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A FACTORIZATION THEOREM FOR ANALYTIC FUNCTIONS OPERATING IN A BANACH ALGEBRA

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Cohen's factorization-theorem asserts that if the Banach algebra $\mathfrak{A}$ has a left approximate identity, then each $y \in \mathfrak{A}$ may be written $y = xz$, $x, z \in \mathfrak{A}$. The vector x may be chosen to be bounded by some fixed constant and z may be chosen arbitrarily close to y. In this setting the theorem below asserts that if F is a holomorphic function defined on a sufficiently large disc about $\zeta = 1$, and satisfying $F(1) = 1$, then each $y \in \mathfrak{A}$ may be written $y = F(x)z$, where $x, z \in \mathfrak{A}$. Again x may be chosen to be bounded by some fixed constant and z may be chosen close to y.

We state and prove our result using the terminology of [2]. The proof is an elaboration of the proof of Theorem 2.2 of [2]. In what follows X is a complex Banach space, $\mathscr{E} = \{E_\alpha\}$ is a uniformly bounded subset of $B(X)$ which we may assume to be directed and which satisfies $\lim_\alpha E_\alpha E = E$ for each $E \in \mathscr{E}$. Convergence is in the norm topology of $B(X)$. Let

$$Y = \{x \in X\colon \lim_\alpha E_\alpha x = x\} ,$$

and let $\mathfrak{A}$ be the closed subalgebra of $B(X)$ generated by $\mathscr{E}$.

For further extensions of Cohen's theorem we refer the reader to Chapter 8 of [3].

THEOREM. *Let F be a holomorphic complex-valued function with $F(1) = 1$, defined on a neighbourhood of $\{z \in \boldsymbol{C} \mid |z - 1| \leqq M\}$, $M > 1$, where $\|E - I\| \leqq M$ for all $E \in \mathscr{E}$.*

Then to every $y \in Y$ and $\delta > 0$ there exist $z \in Y$ and $U \in \mathfrak{A}$ such that

$$y = F(U)z \text{ and } \|y - z\| < \delta .$$

If furthermore F has no zeros in the open interval $]0, 1[$, then U may for some $a \in]0, 1[$ be written in the form

$$U = \sum_1^\infty a(1 - a)^{k-1} E_k ,$$

where $E_k \in \mathscr{E}$ for $k = 1, 2, \cdots$.

Proof. It suffices to prove the theorem in the case where F has no zeros in $]0, 1[$, since we otherwise simply use the function

$$G(z) = F(e^{i\theta} z) F(e^{i\theta})^{-1}$$

for θ small, instead of F.

Let $\{\lambda_1, \cdots, \lambda_m\}$ denote the zeros of F in the disc $\{z \in C \mid |z-1| \leqq M\}$.

Let finally $y \in Y$ and $\delta > 0$ be given.

To proceed we need

LEMMA 1. *Let $0 < a < 1$; $E_1, \cdots, E_n \in \mathscr{E}$ and set*

$$U_n = \sum_1^n a(1-a)^{k-1} E_k + (1-a)^n I .$$

Assume that no λ_i belongs to the spectrum $\sigma(U_n)$ of U_n, and that

$$R(\lambda_i, U_n) Y \subseteqq Y \qquad \text{for } i = 1, \cdots, m ;$$

where

$$R(\lambda_i, U_n) = (\lambda_i I - U_n)^{-1} .$$

Then $F(U_n)$ and $W_n \equiv F^{-1}(U_n)$ belong to $B(X)$ and both map Y into Y.

Proof. We assert first that $\sigma(U_n) \subseteqq \{|z-1| < M\}$. Indeed,

$$U_n - I = \sum_{k=1}^n a(1-a)^{k-1} E_k + (1-a)^n I - I = \sum_{k=1}^n a(1-a)^{k-1}(E_k - I) ,$$

so that

$$\| U_n - I \| \leqq M \sum_{k=1}^n a(1-a)^{k-1} = M(1-(1-a)^n) < M .$$

Now

$$Y = \{x \in X \mid \lim_\alpha E_\alpha x = x\} ,$$

and consequently $EY = Y$ for every $E \in \mathscr{E}$, so that $U_n Y \subseteqq Y$. For $|\zeta - 1| = M$ we have

$$\begin{aligned} R(\zeta, U_n) &= (\zeta-1)^{-1}(I - (\zeta-1)^{-1}(U_n - I))^{-1} \\ &= (\zeta-1)^{-1} \sum (\zeta-1)^{-k}(U_n - I)^k , \end{aligned}$$

which converges absolutely, so that

$$R(\zeta, U_n) Y \subseteqq Y .$$

Since the integral

$$F(U_n) = \frac{1}{2\pi i}\int_{|\zeta-1|=M} F(\zeta)R(\zeta,\, U_n)d\zeta \in B(X)$$

is a limit of Riemann sums,

$$F(U_n)Y \subseteqq Y\,.$$

Since F is holomorphic and does not vanish on $\sigma(U_n)$ we have

$$W_n \equiv F^{-1}(U_n) \in B(X)\,.$$

To show $W_n Y \subseteqq Y$, write

$$F(z) = \prod_{i=1}^{m} (\lambda_i - z)^{k_i} H(z)\,,$$

where H does not vanish on $\{|z-1| < M.\}$ The above argument shows $H^{-1}(U_n)Y \subseteqq Y$. Finally,

$$F^{-1}(U_n) = H^{-1}(U_n)\prod_{i=1}^{m} R(\lambda_i,\, U_n)^{k_i}\,,$$

and

$$R(\lambda_i,\, U_n)Y \subseteqq Y$$

by hypothesis.

LEMMA 2. *If in addition U_n may be chosen so that*

$$||(W_n - W_{n-1})y|| < \frac{\delta}{2^n} \qquad \textit{for } n = 1, 2, \cdots,$$

then the theorem follows.

Proof. Set $z_n = W_n y$. Then $\{z_n\}$ is a Cauchy-sequence. With $z = \lim_n z_n$ we have $||z - y|| \leqq \delta$.

Further, if

$$U = \sum_{1}^{\infty} a(1-a)^{k-1} E_k\,,$$

then

$$\begin{aligned} ||F(U)z - y|| &= ||F(U)z - F(U_n)z + F(U_n)(z - z_n) + F(U_n)z_n - y|| \\ &\leqq ||F(U) - F(U_n)||\,||z|| + ||F(U_n)||\,||z - z_n||\,, \end{aligned}$$

from which the lemma follows.

We will need the following technical lemma in the induction step below, where we use the notation

$$T(a) = \{\mu(1-a)^{-n} \,|\, n = 0, 1, \cdots \text{ and } \mu \in \{\lambda_1, \cdots, \lambda_m\} \cup \{z \,|\, |z-1| = M\}\} \quad \text{for } 0 < a < 1\,.$$

LEMMA 3. *There exists $b \in]0, 1[$ such that*

$$|a(\tau - 1)^{-1}| < \frac{1}{2M} \text{ for all } a \in]0, b] \text{ and all } \tau \in T(a) .$$

Let $A_\alpha = aE_\alpha + (1 - a)I$ for some $a \in]0, b]$. Then for $\tau \in T(a)$ we have that $R(\tau, A_\alpha)$ exists in $B(X)$, maps Y into Y and has $\|R(\tau, A_\alpha)\| \leqq C < \infty$, where C only depends on F and M.

Furthermore, for fixed $E \in \mathscr{E}$ and $x \in Y$,

$$\lim_\alpha R(\tau, A_\alpha)E = (\tau - 1)^{-1}E$$

and

$$\lim_\alpha R(\tau, A_\alpha)x = (\tau - 1)^{-1}x ,$$

both uniformly for $\tau \in T(a)$.

Proof. The first assertion is an easy consequence of the fact that F has no zeros in $]0, 1[$, so that

$$|\tau - 1| \geqq c > 0 \text{ for all } \tau \in T(a) \text{ and all } a \in]0, 1[.$$

Since

$$\tau I - A_\alpha = (\tau - 1)\left(I - \frac{a}{\tau - 1}(E_\alpha - I)\right) ,$$

we have that

$$R(\tau, A_\alpha) = (\tau - 1)^{-1} \sum_0^\infty \left(\frac{a}{\tau - 1}\right)^k (E_\alpha - I)^k ,$$

converges by our choice of a, maps Y into Y and finally that

$$\|R(\tau, A_\alpha)\| \leqq \max |\tau - 1|^{-1} \sum_0^\infty 2^{-k} = 2 \max \{|\tau - 1|^{-1} \mid \tau \in T(a)\} .$$

By an easy calculation

$$R(\tau, A_\alpha) - (\tau - 1)^{-1}I = \frac{a}{(\tau - 1)^2} R(\tau, A_\alpha)(E_\alpha - I) ,$$

which yields the lemma.

We will also need the following trivial lemma.

LEMMA 4. *Let $\{T_{i\,\alpha} \mid \alpha \in A, i = 1, \cdots, n\} \subseteqq B(Y)$ satisfy*

$$\|T_{i\,\alpha}\| \leqq C < \infty \qquad \text{for all } \alpha \in A, i = 1, \cdots, n .$$

If $T_{i\,\alpha} \to T_i$ strongly for $i = 1, \cdots, n$, then

$$T_{1\,\alpha}\, T_{2\,\alpha} \cdots T_{n,\alpha} \to T_1 T_2 \cdots T_n$$

strongly.

We will now by induction find a sequence $\{E_k\} \subseteq \mathscr{E}$ such that for any fixed $a \in]0, b]$,

$$R(\lambda_i, U_n) \in B(X) \quad \text{for all} \quad i = 1, \cdots, m \quad \text{and} \quad n = 0, 1, \cdots,$$

($*$) maps Y into Y and such that

$$\|(W_n - W_{n-1})y\| < \frac{\delta}{2^n} \quad \text{for} \quad n = 1, 2, \cdots .$$

The theorem then follows from Lemma 2. For $n = 0$ we may take $U_0 = I$.

Now suppose we have found $U_0, U_1, \cdots, U_n$ satisfying ($*$).

Let $A_{n+1} = aE_{n+1} + (1 - a)I$, where $E_{n+1} \in \mathscr{E}$ is to be chosen.

Since $R(\tau, A_{n+1})$ makes sense for $\tau \in T(a)$ by Lemma 3, we may define

$$U'_n(\lambda) = -\sum_1^n a(1 - a)^{k-1} R(\lambda(1 - a)^{-n}, A_{n+1})E_k + (1 - a)^n I$$

for $\lambda \in T(a)$. We note that $U'_n(\lambda)$ may be chosen arbitrarily close to $(1 - \lambda(1 - a)^{-n})^{-1}(U_n - \lambda I)$ uniformly for $\lambda \in T(a)$ if we just take E_{n+1} large. Therefore, $U'_n(\lambda)^{-1}$ exists in $B(X)$, maps Y into Y and is uniformly bounded in $T(a)$ and $\mathscr{E}$ for E_{n+1} large.

By an easy calculation

$$\lambda I - U_{n+1} = (\lambda(1 - a)^{-n} I - A_{n+1}) U'_n(\lambda)$$

so

$$R(\lambda, U_{n+1}) = U'_n(\lambda)^{-1} R(\lambda(1 - a)^{-n}, A_{n+1})$$

exists in $B(X)$, maps Y into Y and is uniformly bounded in $T(a)$ and $\mathscr{E}$.

Since

$$F^{-1}(U_k) = H^{-1}(U_k) \prod_1^m R(\lambda_i, U_k)^{k_i},$$

it is by an easy application of Lemma 4 left to show that for each $x \in Y$,

$$R(\lambda, U_{n+1})x \to R(\lambda, U_n)x$$

uniformly for $\lambda \in T(a)$ as E_{n+1} increases in $\mathscr{E}$.

Now

$$\begin{aligned}R(\lambda, U_{n+1})x &- R(\lambda, U_n)x \\ &= U'_n(\lambda)^{-1}R(\lambda(1-a)^{-n}, A_{n+1})x - R(\lambda, U_n)x \\ &= U'_n(\lambda)^{-1}[R(\lambda(1-a)^{-n}, A_{n+1})x - (\lambda(1-a)^{-n}-1)^{-1}x] \\ &\quad + [(\lambda(1-a)^{-n}-1)^{-1}U'_n(\lambda)^{-1} - R(\lambda, U_n)]x .\end{aligned}$$

The first term can be made arbitrarily small by Lemma 3. The second term can be made arbitrarily small too, for we have already observed that

$$U'_n(\lambda) \to (1 - \lambda(1-a)^{-n})^{-1}(U_n - \lambda I) \quad \text{unif. in} \quad T(a)$$

so

$$U'_n(\lambda)^{-1} \to (\lambda(1-a)^{-n} - 1)R(\lambda, U_n) \quad \text{unif. in} \quad T(a) .$$

That finishes the proof.

REMARK. If K is a compact subset of Y then we can use the same U for all $y \in K$. That is proved as in [5].

Similarly, if $y_i \to 0$, then there exist U and x_i, $x_i \to 0$ such that $y_i \in F(U)x_i$.

COROLLARY. *Let A be a commutative self-adjoint semi-simple Banach algebra with a bounded approximate identity $\{e_\alpha\}$. Let $\mathscr{M}_A$ be the maximal ideal space for A. If $f \in C_0(\mathscr{M}_A)$, the continuous complex functions on $\mathscr{M}_A$ vanishing at ∞, and $f \geqq 0$, then there exists a $g \in C_0(\mathscr{M}_A)$, $g \geqq 0$ such that $\sqrt{f/g} \in A$. If $f \in A$, then g may be chosen to be in A.*

Proof. $f \to \bar{f}$ is continuous since A is commutative and semi-simple. $\{f_\alpha\} = \{e_\alpha \bar{e}_\alpha\}$ is an approximate identity consisting of non-negative functions. Let $F(z) = z^2$ and write $f \in C_0(\mathscr{M}_A)$, $f \geqq 0$ as $f = h^2 g$, where $h = \sum a(1-a)^{n-1}f_n$ with $\{f_n\} \subseteq \{f_\alpha\}$ is in A. Then $h = \sqrt{f/g}$, and we are done.

This Corollary may be contrasted with a theorem of Katznelson [4] which asserts that if $\sqrt{f} \in A$ for each nonnegative f in A then $A = C(\mathscr{M}_A)$.

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