{\Phi(q) \mid q \in Z - E^+, p \notin \Phi(q)\}$$

and

$$F = \{z \in E^+ \mid \operatorname{dist}(z, \psi(p)) \geqq \varepsilon\} \cup A .$$

Let x be a nonzero element of $\bar{F}$. Suppose

$$x \in \overline{\{z \in E^+ \mid \operatorname{dist}(z, \psi(p)) \geqq \varepsilon\}} .$$

Then $\operatorname{dist}(x, \psi(p)) \geqq \varepsilon$. If $x \in E^+$, then $\{x\} = \Phi(x) \subseteqq F$. If $x \in Z - E^+$, then, since, $x \notin \psi(p)$, we have $\Phi(x) \subseteqq A$. On the other hand, suppose $x \in \bar{A}$. As each $\Phi(q)$ is structurally-closed and cardinality $(Z - E^+) < \infty$, we have that A is structurally closed. Thus, $\Phi(x) \subseteqq A$ [Proposition 1.1 (C)]. Hence, we see that F is structurally closed [Proposition 1.1 (C)] and so $E^+ - F \subseteqq U_\varepsilon$. As $p \notin F$, we conclude that $p \in U_\varepsilon$.

To show the second statement, we merely note that (1) $\Rightarrow$ (2) is [6, Lemma 2.3] and that (2) $\Rightarrow$ (1) is implied by Φ having a closed graph.

Specializing Theorem 3.3 to the case of M-spaces, we have the following.

THEOREM 3.5. *Let V be a separable M-space. Then the following are equivalent*:

(1) *Suppose $\{p_n\} \subseteqq E^+$ and $\{\lambda_n p_n\} \subseteqq Z$. If $\{p_n\}$ converges and if $\lambda_n p_n \rightarrow 0$, then $p_n \rightarrow 0$.*

(2) *Φ has a closed graph.*

(3) max *V is first countable.*

(4) max *V is second countable.*

(5) max *V is locally compact.*

(6) max *V is metrizable.*

Proof. Since max V for a separable M-space is normal [5, Th. 3.8], the equivalence of (2) through (6) follows from Theorem 3.3 and Urysohn's metrization theorem.

(2) $\Rightarrow$ (1). Since $\Phi(0) = \varnothing$, (1) is merely the statement that Φ has a closed graph at 0, and so the implication is clear. In more detail, suppose $p_n \rightarrow y$ and $y \neq 0$. Let $z \in \Phi(y)$. Then $\{p_n\}$ tends structurally to z [Proposition 1.4]. The closed graph condition implies that $z \in \Phi(0) = \varnothing$, which is a contradiction.

(1) $\Rightarrow$ (5). The map $\Phi: Z^+ \rightarrow E^+$ by

$$\lambda p \rightarrow p$$

is the factor map of Z^+, with the weak* topology, onto E^+, with the structure topology [5, Formula 3.2 and Th. 3.6]. Let K be any compact set in Z^+. If $\Phi^{-1}\Phi(K)$ is compact for all such K, then E^+ is locally compact [3, Proposition 9, Ch. I, § 10]. To show that $\Phi^{-1}\Phi(K)$ is compact, we note that

$$\Phi^{-1}\Phi(K) = \{\lambda x \in Z \mid 0 < \lambda \leqq 1, x \in K\} .$$

As Z is compact, it suffices to show that $\Phi^{-1}\Phi(K)$ is closed in Z. To show this, let $\{q_n\} \subseteqq \Phi^{-1}\Phi(K)$ and $q \in Z$ satisfy $q_n \to q$. By definition, $q_n = \lambda_n x_n$ for some $x_n \in K$ and $0 < \lambda_n \leqq 1$. Since K and $[0, 1]$ are compact metric, there is a subsequence $\{q_{n_j}\}$, an $x \in K$, and a $\lambda \in [0, 1]$ such that $x_{n_j} \to x$ and $\lambda_{n_j} \to \lambda$. Hence, $q = \lambda x$. Since $x \in K$, if we show that $\lambda \neq 0$ then we have that $q \in \Phi^{-1}\Phi(K)$. However, if $\lambda = 0$, then $q = 0$ and so $q_{n_j} \to 0$. By (1), $x_{n_j} \to 0$. Since $K \subseteqq Z^+$, this is impossible.

The results $(3) \Leftrightarrow (4) \Leftrightarrow (6) \Rightarrow (5)$ are contained (tacitly, if not explicitly) in [5, Corollary 3.9; 12, Th. 3].

We may now answer a question posed in [5]. Poulsen [11] has constructed a metrizable compact simplex K for which E^+ is dense in K, i.e., $Z = K$. Taking V to be the affine functions on K vanishing at 0, we have a separable simplex space. Then max V cannot be locally compact. In fact, if $p \in E^+$, then $\lambda p \in Z$ for $0 < \lambda \leqq 1$ and so $0 \in$ closure $(\psi(p))$. Hence, $\psi(p)$ is not compact and so max V is not locally compact at M_p [Th. 3.3 and Th. 2.5]. More careful analysis shows that max V cannot contain even one compact set with interior.

It was conjectured in [5] that local compactness, first countability, second countability, and standard Borel structure were equivalent for separable simplex spaces. We have shown that the first three are indeed equivalent. As for the latter, we say that a Borel space has a *standard Borel structure* whenever it is Borel isomorphic to the Borel space associated with a Borel subset of a complete metric space [9, p. 138]. Since the extreme points of a metrizable compact convex set in a topological vector space form a G_δ-set [10, Proposition 1.3], $EP_1(V)$ may be metrized by a complete separable metric [8, § 29, VI]. As E^+ is a Borel subset of $EP_1(V)$, E^+ is standard. The map $M: p_M \to M$ is one-to-one, onto and continuous. Clearly, if M^{-1} is a Borel function then max V is standard.

PROPOSITION 3.6. *If* max V *has a countable base, then* max V *is standard. Further, there is a separable M-space V for which* max V *is standard but* max V *is not locally compact.*

Proof. The first statement follows immediately from [9, Th. 3.2]. As for the second, let

$$V = \left\{ f \in C[0, 1] \,\middle|\, f\left(\frac{1}{n}\right) = \frac{1}{n} f(1),\ n = 2, \cdots \right\}.$$

V is a separable M-space for which max V is not locally compact [5, Th. 4.3]. By [5, Proposition 4.1],

$$E^+ = \left\{ \delta_y \,\middle|\, 0 < y \leqq 1 \text{ and } y \neq \frac{1}{n} \text{ for } n \geqq 2 \right\}$$

and

$$Z = \{\delta_y \mid 0 \leqq y \leqq 1\}.$$

Therefore $Z - E^+$ is a closed set. Hence, E^+ is an open set in a compact metric space and so can be written as the countable union of compact sets, i.e., there are compact sets $K_i \subseteqq Z$ such that

$$E^+ = \bigcup_i K_i.$$

For any set $A \subseteqq E^+$, let

$$M(A) = \bigcup \{M(p) \mid p \in A\}.$$

Then $M(A) = (M^{-1})^{-1}(A)$. Let F be any closed set in E^+. Hence $M(F) = \bigcup_i M(K_i \cap F)$. Since $K_i \cap F$ is a compact set in E^+, $M(K_i \cap F)$ is closed [5, Corollary 3.5]. Thus, $M(F)$ is a countable union of closed sets in max V and so is Borel. Therefore, M maps Borel sets to Borel sets, i.e., M^{-1} is a Borel map. From above, this implies that max V is standard.

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