PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 2, 1971

GENERALIZED FINAL RANK FOR ARBITRARY LIMIT ORDINALS

DOYLE O. CUTLER AND PAUL F. DUBOIS

Let G be a p-primary Abelian group. The final rank of G can be obtained in two equivalent ways: either as $\inf_{n\in\omega}\{r(p^nG)\}$ where $r(p^nG)$ is the rank of p^nG; or as $\sup\{r(G/B) \mid B$ is a basic subgroup of $G\}$. In fact it is known that there exists a basic subgroup of G such that $r(G/B)$ is equal to the final rank of G. In this paper are displayed two appropriate generalizations of the above definitions of final rank, $r_\alpha(G)$ and $s_\alpha(G)$, where α is a limit ordinal. It is shown that the two cardinals $r_\alpha(G)$ and $s_\alpha(G)$ are indeed the same for any limit ordinal α. In this context one can think of the usual final rank as "ω-final rank".

The final rank of a p-primary Abelian group G is $\inf_{n<\omega}\{r(p^nG)\}$ where $r(p^nG)$ means the rank of p^nG. The same cardinal number is obtained by taking $\sup_{B\in\Gamma} r(G/B)$ where Γ is the set of all basic subgroups of G. In [1] we defined for limit ordinals α, $s_\alpha(G) = \inf_{\beta<\alpha} r(p^\beta G)$ and $r_\alpha(G) = \sup_{H\in\Gamma} r(G/H)$ where Γ is the set of all p^α-pure subgroups H of G such that G/H is divisible; it was shown that for accessible ordinals α that $r_\alpha(G) = s_\alpha(G)$. The proof given there strongly depended on the accessibility of α. In this paper it is proved that $r_\alpha(G) = s_\alpha(G)$ for any limit ordinal α, at the cost of a considerably more difficult argument.

Throughout we consider a reduced p-primary Abelian group G. We consider cardinal and ordinal numbers in the sense of von Neumann; that is, an ordinal number is a set, namely, the set of all smaller ordinals. Cardinal numbers are ordinal numbers that are not equivalent to any smaller ordinal. The cardinal number of a set Γ is denoted by $|\Gamma|$. The symbol ω denotes the first infinite ordinal. In general the notation and terminology is that of [2] or [3].

1. The lemmas.

Let τ be a limit ordinal. We define the final τ-rank of G in two ways, which we will then show are equivalent. Ordinary final rank as defined in [2] corresponds to final ω-rank.

DEFINITION.
(1) $s_\tau(G) = \inf_{\beta<\tau} r(p^\beta G[p])$.
(2) $r_\tau(G) = \sup\{r(G/H)\colon H \subseteqq G,\ G/H$ is divisible, and $0 \to H \to G \to G/H \to 0$ represents an element of $p^\tau \operatorname{Ext}(G/H, H)\}$.

In [1] it is shown that $r_\tau(G) \leqq s_\tau(G)$. To show the converse we will construct a p^τ-pure subgroup H of G with G/H divisible and $r(G/H) = s_\tau(G)$.

We prove the following lemma to simplify the problem and to illustrate some techniques of construction which we will sometimes use later in the paper without explicit proofs.

LEMMA 0. (a) *$r_\tau(G) = r_\tau(G_\tau) + r(p^\tau G)$ for any $p^\tau G$-high subgroup G_τ.*

(b) *$s_\tau(G) = s_\tau(G_\tau) + r(p^\tau G)$ for any $p^\tau G$-high subgroup G_τ.*

(c) *$r_\tau(G) \geqq s_\tau(G)$ holds for all G if it holds for all G satisfying $p^\tau G = 0$.*

Proof. (a) and (b) together show (c). A $p^\tau G$-high subgroup G_τ satisfies $G[p] = G_\tau[p] \oplus (p^\tau G)[p]$, ([4]) and hence is $p^{\tau+1}$-pure (Th. 2.9 of [5]). It is easy to see that for $\alpha < \tau$,

$$(p^\alpha G)[p] = (p^\alpha G_\tau)[p] \oplus (p^\tau G)[p]$$

and (b) follows.

To prove (a), suppose H is a p^τ-pure subgroup of G_τ with G_τ/H divisible. Then H is p^τ-pure in G and $G/H \cong (G/G_\tau)/(G_\tau/H)$ is divisible since G/G_τ is divisible. For H a pure subgroup of G,

$$r(G/H) = r((G/H)[p]) = r(G[p]/H[p]) .$$

Hence in this case

$$r(G/H) = r(G_\tau[p] \oplus (p^\tau G)[p]/H[p]) = r(G_\tau/H) + r(p^\tau G[p]) .$$

Hence $r_\tau(G) \geqq r_\tau(G_\tau) + r(p^\tau G)$.

Now let H be a p^τ-pure subgroup of G with G/H divisible. Let $H[p] = S \oplus (p^\tau G \cap H)[p]$. Let K be a $p^\tau G$-high subgroup containing S, and let $\pi: G \to G/p^\tau G$ be the natural map. Then $(\pi(K))[p] = \pi(K[p]) = \pi(G_\tau[p])$. Choose $S' \subseteqq G_\tau[p]$ such that $\pi(S') = \pi(S)$. We will then have that $r(G_\tau[p]/S') = r(\pi(G_\tau)/\pi(S)) = r(K[p]/S)$. Note that $\{S', (p^\tau G)[p]\} = \{S, (p^\tau G)[p]\}$ and so the p^τ-purity of H and the divisibility of G/H yield, for every $\alpha < \tau$,

$$\begin{aligned} \{p^\alpha G_\tau[p], S'\} &= \{(p^\alpha G \cap G_\tau)[p], S'\} \\ &= \{\{p^\alpha G[p], S'\} \cap G_\tau[p]\} \\ &= \{\{p^\alpha G[p], S\} \cap G_\tau[p]\} \\ &= \{G[p] \cap G_\tau[p]\} = G_\tau[p] . \end{aligned}$$

We let L be such that $G_\tau[p] = L \oplus S'$ and let M be L-high containing S'. Then $M[p] = S'$, M is neat in $G_\tau[p]$ and by Th. 2.9 of [5], M is p^τ-pure in G_τ. Then

$$\begin{aligned} r(G/H) &= r((K[p] \oplus (p^\tau G)[p])/(S \oplus (p^\tau G \cap H)[p])) \\ &= r(K[p]/S) + r(p^\tau G[p]/(p^\tau G \cap H)[p]) \\ &\leqq r(G_\tau[p]/S') + r(p^\tau G[p]) \\ &= r(G_\tau/M) + r(p^\tau G) \\ &\leqq r_\tau(G_\tau) + r(p^\tau G) \end{aligned}$$

and (a) is proved.

Hence we consider only groups G with $p^\tau G = 0$. We will need the following four technical lemmas.

LEMMA 1. *Let G be a p-primary Abelian group of length τ, a limit ordinal. Let $S \subseteqq G[p]$ be such that $S \cap (p^\gamma G)[p] \neq 0$ for all $\gamma < \tau$. Then there exists $S' \subseteqq S$ such that $r(S/S') \geqq 1$ and $\{S', (p^\gamma G)[p]\} = \{S, (p^\gamma G)[p]\}$ for all $\gamma < \tau$.*

Proof. Let $a \in S (a \neq 0)$. We define a family $\{R_j\}_{j<\tau}$ inductively as follows:

Write $S = L_1 \oplus pG \cap S$. If $a \notin L_1$, let $R_1 = L_1$. If $a \in L_1$, let $\{y_\alpha\}_{\alpha \in \Gamma}$ be a basis for L_1. Then $a = \sum_{\alpha \in \Gamma} a_\alpha y_\alpha$ where $0 \leqq a_\alpha < p$ and $a_\alpha = 0$ for all but finitely many α. Choose $\alpha_0 \in \Gamma$ so that $a_{\alpha_0} \neq 0$. Let $R_1 = \sum_{\alpha \in \Gamma - \{\alpha_0\}} \langle y_\alpha \rangle \oplus \langle y_{\alpha_0} - b \rangle$ where $b \in pG \cap S\, (b \neq 0)$. Then $S = R_1 \oplus pG \cap S$ and $a \notin R_1$. Inductively, suppose $\{R_i\}_{i<\gamma}$ has been defined such that $\sum_{i \leqq k < \gamma} R_i \oplus p^k G \cap S = S$ for each $k < \gamma$ and $a \notin \sum_{i<\gamma} R_i$. If $\gamma - 1$ exists we have $\sum_{i<\gamma} R_i \oplus p^{\gamma-1} G \cap S = S$. We choose L_γ so that $L_\gamma \oplus p^\gamma G \cap S = p^{\gamma-1} G \cap S$. If $a \notin \sum_{i<\gamma} R_i \oplus L_\gamma$ we let $R_\gamma = L_\gamma$. Otherwise, let $\{y_\lambda\}_{\lambda \in \Gamma}$ be a basis of L_γ. Then $a = x + \sum_{\lambda \in \Gamma} a_\lambda y_\lambda (0 \leqq a_\lambda < p$, $x \in \sum_{i<\gamma} R_i)$. By the induction hypothesis not all a_λ are zero. Let $\lambda_0 \in \Gamma$ such that $a_{\lambda_0} \neq 0$, and let $R_\gamma = \sum_{\lambda \in \Gamma - \{\lambda_0\}} \langle y_\lambda \rangle \oplus \langle y_{\lambda_0} - b \rangle (b \in p^\gamma G \cap S, b \neq 0)$. It follows that $a \notin \sum_{i<\gamma+1} R_i$ and $\sum_{i \leqq k \leqq \gamma} R_i \oplus p^k G \cap S = S$.

If γ is a limit ordinal, note that $\sum_{i<\gamma} R_i \cap p^\gamma G = 0$. Choose L_γ such that $\sum_{i<\gamma} R_i \oplus L_\gamma \oplus p^\gamma G \cap S = S$. Either $a \notin \sum_{i<\gamma} R_i \oplus L_\gamma$ in which case we let $R_\gamma = L_\gamma$, or $a \in \sum_{i<\gamma} R_i \oplus L_\gamma$ and we modify L_γ as above to get R_γ.

By transfinite induction, we obtain a family $\{R_i\}_{i<\tau}$ such that $\sum_{i \leqq k} R_i \oplus p^k G \cap S = S$ for all $k < \tau$ and $a \notin \sum_{i<\tau} R_i$. Let $S' = \sum_{i<\tau} R_i$ and the conditions of the lemma are satisfied.

The general idea of the above proof for S summable was communicated to the authors by Paul Hill.

LEMMA 2. *Let G be a p-primary Abelian group of length τ a limit ordinal. Let $\{R_j\}_{j<\eta}$, η a limit ordinal, be a collection of subsocles of G satisfying the following conditions:*

(1) $\sum_{j<\eta} R_j$ *is direct,*

(2) $r(R_j) = \aleph$ *is fixed, and*

(3) *For each* $\lambda < \tau$, *there exists* $j < \eta$ *such that* $0 \neq R_j \subseteqq p^\lambda G[p]$.

Then there exists $S \subseteqq \sum_{j<\eta} R_j$ *such that*

(a) *For each* $\lambda < \tau$, $\{S, p^\lambda G[p]\} = \{\sum_{j<\eta} R_j, p^\lambda G[p]\}$, *and*

(b) $r((\sum_{j<\eta} R_j)/S) \geqq \aleph$.

Proof. For each $j < \eta$, let $\{x_{j,\alpha}\}_{\alpha \in \Gamma}$ ($|\Gamma| = \aleph$) be a basis of R_j. Let $S_\alpha = \sum_{j<\eta} \langle x_{j\ \alpha} \rangle$. Note that $\sum_{\alpha \in \Gamma} S_\alpha$ is direct and $\sum_{\alpha \in \Gamma} S_\alpha = \sum_{\alpha<\eta} R_j$. Let $\lambda < \tau$. Then $S_\alpha \cap p^\lambda G[p] \neq 0$ by hypothesis (3). Hence by Lemma 1, there exists, for each $\alpha \in \Gamma$, $T_\alpha \subseteqq S_\alpha$ such that

$$\{S_\alpha, p^\lambda G[p]\} = \{T_\alpha, p^\lambda G[p]\}$$

for all $\lambda < \tau$ and $r(S_\alpha/T_\alpha) \geqq 1$. Let $S = \sum_{\alpha \in \Gamma} T_\alpha$. Then $\{S, p^\lambda G[p]\} = \{\sum S_\alpha, p^\lambda G[p]\} = \{\sum_{j<\eta} R_j, p^\lambda G[p]\}$ for all $\lambda < \tau$ and $r((\sum_{j<\eta} R_j)/S) = \sum_{\alpha \in \Gamma} r(S_\alpha/T_\alpha) \geqq \aleph$.

LEMMA 3. *Let G be a p-primary Abelian group of length τ a limit ordinal. Let σ be an infinite initial ordinal such that $\sigma \leqq \tau$. Let $\{R_j\}_{j<\sigma}$ be a collection of subsocles of G satisfying:*

(1) $\sum_{j<\sigma} R_j$ *is direct.*

(2) *For each* $\lambda < \tau$ *there exists* $j < \sigma$ *such that for all* $i \geqq j$, $R_i \subseteqq p^\lambda G[p]$.

(3) $|\{j \mid R_j \neq 0\}| = \sigma$.

Then there exists $S \subseteqq \sum_{j<\sigma} R_j$ *such that*

(a) $\{S, p^\lambda G[p]\} = \{\sum_{j<\sigma} R_j, p^\lambda G[p]\}$ *for all* $\lambda < \tau$, *and*

(b) $|(\sum_{j<\sigma} R_j)/S| \geqq \sigma$.

Proof. Let $x_j \in R_j$ ($x_j \neq 0$) for each $j \in \{j \mid R_j \neq 0\} = \Gamma$. Then we may write Γ as the disjoint union $\Gamma = \bigcup_{i<\sigma} \Gamma_i$ such that $|\Gamma_i| = \sigma$ for each $i < \sigma$. Since σ is an initial ordinal, $\Gamma_i \not\subseteqq \beta$ for any $\beta < \sigma$. Hence $\sum_{j \in \Gamma_i} \langle x_j \rangle$ satisfies the conditions of Lemma 1. Hence there exists a subgroup $S_i \subseteqq \sum_{j \in \Gamma_i} \langle x_j \rangle$ such that $\{S_i, p^\lambda G[p]\} = \{\sum_{j \in \Gamma_i} \langle x_j \rangle, p^\lambda G[p]\}$ for all $\lambda < \tau$, and $r((\sum_{j \in \Gamma_i} \langle x_j \rangle)/S_i) \geqq 1$. Let Q be such that $\sum_{i<\sigma} \sum_{\Gamma_i} \langle x_j \rangle \oplus Q = \sum_{j<\sigma} R_j$, and define $S = \sum_{i<\sigma} S_i \oplus Q$. Then S satisfies the desired conditions.

LEMMA 4. *Let G be a p-primary Abelian group of length τ a limit ordinal. Let $\{R_j\}_{j \in \sigma}$ (σ a limit ordinal, $\sigma \leqq \tau$) be a collection of subsocles of G satisfying:*

(1) $\sum_{j<\sigma} R_j$ *is direct;*

(2) *For each* $\lambda < \tau$, *there exists* $j < \sigma$ *such that for all* $j < i < \sigma$, $R_i \subseteqq p^\lambda G[p]$; *and*

(3) *For all* $i < j < \sigma$, $r(R_j) \geqq r(R_i) \geqq |\sigma|$.

Then there exists a subgroup $S \subseteqq \sum_{j<\sigma} R_j$ *satisfying*:

(a) $\{S, p^\lambda G[p]\} = \{\sum_{j<\sigma} R_j, p^\lambda G[p]\}$ *for all* $\lambda < \tau$.

(b) $|(\sum_{j<\sigma} R_j)/S| = |\sum_{j<\sigma} R_j|$.

Proof. Define Q_β^α for all $(\alpha, \beta) \in \sigma x \sigma$ as follows: Define $Q_0^0 = R_0$ and $Q_0^\alpha = 0$ for all $\alpha < \sigma, \alpha > 0$. We induct on the lower index. Suppose Q_β^α has been defined for all $\beta < \gamma < \sigma$ satisfying:

(1) For all $\alpha \leqq \beta < \gamma$, $r(Q_\beta^\alpha) = r(Q_\alpha^\alpha)$;

(2) $Q_\beta^\alpha = 0$ if $\beta < \alpha < \sigma$;

(3) For $\beta < \sigma$, $r(Q_\beta^\beta) \neq 0$ if and only if $r(R_\beta) > r(R_\alpha)$ for all $\alpha < \beta$; and

(4) $R_\beta = \sum_{\alpha \in \sigma} Q_\beta^\alpha$.

Suppose $\gamma - 1$ exists. If $r(R_\gamma) = r(R_{\gamma-1})$, let $\varphi: R_{\gamma-1} \to R_\gamma$ be an isomorphism and define $Q_\gamma^\alpha = \varphi(Q_{\gamma-1}^\alpha)$ for all $\alpha \in \sigma$. If $r(R_{\gamma-1}) < r(R_\gamma)$ we first write $R_\gamma = S \oplus R$ where $R \cong R_{\gamma-1}$ (under an isomorphism φ). Let $Q_\gamma^\alpha = \varphi(Q_{\gamma-1}^\alpha)$ for $\alpha < \gamma$, $Q_\gamma^\gamma = S$, and $Q_\gamma^\alpha = 0$ for $\alpha > \gamma$.

Suppose γ is a limit ordinal. If $r(R_\gamma) = r(R_\beta)$ for some $\beta < \gamma$, then $R_\gamma \cong R_\alpha$ for all $\beta \leqq \alpha < \gamma$. Let φ be an isomorphism from R_β onto R_γ and let $Q_\gamma^\alpha = \varphi(Q_\beta^\alpha)$ for all $\alpha < \sigma$.

If for some $\beta < \gamma$, $r(R_\gamma) > r(R_\beta) \geqq r(R_\alpha)$ for all $\alpha < \gamma$ we write $R_\gamma = R \oplus S$ where $R \cong R_\beta$ and proceed as in the case of the non-limit ordinal.

Finally suppose $r(R_\gamma) > r(R_\beta)$ for all $\beta < \gamma$ and that there does not exist $\delta < \gamma$ such that $r(R_\delta) \geqq r(R_\beta)$ for all $\beta < \gamma$. Let $\pi = \sum_{\beta<\gamma} r(R_\beta)$. Since $r(R_\gamma) > r(R_\beta) \geqq |\sigma|$, $\beta < \gamma$, we have $\pi \leqq r(R_\gamma)$ and both of these cardinals are infinite. We may write R_γ as $S \oplus R$ where $r(S) = \pi$. Divide a basis of S into two sets, $\{y_\lambda\}_{\lambda \in \pi}$ and $\{z_\lambda\}_{\lambda \in \pi}$. Let $Q_\gamma^\gamma = R \oplus \langle\{z_\lambda\}_{\lambda \in \pi}\rangle$, and noting that $\pi = \sum_{\alpha<\gamma} r(Q_\alpha^\alpha)$, write π as the disjoint union $\pi = \bigcup_{\alpha<\gamma} \pi_\alpha$ such that $|\pi_\alpha| = r(Q_\alpha^\alpha)$.

Let $Q_\gamma^\alpha = \langle\{y_\lambda | \lambda \in \pi_\alpha\}\rangle$, and we complete the induction. Note that by the construction that if R_γ is the first to have rank ρ, then $r(Q_\gamma^\gamma) = \rho$.

Let $\Lambda = \{r(R_j) | j < \sigma\}$. For each $\rho \in \Lambda$ let j_ρ be the least element of σ such that $r(R_{j_\rho}) = \rho$. Then $Q_{j_\rho}^{j_\rho} \neq 0$ by construction. For each $\rho \in \Lambda$ consider the collection $\{Q_\alpha^{j_\rho}\}_{\alpha \in \Gamma_\rho}$ where $\Gamma_\rho = \{j | j_\rho \leqq j < \sigma\}$. Note that this collection satisfies the hypothesis of Lemma 2. Thus there exists a subgroup $S_\rho \subseteqq \sum_{\alpha \in \Gamma_\rho} Q_\alpha^{j_\rho}$ such that $|(\sum_{\alpha \in \Gamma_\rho} Q_\alpha^{j_\rho})/S_\rho| \geqq \rho$, and for each $\lambda < \tau$, $\{S_\rho, p^\lambda G[p]\} = \{\sum_{\alpha \in \Gamma_\rho} Q_\alpha^{j_\rho}, p^\lambda G[p]\}$. Note that

$$\sum_{\rho \in \Lambda} \sum_{\alpha \in \Gamma_\rho} Q_\alpha^{j_\rho} = \sum_{j<\sigma} R_j$$

since each nonzero Q_α^α is a $Q_{j_\rho}^{j_\rho}$ for some $\rho \in \Lambda$. Let $S = \sum_{\rho \in \Lambda} S_\rho$. Then

$$|(\sum_{j<\sigma} R_j)/S| = \sum_{\rho \in \Lambda} |(\sum_{\alpha \in \Gamma} Q_\alpha^{j_\rho})/S_\rho| \geqq \sum_{\rho \in \Lambda} \rho = \sum_{j<\sigma} r(R_j) = |\sum_{j<\sigma} R_j| .$$

(Note that we use the last part of condition (3) for the second equality). Also for each $\lambda < \tau$, $\{S, (p^\lambda G)[p]\} = \{\sum_{j<\sigma} R_j, (p^\lambda G)[p]\}$.

2. The Theorem.

THEOREM. *Let G be a reduced p-primary Abelian group. Then $r_\tau(G) = s_\tau(G)$.*

Proof. As indicated in the introduction we may assume that the length of G is τ. Let $\lambda < \tau$ be such that $|p^\lambda G| = s_\tau(G)$. Then there exists an ordinal β such that $\tau = \lambda + \beta$, and the length of $p^\lambda G$ is β. Now $r_\tau(G) \geqq r_\beta(p^\lambda G)$ (Use [5, Th. 2.9]) and $s_\tau(G) = s_\beta(p^\lambda G)$. Hence we need only show $r_\beta(p^\lambda G) = s_\beta(p^\lambda G)$. Thus we may consider only those groups G with length τ and $r(G) = s_\tau(G)$.

Let Γ be the set of all ordinals β such that there exists a one-to-one order preserving map f_β from β into τ such that $\bigcup_{\alpha<\beta} f_\beta(\alpha) = \tau$. Let σ be the least element of Γ and $f = f_\sigma$. It follows easily from Theorem 13.4.4 of [6] that σ is an initial ordinal.

Define a set of subgroups $\{P_\alpha\}_{\alpha<\sigma}$ of $G[p]$ as follows: Let P_0 be such that $G[p] = P_0 \oplus (p^{f(0)}G)[p]$. Assuming that P_α has been defined for all $\alpha < \beta < \sigma$, define P_β such that $G[p] = \sum_{\alpha<\beta} P_\alpha \oplus P_\beta \oplus (p^{f(\beta)}G)[p]$. This procedure is inspired by [3].

Choose λ_0 such that $|\sum_{\lambda_0 \leqq i<\sigma} P_i| = \inf_{\lambda<\sigma} |\sum_{\lambda \leqq i<\sigma} P_i|$. By the choice of σ, we have that $[\lambda_0, \sigma) = \sigma$. Hence we assume henceforth that $|\sum_{i<\sigma} P_i| = \inf_{\lambda<\sigma} |\sum_{\lambda \leqq i<\sigma} P_i|$. We may, in fact, assume each P_α, $\alpha < \sigma$ is nonzero, again because σ is regular (see [6]).

Let Q be such that $\sum_{i<\sigma} P_i \oplus Q = \sum_{i<\sigma}$ (the original P_i). Then note that for each $\lambda < \tau$, $\{\sum_{i<\sigma} P_i \oplus Q, (p^\lambda G)[p]\} = G[p]$. Let $M = \sum_{i<\sigma} P_i$.

Case I. Suppose $|M| < s_\tau(G)$. Then [5, Th. 2.9] $M \oplus Q$ supports a p^τ-pure subgroup K of G with G/K divisible and $|G/K| \geqq s_\tau(G)$. So we assume $|M| \geqq s_\tau(G)$.

Case II (A). Suppose $|M| = \sigma$. Then by Lemma 3 there exists a subsocle S of M such that $|M/S| \geqq \sigma \geqq s_\tau(G)$, and $\{S, (p^\lambda G)[p]\} = \{M, (p^\lambda G)[p]\}$ for all $\lambda < \tau$. Thus $S \oplus Q$ supports a p^τ-pure subgroup K of G with G/K divisible and $|G/K| \geqq s_\tau(G)$.

(B). Suppose $|M| > \sigma$. Then construct a family of subsocles $\{R_i\}_{i \leqq \sigma}$ inductively as follows: Let $R_0 = \sum_{i<\lambda_1} P_i$ where λ_1 is the least ordinal such that $|\sum_{i<\lambda_1} P_i| \geqq \sigma$. Assuming R_α has been defined for all $\alpha < \beta$, define $R_\beta = \sum_{\lambda_\beta \leqq i < \lambda_{\beta+1}} P_i$ where λ_β is the least element of σ such that $P_{\lambda_\beta} \cap R_\alpha = 0$ for all $\alpha < \beta$, and $\lambda_{\beta+1}$ is the least element of $\sigma + 1$ such that $|\sum_{\lambda_\beta \leqq i < \lambda_{\beta+1}} P_i| \geqq |R_\alpha|$ for all $\alpha < \beta$.

If λ_β does not exist set $R_\beta = 0$. It will be seen below that if λ_β exists, $\lambda_{\beta+1}$ exists.

Note that $R_\sigma = 0$ since σ is an initial ordinal such that if $\beta < \sigma$, β is not cofinal with σ (i.e., σ is regular). Note that $\sum_{i<\sigma} P_i = \sum_{i<\sigma} R_i$. If not, let η be the least element of $\sigma + 1$ such that $R_\eta = 0$. If $\lambda_\eta = \sigma$ then $\sum_{i<\sigma} P_i = \sum_{i<\eta} R_i$. If $\lambda_\eta < \sigma$, then $|\sum_{\lambda_\eta \leqq i<\sigma} P_i| = |\sum_{i<\sigma} P_i| \geqq |R_\alpha|$ for all $\alpha < \eta$. Hence $R_\eta \neq 0$, a contradiction. Hence $\sum_{i<\sigma} R_i = \sum_{i<\sigma} P_i$.

Let η be the least element of $\sigma + 1$ such that $R_\eta = 0$. Suppose η is not a limit ordinal. Let $\eta = \gamma + 1$. Then $R_\gamma = \sum_{\lambda_\gamma \leqq i<\sigma} P_i$ and $|R_\gamma| \geqq |R_\alpha|$ for all $\alpha < \gamma$. Construct a family $\{R_\gamma^i\}_{i \leqq \sigma}$ as above replacing 0 by λ_γ. Let η_1 be the least ordinal such that $R_\gamma^{\eta_1} = 0$. Suppose η_1 is not a limit ordinal. Let $\eta_1 = \gamma_1 + 1$. Then $|R_\gamma^{\gamma_1}| > |R_\gamma^\alpha|$ for all $\alpha < \gamma_1$. In fact, $|R_\gamma^{\gamma_1}| > \sup_{\alpha<\gamma_1} |R_\gamma^\alpha|$ since $|R_\gamma^{\gamma_1}| = |R_\gamma|$ and assuming otherwise we would have $R_\gamma = (\sum_{\alpha<\gamma_1} R_\gamma^\alpha) \oplus R_\gamma^{\gamma_1}$ with $|\sum_{\alpha<\gamma_1} R_\gamma^\alpha| = |R_\gamma|$ contradicting the construction of R_γ. Hence there exists i, $\lambda_{\gamma_1} \leqq i < \sigma$ such that $|P_i| \geqq \sup_{\alpha<\gamma_1} |R_\gamma^\alpha|$. This contradicts the construction of $R_\gamma^{\gamma_1}$. Hence η_1 is a limit ordinal.

Hence in either case (η or η_1 a limit ordinal) there exists a family of subsocles $\{R_i\}_{i<\eta}$ of $G[p]$ such that $|\sum_{i<\eta} R_i| = |\sum_{i<\sigma} P_i|$ and satisfying the conditions of Lemma 4. Thus there exists a subsocle S of $\sum_{i<\eta} R_i$ satisfying conditions (a) and (b) of Lemma 4. Now $\sum_{i<\eta} R_i$ may not be all of $\sum_{i<\sigma}$ (the original P_i) and so we let Q' be such that $\sum_{i<\sigma}$ (the original P_i) $= \sum_{i<\eta} R_i \oplus Q'$. We then have

$$G[p] = \{\sum_{i<\eta} R_i \oplus Q', (p^\lambda G)[p]\} = \{S \oplus Q', p^\lambda G[p]\}$$

for each $\lambda < \tau$. Further,

$$|G[p]/S \oplus Q'| \geqq |\sum_{i<\eta} R_i/S| \geqq |\sum_{i<\eta} R_i| = |M| = s_\tau(G) .$$

Hence $S \oplus Q'$ supports a p^τ-pure subgroup H of G such that $|G/H| \geqq s_\tau(G)$. Thus $r_\tau(G) \geqq s_\tau(G)$. An application of this theorem appears in [1].

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Received January 20, 1970.

UNIVERSITY OF CALIFORNIA, DAVIS