PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 1, 1971

ON BERGMAN OPERATORS FOR PARTIAL DIFFERENTIAL EQUATIONS IN TWO VARIABLES

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Bergman operators are linear integral operators that map complex analytic functions into solutions of linear partial differential equations with analytic coefficients. In this way methods and results of complex analysis can be used for characterizing general properties of classes of those solutions. For example, this approach yields theorems about the location and type of singularities, the growth, and the coefficient problem for series developments of solutions.

A partial differential equation being given, there exist various types of Bergman operators, and for that purpose it is essential to select an operator whose generating function is as simple as possible. The present paper considers differential equations in two independent variables, introduces a class of Bergman operators satisfying that requirement, and determines the corresponding class of differential equations in an explicit fashion. In fact, necessary and sufficient conditions are obtained in order that the solutions of a partial differential equation can be obtained by means of a Bergman operator of that class. It is also shown that the set of these equations includes several equations of practical importance.

2. Bergman operators of class P_0. We consider partial differential equations of the form

$$\Delta\psi + \alpha(x, y)\psi_x + \beta(x, y)\psi_y + \gamma(x, y)\psi = 0$$

assuming that α, β and γ are real analytic functions in some neighborhood of the origin. Setting $z_1 = x + iX$, $z_2 = y + iY$, we may continue the coefficients to complex values of the variables. We now introduce the variables

$$z = z_1 + iz_2 \quad \text{and} \quad z^* = z_1 - iz_2 .$$

(Note that $z^* = \bar{z}$ if z_1 and z_2 are real.) Transforming the given equation and eliminating one of the two first partial derivatives, we obtain

$$(2.1) \qquad Lu := u_{zz^*} + b(z, z^*)u_{z^*} + c(z, z^*)u = 0 .$$

A Bergman operator B corresponding to (2.1) may be defined by means of

$$(2.2)\qquad (Bf)\ (z, z^*) := \int_{-1}^{1} g(z, z^*, t) f\left(\frac{z}{2}(1-t^2)\right)(1-t^2)^{-1/2}\, dt$$

(t real). S. Bergman [4] has shown that if $g(z, z^*, t)$ is a solution of

$$(2.3)\qquad (1-t^2)g_{z^*t} - t^{-1} g_{z^*} + 2ztLg = 0$$

and g_{z^*}/tz is continuous, then $u(z, z^*) = (Bf)(z, z^*)$ is a solution of (2.1); here f is any analytic function. g is called the *generating function* of the operator B, and f is called a *B-associated function* of that solution u.

If L is such that (2.3) has a solution of the form

$$(2.4)\qquad g(z, z^*, t) = \sum_{\mu=0}^{m} q_{2\mu}(z, z^*) t^{2\mu}$$

(g_{z^*}/tz continuous), then L and B (with g given by (2.4)) are said to be of *class P*. Note that in this polynomial (2.4) we have omitted odd powers of t without loss of generality, as can be seen from (2.2).

If in (2.4), the functions $q_{2\mu}(z, z^*)$ can be represented in the form

$$(2.5)\qquad q_{2\mu}(z, z^*) = \lambda_{2\mu} q(z, z^*)^{\mu}$$

where $\mu = 0, \cdots, m$ and $\lambda_0 = 1,\ \lambda_2, \cdots, \lambda_{2m}$ are constants, then L and B (with g given by (2.4), (2.5)) are said to be of *class* P_0.

L is of class P if and only if there is a natural number m such that

$$h_{2m+2}(z, z^*) = 0$$

where $h_2(z, z^*) = c(z, z^*)q_0(z)$ with arbitrary analytic $q_0(z)$ and

$$h_{2\mu+2}(z, z^*) = (M_{2\mu} h_{2\mu})(z, z^*) \qquad (\mu = 1, \cdots, m)$$

with $M_{2\mu}$ defined by

$$(M_{2\mu} w)(z, z^*) = \frac{2z}{2\mu-1}\left(w_z + \left(b(z, z^*) - \frac{\mu-1}{z}\right) w + c(z, z^*)\int w\, dz^*\right).$$

This necessary and sufficient condition for $L \in P$ was obtained in [9]. An equation (2.1) being given, it is clear that this condition may readily by used to find out whether or not $L \in P$. However, the condition does not yield an explicit characterization of the class P, that is, it does not give *explicit* expressions for the coefficients $b(z, z^*)$ and $c(z, z^*)$ of all the differential equations (2.1) with $L \in P$.

Such explicit expressions are desirable, for instance, in connection with generalizing Bauer's and Peschl's theory [1-3] of the equation

$$L_0 u : = u_{zz^*} + \frac{\varepsilon m(m+1)}{(1+\varepsilon z z^*)^2} u = 0 \quad (m = 1, 2, \cdots ; \ \varepsilon = \pm 1) \tag{2.6}$$

to other equations. L_0 is of class P, and (2.6) is of importance because it is closely related to the Laplace and wave equations.

We shall now see that the class P_0, which is a subclass of the class P, can be characterized by necessary and sufficient conditions in an explicit fashion. Some applications will be given § 4, and it will be shown that $L_0 \in P_0$.

3. Explicit determination of the class P_0. For any equation (2.1) a corresponding Bergman operator B [cf. (2.2)] can be obtained by determining a power series solution of (2.3), as Bergman [4] has shown. Clearly, if we choose a particular class of generating functions $g(z, z^*, t)$, such as (2.4), we impose certain restrictions on the coefficients $b(z, z^*)$ and $c(z, z^*)$ in (2.1), and it is of interest to find corresponding conditions for b and c. A first class of Bergman operators for which this problem was solved is the class of exponential operators, which are suitable for the Helmholtz equation and other equations and have the property that the theory of linear ordinary differential equations can be applied in the study of solutions of partial differential equations represented by means of those operators; cf. [7] and [8]. We shall now obtain a complete solution of that problem for the class P_0.

THEOREM 1. *In* (2.1), $L \in P_0$ *if and only if* (a) *or* (b) *holds*:

(a) *There is a function* $p(z, z^*)$ *such that* b *and* c *in* (2.1) *can be represented in the form*

$$b(z, z^*) = \lambda p - \frac{p_{zz^*}}{p_{z^*}}, \quad c(z, z^*) = -\lambda p_{z^*} \quad (\lambda \textit{ any constant}) . \tag{3.1}$$

(b) *There is an integer* $m > 1$ *and a function* $\sigma(z^*)$ *such that* b *and* c *in* (2.1) *can be represented in the form*

$$b(z, z^*) = \frac{\kappa}{kz + \sigma(z^*)} \qquad (\kappa, k \textit{ any constants}) \tag{3.2a}$$

and

$$c(z, z^*) = -m((m+1)k - \kappa) \frac{\sigma'(z^*)}{(kz + \sigma(z^*))^2} . \tag{3.2b}$$

In Case (a), $m = 1$ *in* (2.4), $\lambda_2 = 2\lambda$ *and* $q(z, z^*) = zp(z, z^*)$ *in* (2.5). *In Case* (b),

$$\lambda_{2\mu} = \frac{(-4)^\mu\, m!}{(m-\mu)!\,(2\mu)!} \prod_{\nu=1}^{\mu} ((m+\nu)k - \kappa) \quad (\mu = 1, \cdots, m) \tag{3.3a}$$

and

$$q(z, z^*) = \frac{z}{kz + \sigma(z^*)} . \tag{3.3b}$$

Proof. If we substitute (2.4) in (2.3) and equate to zero the coefficient of each occurring power of t, we obtain a system of $m + 2$ partial differential equations involving $b(z, z^*)$, $c(z, z^*)$, and the coefficients $q_0(z, z^*), \cdots, q_{2m}(z, z^*)$ and their first and second partial derivatives. Let $\langle n \rangle$ denote the equation corresponding to t^n. Then the system is

$$\begin{array}{ll}
\langle -1 \rangle & q_{0z^*} = 0 \\
\langle 1 \rangle & q_{2z^*} + 2zcq_0 = 0 \\
\langle 2\mu-1 \rangle & (2\mu-1)q_{2\mu z^*} + 2zLq_{2\mu-2} - 2(\mu-1)q_{2\mu-2,z^*} = 0 \\
 & \hfill (\mu = 2, \cdots, m) \\
\langle 2m+1 \rangle & zLq_{2m} - mq_{2mz^*} = 0
\end{array}$$

(where $\langle 2\mu-1 \rangle$ must be ignored if $m = 1$). Setting

$$q_{2\mu}(z, z^*) = z^\mu p_{2\mu}(z, z^*) , \tag{3.4}$$

we see that the $p_{2\mu}(z, z^*)$ satisfy the simpler system

$$\begin{array}{lll}
\text{(a)} & p_{0z^*} = 0 & \\
\text{(b)} & p_{2z^*} + 2cp_0 = 0 & \\
\text{(c)} & (2\mu-1)\, p_{2\mu z^*} + 2Lp_{2\mu-2} = 0 & (\mu = 2, \cdots, m) \\
\text{(d)} & Lp_{2m} = 0 &
\end{array} \tag{3.5}$$

(where (3.5c) must be ignored if $m = 1$). We set $q(z, z^*) = zp(z, z^*)$. Then (2.5) and (3.4) imply

$$p_{2\mu}(z, z^*) = \lambda_{2\mu}\, p(z, z^*)^\mu \qquad (\mu = 0, \cdots, m) . \tag{3.6}$$

We have $\lambda_0 = 1$, thus $q_0 = 1$, and (3.5b) now gives

$$c(z, z^*) = -\frac{\lambda_2}{2}\, p_{z^*} . \tag{3.7}$$

Let $m = 1$. Then the only other equation to be considered is (3.5d) with $m = 1$, and because of (3.6) and (3.7) it takes the form

$$p_{zz^*} + bp_{z^*} - \frac{\lambda_2}{2}\, p_{z^*} p = 0 .$$

From this and (3.7) we obtain (3.1) where $\lambda = \lambda_2/2$.

Let $m > 1$. Then (3.5c) with $\mu = 2$ and (3.7) give

$$b(z, z^*) = \lambda p - \frac{p_{zz^*}}{p_{z^*}} \quad \text{where} \quad \lambda = \frac{\lambda_2}{2} - \frac{3\lambda_4}{\lambda_2} .$$

Substituting this and (3.7) in (3.5c) with any μ, $3 \leqq \mu \leqq m$, and simplifying the resulting equation, we arrive at

$$p_z = -kp^2 \tag{3.8}$$

where

$$k = \frac{\lambda}{\mu-2} + \frac{1}{(\mu-1)(\mu-2)} \left\{ \frac{(2\mu-1)\mu\lambda_{2\mu}}{2\lambda_{2\mu-2}} - \frac{\lambda_2}{2} \right\} . \tag{3.9}$$

By integrating we have

$$p(z, z^*) = \frac{1}{kz+\sigma(z^*)} \qquad (\sigma(z^*) \text{ analytic}) \tag{3.10}$$

and obtain (3.3b). Furthermore, we now see that

$$b(z, z^*) = (\lambda + 2k)p(z, z^*)$$

and, setting $\lambda + 2k = \kappa$, we obtain (3.2a). Substituting (3.2a), (3.6)–(3.8) in (3.5d), we see that a factor $\lambda_{2m}p^m p_{z^*}$ drops out and we are left with an equation for λ_2. The solution is

$$\lambda_2 = -2m((m + 1)k - \kappa) . \tag{3.11}$$

Form (3.9) and (3.11) it follows that

$$\lambda_{2\mu} = -\frac{2(m-\mu+1)}{\mu(2\mu-1)}((m + \mu)k - \kappa)\lambda_{2\mu-2} \quad (\mu = 1, \cdots, m) .$$

The solution is (3.3a). Finally, from $q(z, z^*) = zp(z, z^*)$, (3.10), and (3.11) we obtain (3.2b). This proves that $L \in P_0$ implies (3.1)–(3.3). Conversely, starting from (3.1)–(3.3) we obtain $L \in P_0$, and the proof is complete.

4. Some applications. We first note that if L is such that in (3.2b), $\kappa/k \neq m + n$ (n any natural number), then (3.3a) may be written

$$\lambda_{2\mu} = \frac{(-4k)^\mu m!\, \Gamma(m+\mu+1-\kappa/k)}{(m-\mu)!\,(2\mu)!\,\Gamma(m+1-\kappa/k)} . \tag{4.1}$$

Taking $k = 0$, we see from Theorem 1 that L_1 in the special Delassus equation

$$L_1 u := u_{zz^*} + \frac{\kappa}{\sigma(z^*)} u_{z^*} + \frac{m\kappa\sigma'(z^*)}{\sigma(z^*)^2} u = 0$$

is of class P_0 and a corresponding Bergman operator has the simple generating function

$$g_1(z, z^*, t) = m! \sum_{\mu=0}^{m} \frac{(4\kappa)^\mu}{(m-\mu)!\,(2\mu)!} \left(\frac{zt^2}{\sigma(z^*)}\right)^\mu.$$

If in case (b) of Theorem 1 we require q to be of the form $q(z, z^*) = \tilde{q}(r)$, where $r = zz^*$, then $\sigma(z^*) = 1/z^*$ and the only equation with $L \in P_0$ satisfying that condition is

$$u_{zz^*} + \kappa z^* \omega u_{z^*} + \eta \omega^2 u = 0\,, \qquad \omega = (kzz^* + 1)^{-1}$$

where $\eta = m(m+1)k - \kappa$. This generalizes a theorem for the equation (2.6), recently obtained by W. Watzlawek [12] by entirely different methods.

Taking $\kappa = 0$, we obtain from Theorem 1 the following theorem, which generalizes the main result in [10].

THEOREM 2. *The operator* L_2 *defined by*

$$L_2 u := u_{zz^*} + c(z, z^*)u = 0 \tag{4.2}$$

is of class P_0 *if and only if* (a) *or* (b) *holds*:

(a) $c(z, z^*)$ *can be represented in the form* $c(z, z^*) = \lambda p_{z^*}$ *where* λ *is any constant and* $p(z, z^*)$ *satisfies the differential equation*

$$p_z - \frac{\lambda}{2} p^2 = \phi(z) \qquad (\phi(z) \text{ arbitrary})\,.$$

(b) *There is an integer* $m > 1$ *and a function* $\sigma(z^*)$ *such that*

$$c(z, z^*) = -m(m+1)k \frac{\sigma'(z^*)}{(kz + \sigma(z^*))^2}\,. \tag{4.3}$$

In case (a), $m = 1$ *in* (2.4), $\lambda_2 = 2\lambda$ *and* $q(z, z^*) = zp(z, z^*)$. *In Case* (b),

$$\lambda_{2\mu} = (-4k)^\mu \binom{m+\mu}{m-\mu}$$

and $q(z, z^*)$ *is given by* (3.3b).

If $L \in P$, then (2.2) (with g given by (2.4)) may be converted to a form free of integrals. In fact, $u(z, z^*) = (Bf)\,(z, z^*)$ can then be written

$$u(z, z^*) = \sum_{\mu=0}^{m} \frac{(2\mu)!}{\mu!\,(4z)^\mu} q_{2\mu}(z, z^*) \tilde{f}^{(m-\mu)}(z) \tag{4.4}$$

where

$$\tilde{f}(z) = \sum_{\nu=0}^{\infty} \frac{\nu!\, B(\frac{1}{2}, \nu+\frac{1}{2})}{2^\nu (m+\nu)!} a_\nu z^{m+\nu}$$

where a_ν are the coefficients of the development

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu .$$

This was proved in [6]. For (2.6) this representation is identical with a representation derived by K. W. Bauer and E. Peschl (cf. [1−3]) by means of the theory of automorphic functions and used for developing a function theory of solutions of (2.6). We mention that (2.6) was also investigated by M. Eichler [5] and I. N. Vekua [11], and a special case of (2.6) plays a role in the study of minimal surfaces by H. A. Schwarz.

Furthermore, (4.4) and results by W. Watzlawek [13] imply that the notion of fundamental systems of solutions of *ordinary* differential equations may be generalized to *partial* differential equations (2.1) with $L \in P$.

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Received November 26, 1969.