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ON SPACES WITH REGULAR G_δ-DIAGONALS

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It is the purpose of this note to investigate spaces with regular G_δ-diagonals. Among other things, it is shown that if X is T_1-space, then 1. X admits a development satisfying the 3-link property if and only if X is a $\omega\Delta$-space with a regular G_δ-diagonal and 2. X is metrizable if and only if X is an M-space with a regular G_δ-diagonal.

Recall that a subset H of the space X is a regular G_δ-set if there is a sequence $\{U_n\}$ of open sets in X such that $H = \bigcap_{i=1}^\infty U_i = \bigcap_{i=1}^\infty U_i^-$. We will say that X has a regular G_δ-diagonal if $\Delta X = \{(x, x): x \in X\}$ is a regular G_δ-set in X^2.

In [4], Ceder shows that X has a G_δ-diagonal if and only if there is a sequence $\{G_n\}$ of open covers of X such that if x is a point of X, then $x = \bigcap_{i=1}^\infty \mathrm{st}\,(x, G_i)$. In Theorem 1, we show that there is a similar characterizing property for spaces with regular G_δ-diagonals.

THEOREM 1. *The topological space X has a regular G_δ-diagonal if and only if there is a sequence $\{G_n\}$ of open covers of X such that if x and y are distinct points of X, then there are an integer n and open sets u and v containing x and y respectively such that no member of G_n intersects both u and v.*

Proof. Suppose that X has a regular G_δ-diagonal. Let $\{U_n\}$ be a sequence of open sets in X^2 such that $\Delta X = \bigcap_{i=1}^\infty U_i = \bigcap_{i=1}^\infty U_i^-$. For each n, let $G_n = \{g: g$ is an open subset of X such that $g \times g \subset U_n\}$. Let x and y be distinct points of X. Let n be an integer such that (x, y) is not in U_n^-. Let u and v be open sets in X that contain x and y respectively such that $u \times v$ does not intersect U_n. To see that no member of G_n intersects both u and v, suppose otherwise; that is, suppose that g is a member of G_n, p is a point of $g \cap u$ and q is a point of $g \cap v$. Then (p, q) is a point of $U_n \cap (u \times v)$ which is a contradiction.

Now, suppose that $\{G_n\}$ is a sequence of open covers of X as described in the theorem. For each n, let $U_n = \bigcup \{(g \times g): g \in G_n\}$. Clearly, $\Delta X \subset \bigcap_{i=1}^\infty U_i$. To see that $\Delta X = \bigcap_{i=1}^\infty U_i^-$, let x and y be distinct points of X. Then there are an integer n and open sets u and v containing x and y respectively such that no member of G_n intersects both u and v. It must be the case that U_n does not intersect $u \times v$.

COROLLARY. *If X has a regular G_δ-diagonal, then X is Hausdorff.*

A development $\{G_n\}$ for the space X is said to satisfy the *3-link property* if it is true that if p and q are distinct points of X, then there is an integer n such that no member of G_n intersects both st (x, G_n) and st (y, G_n) (Heath [6]). According to Borges [3], the space X is a *$\omega\Delta$-space* if there is a sequence $\{U_n\}$ of open covers of X such that if x is a point and if, for each n, x_n is a point of st (x, U_n), then the sequence $\{x_n\}$ has a cluster point. Clearly, the class of $\omega\Delta$-spaces includes the class of strict p-spaces, the class of M-spaces, and the class of developable spaces. It is easy to see that the Niemytski plane (page 100 of [11]) is a non-metrizable Moore space that admits a development satisfying the 3-link property. In [6], Heath establishes the existence of Moore spaces that do not admit developments that satisfy the 3-link property. In [5], Cook asserts that a continuously semi-metrizable space is a Moore space that admits a development that satisfies the 3-link property. Cook's result follows as a corollary to the following theorem:

THEOREM 2. *Let X be a topological space. Then the following conditions are equivalent*:

1. *X admits a development satisfying the 3-link property.*
2. *X is a $\omega\Delta$-space with a regular G_δ-diagonal. And*
3. *There is a semi-metric d on X such that*:

a. *If $\{x_n\}$ and $\{y_n\}$ are sequences both converging to x, then $\lim_{n\to\infty} d(x_n, y_n) = 0$, and*

b. *If x and y are distinct points of X and $\{x_n\}$ and $\{y_n\}$ are sequences converging to x and y respectively, then there are integers N and M such that if $n > N$, then $d(x_n, y_n) > 1/M$.*

Proof. It is obvious that a developable space is a $\omega\Delta$-space; thus, that (1) implies (2) is a corollary to Theorem 1.

To see that (2) implies (1), let X be a $\omega\Delta$-space with a regular G_δ-diagonal. Let $\{U_n\}$ be a sequence of open covers of X as given by the fact that X is a $\omega\Delta$-space. According to Theorem 1, there is a sequence $\{V_n\}$ of open covers of X such that if p and q are distinct points, then there are an integer n and open sets u and v containing p and q respectively such that no member of V_n intersects both u and v. For each integer n, let G_n be an open cover of X such that (i) G_n refines both U_n and V_n and (ii) G_{n+1} refines G_n. We will show that $\{G_n\}$ is a development for X that satisfies the 3-link property. First, to see that $\{G_n\}$ is a development, suppose the contrary; that is, suppose that there are a point x and an open set u containing x such that, for each n, there is a point p_n in st $(x, G_n) - u$. Then, for each n, p_n is in st (x, U_n). Thus, $\{p_n\}$ has a cluster point p. Since for each n, G_n refines each of $V_1, \cdots, V_n$, it follows that there are an integer N and open sets v and w containing x and p respectively such that if $j > N$, then no member of G_j intersects both v and w. But this is a contradiction since there is a $j < N$ such that p_j is in w. Thus, $\{G_n\}$ is a development for X. To see that G_n satisfies the 3-link property, let p and q be distinct points, u and v open sets containing p and q respectively, and N an integer such that if $n > N$, then no member of G_n meets both u and v. Let S and T be integers such that $\mathrm{st}(p, G_S) \subset u$ and $\mathrm{st}(q, G_T) \subset v$. Let $M = \max\{N, S, T\}$. Then no member of G_M meets both $\mathrm{st}(p, G_M)$ and $\mathrm{st}(q, G_M)$.

(1) implies (3): Let $\{G_n\}$ be a development satisfying the 3-link property. Assume that for each n, G_{n+1} refines G_n. If x and y are distinct points, define $d(x, y) = 1/N$, where N is the first integer such that y is not in $\mathrm{st}(x, G_N)$. Define $d(x, x) = 0$. It is a standard argument to see that d is a semi-metric on X. To show that (a) is satisfied, suppose that $\{x_n\}$ and $\{y_n\}$ are sequences converging to x. Let N be an integer and let g be a member of G_N that contains x. There is an integer $M > 0$ such that if $n > M$, then both x_n and y_n are in g. It follows that if $n > M$, then $d(x_n, y_n) < 1/N$; and so, $\lim_{n\to\infty} d(x_n, y_n) = 0$. To see that (b) is satisfied, let x and y be distinct points of X and suppose that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y. Let M denote an integer such that if $n \geqq M$, then no member of G_n intersects both $\mathrm{st}(x, G_n)$ and $\mathrm{st}(y, G_n)$. There is an integer N such that if $n > N$, then x_n is in $\mathrm{st}(x, G_M)$ and y_n is in $\mathrm{st}(y, G_M)$. Thus, if $n > \max\{N, M\}$, then $d(x_n, y_n) > 1/M$.

(3) implies (1): Let $G = \{\mathrm{int.}\, D_\varepsilon(x) \colon \varepsilon > 0, x \in X\}$ where $D_\varepsilon(x) = \{y \in X \colon d(x, y) < \varepsilon\}$. For each N, let $G_N = \{g \in G \colon \mathrm{diam.}\, g < 1/N\}$ where $\mathrm{diam.}\, g = \mathrm{lub}\{d(x, y) \colon (x, y) \in g \times g\}$. Clearly, if for each n, G_n convers X, then $\{G_n\}$ is a development for X. Suppose that $x \in X$ and N is an integer such that no member of G_N contains x. Then for each integer j there are points x_j and y_j such that $d(x, x_j) \leqq 1/j$ and $d(x, y_j) \leqq 1/j$ and such that $d(x_j, y_j) > 1/N$. But this says that $\{x_j\}$ and $\{y_j\}$ are sequences converging to x such that the sequence $\{d(x_j, y_j)\}$ does not converge to zero. This is a contradiction from which it follows that $\{G_n\}$ is a development for X.

Now, suppose that x and y are distinct points of X such that for each n there is a member of G_n intersecting both $\mathrm{st}(x, G_n)$ and $\mathrm{st}(y, G_n)$. Then for each n, there are points x_n and y_n in $\mathrm{st}(x, G_n)$ and $\mathrm{st}(y, G_n)$ respectively such that x_n and y_n are in a common member of G_n. But, this means that $\{x_n\}$ converges to x, $\{y_n\}$ converges to y, and $\lim_{n\to\infty} d(x_n, y_n) = 0$ which is a contradiction.

Note. The argument that (3) implies (1) is essentially the argument that H. Cook used when he showed the author how to prove that a continuously semi-metrizable space admits a development satisfying the 3-link property. Also, recall that in [1] it is shown that X is developable if and only if there is a semi-metric satisfying condition (a) and in [7], Hodel defines the notion of a G_δ^*-diagonal and he shows that the space X is a Hausdorff developable space if and only if X is a $\omega\Delta$-space with a G_δ^*-diagonal.

A space X is said to be an M-space if there is a normal sequence $\{G_n\}$ of open covers of X such that if x is a point and $\{x_n\}$ is a sequence of points such that, for each n, x_n is in $\mathrm{st}(x, G_n)$, then $\{x_n\}$ has a cluster point (Morita [10]).

LEMMA. *If X is an M-space, then either X is discrete or there is a countable discrete subspace of X that is not closed in X.*

Proof. Suppose that x_0 is a limit point of X. Let $\{G_n\}$ be a normal sequence of open covers of X as given by the fact that X is an M-space. Let x_1 be a point of $\mathrm{st}(x_0, G_1)$ distinct from x_0 and let u_1 be an open set containing x_1 such that x_0 is not in $\mathrm{cl}\, u_1$. Having $x_1, \cdots, x_n$ and $u_1, \cdots, u_n$, let x_{n+1} be a point of $\mathrm{st}(x_0, G_{n+1}) - U_{i=1}^n \mathrm{cl}\, u_i$ distinct from x_0. Let u_{n+1} be an open set containing x_{n+1} such that x_0 is not in $\mathrm{cl}\, u_{n+1} \cdot \{x_1, x_2, \cdots\}$ is a countable discrete subspace of X that is not closed in X.

THEOREM 3. *Let X be a topological space. The following statements are equivalent:*

1. *X is metrizable.*
2. *X is a Hausdorff M-space such that X^2 is perfectly normal.*
3. *X is an M-space with a regular G_δ-diagonal.*
4. *X is a Hausdorff M-space such that X^3 is hereditarily normal.*
5. *X is a Hausdorff M-space such that X^3 is hereditarily countable paracompact.*

Proof. That (1) implies each of the other conditions is obvious. Also, it is clear that (2) implies (3). That (4) implies (2) follows from our Lemma and Corollary 1 of [8] and that (5) implies (2) follows from our Lemma and Theorem B of [12]. It remains to show that (3) implies (1). To this end, it follows from Theorem 2 that X is developable and Hausdorff. According to Theorem 6.1 of [10], there is a closed mapping f taking X onto a metric space Y such that $f^{-1}(y)$ is countably compact for each y in Y. Since X is developable, $f^{-1}(y)$ is compact for each y in Y; thus, f is a perfect map. It is a well known consequence of Theorem 1 of [9] that the preimage of a metric space under a perfect map is paracompact. But, it is shown in [2] that a paracompact developable space is metrizable.

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