PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 2, 1971

INVERSION OF THE HANKEL POTENTIAL TRANSFORM

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We consider the Hankel potential transform

$$f(x) = \int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} \phi(t) d\mu(t),\ \nu > 0\ , \tag{1.1}$$

where

$$d\mu(x) = \frac{1}{2^{\nu-1/2}\Gamma(\nu+1/2)} x^{2\nu} dx\ . \tag{1.2}$$

This transform is intimately related to the Hankel transform. Indeed, its kernel is the Hankel transform

$$\frac{t}{(x^2+t^2)^{\nu+1}} = \frac{\sqrt{\pi}}{2^{\nu+1/2}\Gamma(\nu+1)} \int_0^\infty \mathscr{J}(xy) e^{-tx} d\mu(y)\ , \tag{1.3}$$

where

$$\mathscr{J}(z) = 2^{\nu-1/2}\Gamma\left(\nu+\frac{1}{2}\right) z^{-\nu+1/2} J_{\nu-1/2}(z)\ ,$$

$J_\gamma(z)$ **being the ordinary Bessel function of order** γ**. Our object is to develop an inversion theory for** (1.1) **and to exploit the relationship of** (1.1) **to the Hankel transform.**

When $\nu = 0$, some of our results reduce, modulo a constant, to those of D. V. Widder in [6], a paper on which the present work is closely based. In [7], Widder derived an inversion theory for the general transform

$$f(x) = \int_0^\infty K\left(\frac{x}{t}\right) \frac{\phi(t)}{t} dt\ , \tag{1.4}$$

which includes his result in [6] as a special case; however, the transform (1.1) for $\nu > 0$ is not covered by that development.

2. Preliminaries. The differential operator L_x which is to effect the desired inversion of (1.1) is defined as follows:

$$L_x[f(x)] = \lim_{n\to\infty} L_{n,x}[f(x)]\ , \tag{2.1}$$

where, with D denoting differentiation with respect to x,

$$L_{1,x}[f] = \frac{\sqrt{\pi}\,\Gamma(2\nu+1)}{2^{5/2+\nu}[\Gamma(2+\nu/2)]^2 x^{1+2\nu}} D x^{5+2\nu} D \frac{1}{x} D[f]\ .$$

$$\begin{aligned} L_{n,x}[f] = {} & \frac{(-1)^{n+1}\sqrt{\pi}\,\Gamma(2\nu+1)}{2^{2n+\nu+1/2}[\Gamma(n+\nu/2+1)]^2 x^{2n+2\nu-1}} D x^{4n+2\nu+1} \\ & \times \left[\prod_{k=1}^{n-1} D \frac{1}{x^{4k+2\nu+1}} D x^{4k+2\nu+1}\right] D \frac{1}{x} D[f],\ n = 2, 3, \cdots \end{aligned} \tag{2.2}$$

the operations to be performed in order of increasing k, $k = 1, 2, \cdots, n-1$. It may readily be established that

$$(2.3)\quad L_{n+1,x}[f] = \frac{-1}{4(n+\nu/2+1)^2}\frac{1}{x^{2n+2\nu+1}}Dx^{4n+2\nu+5}D\frac{1}{x^{2n+2}}L_{n,x}[f]\ .$$

Hence, an induction argument establishes the validity of

$$(2.4)\quad L_{n,x}\left[\frac{1}{(x^2+t^2)^{\nu+1}}\right] = \frac{2^{\nu+1/2}\Gamma(\nu+1/2)\Gamma(2n+\nu+2)}{[\Gamma(n+\nu/2+1)]^2}\frac{x^{2n+2}t^{2n}}{(x^2+t^2)^{2n+\nu+2}}\ .$$

We note that an alternative form for the operator $L_{n,x}$ is given by

$$(2.5)\quad L_{n,x} = -\frac{2^{\nu-1/2}\Gamma(\nu+1/2)\Gamma(n+\nu+1)n!}{[\Gamma(n+\nu/2+1)]^2}\theta\prod_{k=1}^{n}\left(1-\frac{\theta}{2k}\right)\left(1+\frac{\theta}{2k+2\nu}\right),$$

where

$$(2.6)\quad \theta = xD \quad\text{and}\quad \theta^2 = xD(xD)\ .$$

Since $\theta[x^\alpha] = \alpha x^\alpha$ and $\theta^2[x^\alpha] = \alpha^2 x^\alpha$, we have, from (2.5), that

$$(2.7)\quad \begin{aligned} L_{n,x}[x^\alpha] = &-\frac{2^{\nu-1/2}\Gamma(\nu+1/2)\Gamma(n+\nu+1)n!}{[\Gamma(n+\nu/2+1)]^2} \\ &\times\alpha\prod_{k=1}^{n}\left(1-\frac{\alpha}{2k}\right)\left(1+\frac{\alpha}{2k+2\nu}\right)x^\alpha\ , \end{aligned}$$

so that it is clear that the operator $L_{n,x}$ annihilates positive even powers of x. Letting $n\to\infty$ in (2.7), we find that

$$(2.8)\quad L_x[x^\alpha] = -\left[\frac{\sqrt{\pi}\,\Gamma(2\nu+1)}{2^{\nu+1/2}\Gamma(\nu+1+\alpha/2)\Gamma(1-\alpha/2)}\right]\alpha x^\alpha\ .$$

See p. 5 of [1].

A function $\phi(x)$ will belong to L for $x\in[0, R)$, $0 < R \leqq \infty$, if

$$(2.9)\quad \int_0^R |\phi(x)|\,d\mu(x) < \infty\ .$$

3. Inversion. We establish our principal inversion formula for the transform (1.1).

THEOREM 3.1. *Let $\phi(t)$ belong to L for $t\in[0, R]$ for every positive R, and let the integral*

$$(3.1)\quad f(x) = \int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}}\phi(t)d\mu(t)$$

converge for some $x \neq 0$. Then

$$\lim_{n\to\infty} L_{n,x}[f(x)] = \phi(x)\ , \tag{3.2}$$

at all points x of the Lebesgue set for the function $\phi(x)$.

Proof. We seek to prove (3.2) for all t for which

$$\int_t^x |\phi(u) - \phi(t)|\, d\mu(u) = o(|x - t|)\ , \qquad x \longrightarrow t\ .$$

An appeal to Theorem 2c, p. 328, of [5] establishes that the integral (3.1) converges for all $x \neq 0$ and that differentiation under the integral sign is valid. Hence, using (2.4), we find that

$$\begin{aligned} L_{n,x}[f(x)] &= \frac{2^{\nu+1/2}\Gamma(\nu+1/2)\Gamma(2n+\nu+2)}{[\Gamma(n+\nu/2+1)]^2}\int_0^\infty \frac{x^{2n+2}t^{2n+1}}{(x^2+t^2)^{2n+\nu+2}}\phi(t)d\mu(t) \\ &= \frac{2\Gamma(2n+\nu+2)}{[\Gamma(n+\nu/2+1)]^2}\int_0^\infty \left(\frac{t}{1+t^2}\right)^{2n+2}\frac{\phi(xt)}{t^{1-2\nu}(1+t^2)^\nu}dt\ . \end{aligned} \tag{3.3}$$

We may now apply the asymptotic estimate of Corollary 2b.2, p. 279, of [5] to

$$\int_0^1 \left(\frac{t}{1+t^2}\right)^{2n+2}\frac{\phi(xt)}{t^{1-2\nu}(1+t^2)^\nu}dt\ ,$$

and that of Theorem 2b, p. 278, of [5] to

$$\int_1^\infty \left(\frac{t}{1+t^2}\right)^{2n+2}\frac{\phi(xt)}{t^{1-2\nu}(1+t^2)^\nu}dt\ ,$$

to obtain

$$L_{n,x}[f(x)] \sim \frac{\Gamma(2n+\nu+2)\sqrt{\pi}}{2^{2n+\nu+1}[\Gamma(n+\nu/2+1)]^2\sqrt{n}}\phi(x)\ , \qquad n \longrightarrow \infty\ . \tag{3.4}$$

Using Stiring's formula, we find that

$$\lim_{n\to\infty} L_{n,x}[f(x)] = \phi(x)\ ,$$

and the proof is complete.

COROLLARY 3.2. (3.2) *holds for almost all x, $x \in (0, \infty)$.*

COROLLARY 3.3. (3.2) *holds for all x for which $\phi(x)$ is continuous.*

COROLLARY 3.4. *At all points x in a neighborhood of which $\phi(x)$ is of bounded variation, we have*

$$L_x[f(x)] = \frac{\phi(x^+) + \phi(x^-)}{2}\ .$$

An additional result which follows by the method of Theorems 3b, 3c of [9], pp. 282–283, is the following.

COROLLARY 3.5. *Let $\phi(t)$ belong to L for $t \in (\varepsilon, R)$ for every ε, R, $0 < \varepsilon < R$, and let the integral*

$$f(x) = \int_{\varepsilon}^{R} \frac{t}{(x^2 + t^2)^{\nu+1}} \phi(t) d\mu(t)$$

converge for some $x \neq 0$, and the integral

$$\int_{0+}^{1} \phi(x) x^r dx$$

converge for some fixed r. Then

$$\lim_{n \to \infty} L_{n,x}[f(x)] = \phi(x)$$

at all points x of the Lebesgue set for the function $\phi(x)$.

As an example illustrating the theorem, consider the Hankel transform

$$\int_0^\infty \frac{\mathscr{J}(ty)}{t} \frac{t}{(x^2 + t^2)^{\nu+1}} d\mu(t) = \frac{\sqrt{\pi}}{2^{\nu+1/2}\Gamma(\nu + 1)} \frac{e^{-xy}}{x} . \tag{3.5}$$

Now

$$\begin{aligned} L_{n,x}\left[\frac{e^{-xy}}{x}\right] = &-\frac{2^{\nu-1/2}\Gamma(\nu + 1/2)\Gamma(n + \nu + 1)n!}{[\Gamma(n + \nu/2 + 1)]^2} \sum_{j=0}^{\infty} \frac{(-1)^j y^j}{j!} \\ &\times (j - 1) \prod_{k=1}^{n} \left(1 - \frac{j-1}{2k}\right)\left(1 + \frac{j-1}{2k + 2\nu}\right) x^{j-1} . \end{aligned} \tag{3.6}$$

A straightforward computation gives

$$L_x\left[\frac{e^{-xy}}{x}\right] = \frac{2^{\nu+1/2}\Gamma(\nu + 1)}{\sqrt{\pi}} \frac{\mathscr{J}(xy)}{x} , \tag{3.7}$$

since taking the limit under the summation sign in (3.6) is valid due to the fact that the series of (3.6) is dominated by the series

$$\sum_{j=0}^{\infty} \frac{1}{j!} (xy)^j e^{\pi/2|j-1|}$$

which converges for all x. Hence the result predicted by the theorem is derived.

Other examples where the validity of the theorem may likewise be verified are

$$f(x) = \int_0^\infty \frac{\cos t}{t^{1+2\nu}} \frac{t}{(x^2+t^2)^{\nu+1}} d\mu(t) = \frac{K_{\nu+1/2}(x)}{x^{\nu+1/2}\Gamma(2\nu+1)}, \tag{3.8}$$

where

$$L_x[x^{-\nu-1/2}K_{\nu+1/2}(x)] = \Gamma(2\nu+1)x^{-2\nu-1}\cos x ;$$

and

$$\begin{aligned} f(x) &= \int_0^\infty t^{-2}J_{2\nu}(t)\frac{t}{(x^2+t^2)^{\nu+1}}d\mu(t) \\ &= \frac{\Gamma(1-\nu)}{2^{\nu+2}\sqrt{\pi}\,\nu\Gamma(2\nu+1)}\,{}_1F_2\Big(2\nu;\ \nu,\ 2\nu+1;\ \frac{x^2}{4}\Big) \\ &\quad + \frac{\Gamma(\nu-1)}{2^{-\nu-7/2}\sqrt{\pi}\,(\nu^2-1)\Gamma(2\nu)}\,{}_1F_2\Big(1+\nu;\ 2+\nu,\ 2+\nu;\ \frac{x^2}{4}\Big), \end{aligned} \tag{3.9}$$

where

$$L_x[f(x)] = x^{-2}J_{2\nu}(x) .$$

4. Relation to the Hankel transform. The relation of the Hankel potential transform to the Hankel transform enables us to derive an inversion of the latter in terms of the operator L_t. We have the following result.

THEOREM 4.1. *If $\phi(u)$ belongs to L for $0 \leqq t < \infty$, and if*

$$\phi^\wedge(x) = \int_0^\infty \mathscr{J}(xu)\phi(u)d\mu(u) , \tag{4.1}$$

then, for almost all t,

$$\phi(t) = \frac{\sqrt{\pi t}}{2^{\nu+1/2}\Gamma(\nu+1)}L_t\Big[\frac{R(t)}{t}\Big],\ 0 < t < \infty , \tag{4.2}$$

where

$$R(t) = \int_0^\infty e^{-tu}\phi^\wedge(u)d\mu(u) . \tag{4.3}$$

Proof. An appeal to Fubini's theorem establishes that

$$\begin{aligned} R(t) &= \int_0^\infty e^{-ut}d\mu(u)\int_0^\infty \mathscr{J}(uy)\phi(y)d\mu(y) \\ &= \int_0^\infty \phi(y)d\mu(y)\int_0^\infty e^{-ut}\mathscr{J}(uy)dd\mu(u) \\ &= \frac{2^{\nu+1/2}\Gamma(\nu+1)}{\sqrt{\pi}}t\int_0^\infty \frac{1}{(y^2+t^2)^{\nu+1}}\phi(y)d\mu(y) , \end{aligned}$$

and the proof is complete on application of Theorem 3.1.

If in addition to the conditions of the theorem, we assume that

$\phi^\wedge(x) \in L$ for $0 \leqq x < \infty$, we obtain the familiar inversion of the Hankel transform given in the following corollary.

COROLLARY 4.2. *If $\phi(u)$ belongs to L for $0 \leqq u < \infty$, and if*

$$\phi^\wedge(x) = \int_0^\infty \mathscr{J}(xu)\phi(u)d\mu(u) \tag{4.5}$$

with $\phi^\wedge(x) \in L$ for $0 \leqq x < \infty$, then

$$\phi(u) = \int_0^\infty \mathscr{J}(xu)\phi^\wedge(x)d\mu(x) . \tag{4.6}$$

Proof. We have, with the notation of the theorem,

$$L_t\left[\frac{R(t)}{t}\right] = \lim_{n\to\infty} L_{n,t}\int_0^\infty \frac{e^{-ty}}{t}\phi^\wedge(y)d\mu(y) . \tag{4.7}$$

Since $\phi^\wedge(x) \in L$, the operator $L_{n,t}$ may be taken under the integral sign, and, by the argument used to establish (3.7), the limit in (4.7) may be applied to the integrand directly. Hence we have, on taking note of (3.7),

$$L_t\left[\frac{R(t)}{t}\right] = \frac{2^{\nu+1/2}\Gamma(\nu+1)}{\sqrt{\pi}}\,\frac{1}{t}\int_0^\infty \mathscr{J}(ty)\phi^\wedge(y)d\mu(y) . \tag{4.8}$$

Substituting (4.8) in (4.2), we derive the desired inversion (4.6).

Since the Hankel transform of the Hankel potential transform is a Laplace transform, we may use the inversion algorithm of the latter to obtain another inversion formula for the Hankel potential transform, as in the following result.

THEOREM 4.3. *Let $\phi(u)$ belong to L for $0 \leqq u < \infty$, and let*

$$f(x) = \int_0^\infty \frac{t}{(t^2+x^2)^{\nu+1}}\phi(t)d\mu(t) . \tag{4.9}$$

Then, for almost all t,

$$\phi(t) = \Gamma(2\nu+1)t^{-2\nu}\lim_{k\to\infty}\mathscr{L}_{k,t}f^\wedge(t) , \tag{4.10}$$

where

$$\mathscr{L}_{k,t}f(t) = \frac{(-1)^k}{k!}f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1} , \tag{4.11}$$

and

$$f^\wedge(t) = \int_0^\infty \mathscr{J}(tu)f(u)d\mu(u) . \tag{4.12}$$

Proof. Since $\phi(u) \in L$, it follows that $f(x) \in L$ for $0 < x < \infty$. Hence, clearly, $f^\wedge(t)$ exists, and by Fubini's theorem, we have

$$\begin{aligned} f^\wedge(z) &= \int_0^\infty \mathscr{J}(zu)d\mu(u) \int_0^\infty \frac{t}{(t^2 + u^2)^{\nu+1}} \phi(t)d\mu(t) \\ &= \int_0^\infty \phi(t)d\mu(t) \int_0^\infty \frac{t\mathscr{J}(zu)}{(t^2 + u^2)^{\nu+1}} d\mu(u) \\ &= \frac{\pi^{1/2}}{2^{\nu+1/2}\Gamma(\nu + 1)} \int_0^\infty e^{-tz}\phi(t)d\mu(t) . \end{aligned}$$

But the right hand side is a Laplace transform, and by Theorem 6a, p. 288 of [5], (4.10) follows.

The following example illustrates the theorem. Let

$$f(x) = \int_0^\infty \frac{t}{(t^2 + x^2)^{\nu+1}} e^{-t}d\mu(t) . \tag{4.13}$$

Hence

$$f^\wedge(t) = \frac{1}{(t + 1)^{2\nu+1}}$$

and

$$\begin{aligned} \mathscr{L}_{k,t}[f^\wedge(t)] &= \frac{\Gamma(2\nu + 1 + k)}{k!\, \Gamma(2\nu + 1)} \left(\frac{k}{k + t}\right)^{k+1} \left(\frac{t}{k + t}\right)^{2\nu} \\ &\sim \frac{t^{2\nu}}{\Gamma(2\nu + 1)} e^{-t},\ k \longrightarrow \infty . \end{aligned}$$

We thus have

$$\begin{aligned} \Gamma(2\nu + 1)t^{-2\nu} \lim_{k\to\infty} \mathscr{L}_{k,t}[f^\wedge(t)] &= e^{-t} \\ &= \phi(t) \end{aligned}$$

as predicted.

5. Inversion for a function having a power series expansion. If $f(x)$ represented by the transform (1.1) has an expansion in powers of x, the inversion of (3.2) becomes especially simple. Indeed, an elementary, practical algorithm may be illustrated by the following example:

$$\int_0^\infty \frac{t}{(x^2 + t^2)^2} \frac{1}{t(1 + t^2)} \sqrt{\frac{2}{\pi}} t^2 dt = \frac{\sqrt{\pi}}{2^{3/2}} \frac{1}{x(1 + x)^2} . \tag{5.1}$$

To invert this transform, we expand $1/x(1 + x)^2$ in powers of x, cancel the odd powers, change the alternate signs, and replace the absolute value of each coefficient by $2^{3/2}/\sqrt{\pi}$. Illustrating each of the steps in this case, we have, on expanding in a power series

$$\frac{1}{x}\left\{1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots\right\};$$

cancelling odd powers and changing alternate signs, we obtain

$$\frac{1}{x}\left\{1 - 3x^2 + 5x^4 - 7x^6 + \cdots\right\};$$

and, finally, replacing the absolute value of each coefficient by $2^{3/2}/\sqrt{\pi}$, we find

$$\frac{2^{3/2}}{\sqrt{\pi}}\frac{1}{x}\{1 - x^2 + x^4 - x^6 + \cdots\} = \frac{2^{3/2}}{\sqrt{\pi}}\frac{1}{x(1 + x^2)},$$

so that the algorithm applied to $(\sqrt{\pi}/2^{3/2})/x(1 + x^2)$ yields $1/x(1 + x^2)$ as desired.

The general result which describes this algorithm is given in the following.

THEOREM 5.1. *Let*

$$f(x) = \int_0^\infty \frac{t}{(x^2 + t^2)^{\nu+1}}\phi(t)d\mu(t), \tag{5.2}$$

the integral converging for $x \neq 0$, *and let*

$$f(x) = \frac{1}{x^\alpha}\sum_{k=0}^\infty a_k x^k,\ 0 < x < \rho. \tag{5.3}$$

Then

$$\begin{aligned}\phi(x) = {} & \frac{2^{1/2-\nu}\Gamma(2\nu + 1)}{\sqrt{\pi}}x^{-\alpha} \\ & \times \left[\sin\frac{\pi\alpha}{2}\sum_{k=0}^\infty (-1)^k a_{2k}\frac{\Gamma(k + 1 - \alpha/2)}{\Gamma(k + \nu + 1 - \alpha/2)}x^{2k}\right. \\ & \left. - \cos\frac{\pi\alpha}{2}\sum_{k=0}^\infty (-1)^k a_{2k+1}\frac{\Gamma(3/2 - \alpha/2 + k)}{\Gamma(3/2 - \alpha/2 + k + \nu)}x^{2k+1}\right]\end{aligned} \tag{5.4}$$

valid in some interval $0 < x < \rho e^{-\pi/2}$.

Proof. Since termwise differentiation is valid, we have

$$\begin{aligned}L_{n,x}[f(x)] = {} & -\frac{2^{\nu-1/2}\Gamma(\nu + 1/2)\Gamma(n + \nu + 1)n!}{[\Gamma(n + \nu/2 + 1)]^2}\sum_{k=1}^\infty a_k(k - \alpha) \\ & \times \prod_{j=1}^n \left(1 - \frac{k - \alpha}{2j}\right)\left(1 + \frac{k - \alpha}{2j + 2\nu}\right)x^{k-\alpha},\end{aligned} \tag{5.5}$$

or, if we take the limit as $n \to \infty$ under the summation sign, we find that

$$
\begin{aligned}
L_x[f(x)] &= -2^{\nu-1/2}\Gamma\Big(\nu+\frac{1}{2}\Big)\sum_{k=0}^{\infty} a_k \\
&\quad\times \frac{(k-\alpha)\Gamma(\nu+1)}{\Gamma(1+\nu+k/2-\alpha/2)\Gamma(1-k/2-\alpha/2)}x^{k-\alpha} \\
&= -2^{\nu-1/2}\Gamma\Big(\nu+\frac{1}{2}\Big)\Gamma(\nu+1)x^{-\alpha} \\
&\quad\times\Big\{\sum_{k=0}^{\infty} a_{2k}\frac{(2k-\alpha)}{\Gamma(1+\nu+k-\alpha/2)\Gamma(1-k+\alpha/2)}x^{2k} \\
&\quad+\sum_{k=0}^{\infty} a_{2k+1}\frac{(2k+1-\alpha)}{\Gamma(3/2-\alpha/2+\nu+k)\Gamma(1/2+\alpha/2-k)}x^{2k+1}\Big\}\,.
\end{aligned}
\tag{5.6}
$$

The left hand side of (5.6) is equal to $\phi(x)$ by Theorem 3.1, and the right hand side reduces to that given in (5.4) by a simple computation.

The validity of taking the limit in (5.5) and obtaining (5.6) follows from the fact that

$$
\begin{aligned}
|L_{n,x}[f(x)]| &\leqq \frac{2^{\nu-1/2}\Gamma(\nu+1/2)\Gamma(n+\nu+1)n!}{[\Gamma(n+\nu/2+1)]^2} \\
&\quad\times\sum_{k=0}^{\infty}|a_k|\,|k-\alpha|\prod_{j=1}^{\infty}\Big(1+\frac{(k-\alpha)^2}{4j^2}\Big)x^{k-\alpha} \\
&= \frac{2^{\nu+1/2}\Gamma(\nu+1/2)\Gamma(n+\nu+1)n!}{\pi[\Gamma(n+\nu/2+1)]^2}x^{-\alpha} \\
&\quad\times\sum_{k=0}^{\infty}|a_k|\sin h\frac{\pi|k-\alpha|}{2}x^k\,,
\end{aligned}
$$

so that the series of (5.6) converges uniformly in n if

$$\sum_{k=0}^{\infty}|a_k|x^k e^{\pi/2|k-\alpha|}<\infty\,.$$

But this is so for $0<x<\rho e^{-\pi/2}$ and hence the proof is complete.

Note that if ρ is replaced by ∞ in the hypothesis of the theorem, the conclusion holds in $(0,\infty)$. Further, if ϕ is known to be analytic at the start in $(0,\infty)$, then (5.4) will determine ϕ completely by analytic continuation. For example, applying the theorem to (5.1), we note that $\alpha=1$ so that the second sum of (5.4) vanishes and we have

$$
\begin{aligned}
L_x\Big[\frac{\sqrt{\pi}}{3^{3/2}}\frac{1}{x(1+x)^2}\Big] &= \frac{\sqrt{\pi}}{2^{3/2}}L_n\sum_{k=0}^{\infty}(-1)^k(k+1)x^{k-1} \\
&= \frac{1}{x}\sum_{k=0}^{\infty}(-1)^k x^{2k} \\
&= \frac{1}{x(1+x^2)}\,.
\end{aligned}
$$

Here $\phi(x)=1/x(1+x^2)$ is analytic for $0<x<\infty$. Even though

$\sum_{k=0}^{\infty}(-1)^k x^{2k}$ is valid only for $|x| < 1$, it nonetheless determines ϕ for all positive x by analytic continuation.

For $f(x)$ having a power series expansion in terms of negative powers of x, we have the following companion result whose proof is analogous to that of the preceding theorem and hence will be omitted.

THEOREM 5.2. *Let*

$$f(x) = \int_0^\infty \frac{t}{(x^2 + t^2)^{\nu+1}} \phi(t) d\mu(t) , \tag{5.7}$$

the integral converging for $x \neq 0$, and let

$$f(x) = \frac{1}{x^\alpha} \sum_{k=0}^{\infty} a_k x^{-k},\ \rho < x < \infty . \tag{5.8}$$

Then

$$\begin{aligned} \phi(x) = {} & \frac{2^{1/2-\nu}\Gamma(2\nu+1)}{\sqrt{\pi}} x^{-\alpha} \\ & \times \left[\sin \pi\left(\frac{\alpha}{2} - \nu\right) \sum_{k=0}^{\infty} (-1)^k a_{2k} \frac{\Gamma(k + \alpha/2 - \nu)}{\Gamma(k + \alpha/2)} x^{-2k} \right. \\ & \left. + \cos \pi\left(\frac{\alpha}{2} - \nu\right) \sum_{k=0}^{\infty} (-1)^k a_{2k+1} \frac{\Gamma(1/2 + \alpha/2 - \nu + k)}{\Gamma(1/2 + \alpha/2 + k)} x^{-2k-1}\right] , \end{aligned} \tag{5.9}$$

valid in some interval $\rho e^{\pi/2} < x < \infty$.

We may illustrate this theorem with example (5.1) by expanding $1/x(1 + x)^2$ in the series $x^{-3}\sum_{j=0}^{\infty}(-1)^j(j + 1)x^{-j}$. In this case $\alpha = 3$, and the second sum of (5.9) vanishes so that we have

$$\phi(x) = x^{-3} \sum_{k=0}^{\infty} (-1)^k x^{-2k} = \frac{1}{x(1 + x^2)}$$

as expected.

6. Hankel harmonic functions. Formula (1.3) shows that the kernel of the Hankel potential transform (1.1) is the Abel mean of the Bessel function associated with the Hankel transform. Thus, the kernel of (1.1) is the Poisson kernel, in its non-Hankel translated form, associated with the Hankel convolution of Hirschman and Delsarte [2]. The inversion formula for the Hankel potential transform can be applied to the Hankel harmonic functions associated with the Hankel-Poisson kernel. The Hankel harmonic functions also appear as a special case of the generalized axially symmetric potential theory of A. Weinstein [4].

Let $u(x, t)$ be of class C^2 in a region R of $t \geqq 0$ such that, if R

contains a segment of the line $x = 0$, then $u(x, t) = u(-x, t)$ in a neighborhood of such segments. The function $u(x, t)$ is said to be Hankel harmonic in R if and only if

$$\left(\Delta_x + \frac{\partial^2}{\partial t^2}\right)u(x, t) = 0 , \tag{6.1}$$

for all $(x, t) \in R$, where

$$\Delta_x f(x) = f''(x) + \frac{2\nu}{x}f'(x) . \tag{6.2}$$

The Hankel-Poisson kernel is given by

$$P(x, y, t) = \frac{\sqrt{\pi}}{2^{\nu+1/2}\Gamma(\nu + 1)} \int_0^\infty e^{-tu} \mathscr{J}(xu)\mathscr{J}(yu)d\mu(u) . \tag{6.3}$$

Note that, by (1.3), we have

$$P(x, 0, t) = P(x, t) = \frac{t}{(x^2 + t^2)^{\nu+1}} .$$

Since $\Delta_x \mathscr{J}(xu) = -u^2 \mathscr{J}(xu)$, it may readily be verified that $P(x, t)$ is a Hankel harmonic function in the half plane $t > 0$.

We establish the following inversion for Hankel harmonic functions.

THEOREM 6.1. *Let $u(x, t)$ be an even function of x and a bounded Hankel harmonic function in the half plane $t \geqq 0$. Then*

$$L_x\left[\frac{u(0, x)}{x}\right] = \frac{2^{\nu+1/2}\Gamma(\nu + 1)}{\sqrt{\pi}} \frac{u(x, 0)}{x}, \quad x \neq 0 . \tag{6.4}$$

Proof. From [3], it follows that

$$u(x, t) = \frac{2^{\nu+1/2}\Gamma(\nu + 1)}{\sqrt{\pi}} \int_0^\infty P(x, y, t)u(y, 0)d\mu(y), \ t > 0 .$$

Therefore,

$$u(0, t) = \frac{2^{\nu+1/2}\Gamma(\nu + 1)}{\sqrt{\pi}} \int_0^\infty \frac{t}{(y^2 + t^2)^{1+\nu}} u(y, 0)d\mu(y) ,$$

or, equivalently,

$$\frac{u(0, t)}{t} = \frac{2^{\nu+1/2}\Gamma(\nu + 1)}{\sqrt{\pi}} \int_0^\infty \frac{y}{(y^2 + t^2)^{1+\nu}} \frac{u(y, 0)}{y} d\mu(y) ,$$

for all $t > 0$. Applying the inversion formula, we obtain (6.4), and the proof is complete.

As an illustrative example, consider the function

$$u(x, t) = e^{-at} \mathscr{J}(ax) . \tag{6.5}$$

Then $u(x, t)$ is even in x and is a bounded Hankel harmonic function for $t \geqq 0$. An appeal to (3.7) provides the verification of the inversion (6.4).

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Received November 2, 1970 and in revised form January 13, 1971. The research of the first author was supported by the National Science Foundation grant GP-23118 and that of the second by National Science Foundation grant 20536.

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