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FINITELY-VALUED f-MODULES

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Let M be a right f-module over the directed po-ring R (i.e., M is a lattice-ordered R-module that is a subdirect product of a family of totally ordered R-modules), and let g be a nonzero element of M. There is a natural one-to-one correspondence between the set of R-values of g in M and the set of Z-values of g in M. This basic fact enables one to obtain all of the local structure theory for f-modules that Conrad [Czechoslovak Math. J. 15 (1965)] has obtained for ℓ-groups. There is, in addition, the interaction between the two structures. For example, a special element g has the same value in $C_R(g)$, the convex ℓ-submodule generated by g, that it has in $C_Z(g)$. Using this structure theory and the fact that a special element is basic in a Johnson semisimple f-ring, it is shown that a finitely-valued Johnson semisimple f-ring is a direct sum of unital ℓ-simple f-rings.

It is well-known that an abelian lattice-ordered group (ℓ-group) can be represented as a subdirect product of a family of totally ordered groups. In this paper we begin the study of those lattice-ordered modules (ℓ-modules) that can be represented as subdirect products of totally ordered modules. An ℓ-module that can be so represented is called an f-module.[1] We will continue this study in a later paper, considering the problem of supplying the injective hull of an f-module with a lattice order so that it becomes an f-module extension, and considering the related problem of characterizing relative injectives in the category of f-modules (see [16] and [19]). Here we are concerned with the structure of f-modules.

We wish to mention that many of the results in this paper hold in the more general situation where M is a not-necessarily abelian ℓ-group with appropriate operator set R. Appropriate means, here, that the mapping induced by each $r \in R$ preserves all polars and sends positive elements to positive elements.

Throughout Z and Q will denote the totally ordered rings of integers and rational numbers respectively.

1. Characterizations of f-modules. Let R be a partially ordered ring (po-ring). An *(right) ℓ-module over R* is a right R-module

[1] We have recently learned that Bigard [2] has defined an f-module (he calls it an ℓ-module) and has proven the equivalence of (a) and (b) of 1.1 for the case that R is an ℓ-ring with a positive identity element.

M that is also an ℓ-group for which $M^+R^+ \subseteqq M^+$. The ℓ-module M will be called an *f-module* if it is isomorphic to a subdirect product of a family of totally ordered modules. A *convex ℓ-submodule* of the ℓ-module M is a convex ℓ-subgroup N that is also an R-submodule. The set of convex ℓ-submodules of M, partially ordered by inclusion, is a distributive lattice. If M/N is totally ordered, then N is called a *prime submodule* of M. A *minimal prime submodule* of M is a prime submodule that does not contain any other prime submodule of M. When $R = Z$, these definitions agree with the usual definitions for ℓ-groups. The following theorem is just a translation of the theorem about representable ℓ-groups.

THEOREM 1.1. *Let M be an ℓ-module over the directed po-ring R. The following statements are equivalent.*

(a) *M is an f-module.*

(b) *If x and y are in M and r is in R^+, then $x \wedge y = 0$ implies $xr \wedge y = 0$.*

(c) *Every minimal prime subgroup is a submodule.*

(d) *Every polar of M is a submodule.*

Proof. That (a) implies (b) and (c) implies (a) is trivial. That (b) implies (c) follows from the fact that if N is a minimal prime subgroup and $x \in N$, then the polar of x is not contained in N [14, Theorem 6.5]. Finally, (c) and (d) are equivalent since each minimal prime subgroup is the union of principal polars, and each polar is the intersection of minimal prime subgroups.

The proof of (b) implies (c) is essentially the proof of the fact that a polar preserving endomorphism preserves minimal prime subgroups [8].

If R is not directed, then Theorem 1.1 is false. In particular, let $R = D_2$ be the two-by-two matrix ring over a totally ordered division ring D. Then R is a po-ring if its positive cone is defined by $R^+ = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}: x, y \in D^+\right\}$. Let $M = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}: a, b \in D\right\}$ and $M^+ = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}: a, b \in D^+\right\}$. Then (M, M^+) is an ℓ-module over R that satisfies (b), but neither (a), nor (c), nor (d).

If R is a po-ring, then $S = R^+ - R^+$ is the largest directed po-subring of R. Thus, if M is an ℓ-module over R, then M is an f-module over S if and only if M satisfies (b). For this reason and for the sake of simplicity, unless specified otherwise, *all po-rings will be directed* for the remainder of this paper.

It is known that an f-ring can be characterized as an ℓ-ring for which every subdirectly irreducible homomorphic image is totally ordered. An f-module can be characterized in an analogous manner.

For a non-empty subset S of the ℓ-module M_R let $C_R(S)$ be the convex ℓ-submodule generated by S.

PROPOSITION 1.2. *Let M_R be an ℓ-module, and let $a \in M$. Then* $C_R(a) = \{x \in M\colon |x| \leqq n|a| + |a|r$ *for some* $r \in R^+$ *and some* $n \in Z^+\}$.

Proof. Let N be the set defined in the proposition. Since $a \in N \subseteqq C_R(a)$, we only have to verify that N is a convex ℓ-submodule. If $x, y \in N$, then $|x - y| \leqq |x| + |y| \leqq (n|a| + |a|r) + (m|a| + |a|s) = (n + m)|a| + |a|(r + s)$. Therefore N is a subgroup, and hence it is a convex ℓ-subgroup. If $x \in N^+$ and $r \in R^+$, then $xr \in N$, and it follows that N is a convex ℓ-submodule.

COROLLARY 1.3. *If M_R is an f-module, then $x \wedge y = 0$ implies* $C_R(x) \cap C_R(y) = 0$.

Proof. If $0 \leqq a \in C_R(x) \cap C_R(y)$, then $a \leqq (nx + xr) \wedge (my + ys) = 0$, since M is an f-module.

COROLLARY 1.4. *An ℓ-module M is an f-module if and only if each of its subdirectly irreducible homomorphic images is totally ordered.*

Proof. Since a homomorphic image of an f-module is an f-module, a subdirectly irreducible homomorphic image of an f-module is totally ordered, by 1.3.

The converse follows from the fact that an ℓ-module is a subdirect product of its subdirectly irreducible homomorphic images.

It is sometimes convenient to work with unital modules, so we will show that this can always be arranged. For a po-ring R let R_* be the po-ring obtained by freely adjoining an identity to R. Thus, $R_* = R \oplus Z$ as a po-group with multiplication given by $(r, n)(s, m) = (rs + mr + ns, nm)$. Then R_* is a po-ring with a positive identity element $(0, 1)$. If M is an ℓ-module over R, then M becomes a unital ℓ-module over R_* if we define $x(r, n) = xr + nx$ for $x \in M$ and $(r, n) \in R_*$.

An ℓ-module M_R is called a *distributive ℓ-module* if for $x, y \in M$ and $r \in R^+$, $(x \vee y)r = xr \vee yr$. This is, of course, equivalent to saying that multiplication by $r \in R^+$ is a lattice homomorphism of M. Not every distributive ℓ-module is an f-module. The following proposition is well known for ℓ-rings [4, p. 59].

PROPOSITION 1.5. *If M_R is a distributive ℓ-module and if $x \in M$*

and $xR = 0$ implies $x = 0$, then M is an f-module.

Proof. Suppose that $x \wedge y = 0$ in M. Let $a, b \in R^+$ and let $c \in R^+$ with $c \geqq ab, b$. Then $0 \leqq (xa \wedge y)\ b \leqq xab \wedge yb \leqq xc \wedge yc = 0$. Therefore $(xa \wedge y)R = 0$, so $xa \wedge y = 0$.

COROLLARY 1.6. *The following statements are equivalent for the ℓ-module M_R.*

(a) *M is an f-module over R.*

(b) *M is a distributive ℓ-module over R_*.*

(c) *M is an f-module over R_*.*

Proof. The equivalence of (b) and (c) follows from 1.5. That (a) and (c) are equivalent follows from the fact that the R-submodules and the R_*-submodules of M are the same.

We mention next a special type of f-module that arises frequently. Suppose that $\phi\colon M \rightarrow \Pi_{i \in I} M_i$ is a representation of the f-module M_R as a subdirect product of the family of totally ordered R-modules $\{M_i\colon i \in I\}$. The representation is *irredundant* (and M is called an *irredundant f-module*) if, for each $i \in I$, the map $\phi_i\colon M \rightarrow \Pi_{j \neq i} M_j$ that is induced by ϕ has nonzero kernel. Using [12, p. 40] and Theorem 1.1 we immediately get

PROPOSITION 1.7. *The f-module M is an irredundant f-module if and only if its Boolean algebra of polars is atomic. If this is the case, then M is an irredundant subdirect product of the family of totally ordered modules $\{M/N\colon N$ a maximal polar of $M\}$, and this is the unique* (up to isomorphism) *irredundant representation of M.*

We close this section with a negative observation.

PROPOSITION 1.8. *Let R be a* (not necessarily directed) *po-ring with a set of positive matrix units $\{e_{ij}\colon i, j = 1, \cdots, n\}$ of degree $n > 1$. If M is an f-module over R, then $Me_{ij} = 0$ for all i, j.*

Proof. Since M is a subdirect product of totally ordered R-modules we may assume that M is itself totally ordered. If $x \in M$, then $N = R/r(x)$ is isomorphic to xR as an R-module. ($r(x) = \{r \in R\colon xr = 0\}$.) Thus N can be made into a totally ordered R-module.

Let $x_{ij} = e_{ij} + r(x) \in N$. Suppose $x_{ij} > 0$ for some i, j. Then $x_{ik} = x_{ij}e_{jk} > 0$ for all $k = 1, \cdots, n$. Since $x_{i_1} - x_{i_2} \geqq 0$ or $x_{i_1} - x_{i_2} \leqq 0$,

$-x_{i_1} = (x_{i_1} - x_{i_2})e_{21} > 0$ or $-x_{i_1} = (x_{i_1} - x_{i_2})(-e_{11}) > 0$, which is a contradiction. Similarly, $x_{ij} < 0$ for some i, j leads to a contradiction. Therfore $x_{ij} = 0$ for all i, j; i.e., $\{e_{ij}: i, j = 1, \cdots, n\} \subseteqq r(x)$. Since x was arbitrary we are done.

If R is a po-ring, then the n-by-n matrix ring over R, R_n, becomes a po-ring if its positive cone is defined by $R_n^+ = \{(a_{ij}): a_{ij} \in R^+$ for all $i, j\}$. The module M_R is said to be non-trivial if $MR \neq 0$.

COROLLARY 1.9. *If R is a* (not necessarily directed) *po-ring with a positive identity element and $n > 1$, then R_n has no nontrivial f-modules.*

If the matrix units are not positive, then 1.8 is false. For let

$$R = Z_2,\ R^+ = \left\{\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}: a \in Z^+\right\},\ M = \{(a, b): a, b \in Z\},$$

and

$$M^+ = \{(a, b): b > 0 \text{ or } b = 0 \text{ and } a \geqq 0\}.$$

Then M is a totally ordered R-module.

2. Finitely-valued f-modules. In this section we obtain all of the local structure theory for f-modules that Conrad [6] has obtained for ℓ-groups. In addition, it will be seen that there is a strong interaction between the ℓ-group and f-module structures of an f-module. Many of the arguments used in this section are modelled after those used by Conrad for the ℓ-group case.

The f-module M_R is a *lexicographic extension* of its convex ℓ-submodule N if N is a prime submodule of M, and $g \in M^+ \backslash N$ implies $g > N$ [5]. We write $M = \operatorname{lex} N$. The convex ℓ-submodule N of M is called a *lower submodule* if there is an $x \in M$ such that N is a maximal element in the set of convex ℓ-submodules not containing x. N is then called an *R-value of x.* It is clear that N is a lower submodule if and only if it is covered by some convex ℓ-submodule K; i.e., K is the smallest convex ℓ-submodule of M properly containing N. Also, $K = N + C_R(x)$ for any $x \in K \backslash N$, and N is an R-value of $x \in M$ if and only if $x \in K \backslash N$.

If N is a lower submodule covered by K, then K/N is an *ℓ-simple f-module*; i.e., K/N has exactly two convex ℓ-submodules. It is well kown that an ℓ-simple ℓ-group is isomorphic to an ℓ-subgroup of the reals (see [18, p. II 41] or [10, p. 74]). No such nice characterization of an ℓ-simple f-module is available, in general. If R is a commutative totally ordered integral domain with an identity element, Viswanathan [17] has given necessary and sufficient conditions for an ℓ-simple totally ordered R-module to be isomorphic to an ℓ-submodule of the completion of the quotient field of R.

A po-set is said to be *rooted* if the set of all elements which exceed a given element is totally ordered. A maximal totally ordered subset of a rooted po-set is called a *root*. By the *trunk* of a rooted po-set we shall mean the intersection of its roots.

Since a prime submodule is a prime subgroup, the po-set Γ_R of lower submodules of an f-module is rooted. Γ_R will be called the *R-value set* of M. Let $M_0(R)$ be the intersection of all of the elements of the trunk of Γ_R. For Γ_Z and $M_0(Z)$ we will write Γ and M_0, respectively. Weinberg [18, p. II 74] has shown that $M = \text{lex}\, M_0$ and that M_0 is trunkless for any ℓ-group M.

Another description of M_0 has been given by Conrad [6]. An element $g > 0$ of the f-module M is called a *nonunit* if its polar $g^\perp \neq 0$. Let N_0 be the subgroup of M generated by the set of nonunits of M. Then M is a lexicogaphic extension of a convex ℓ-subgroup K if and only if $K \supseteqq N_0$; and N_0 is the smallest convex ℓ-subgroup that is comparable with every convex ℓ-subgroup of M (smallest is to be understood as smallest nonzero if $N_0 \neq 0$). Since M_0 also has these properties, $M_0 = N_0$. It can be shown that M is a lexicographic extension of a convex ℓ-submodule K if and only if $K \supseteqq M_0(R)$. Since N_0 is a submodule of M, $M_0 = M_0(R)$. We collect this information in the following theorem.

THEOREM 2.1. *If M is an f-module over the po-ring R, then $M_0(R) = M_0 = N_0$. If K is a convex ℓ-submodule of M, then $M = \text{lex}\ K$ if and only if $K \supseteqq M_0$. If K is a nonzero convex ℓ-subgroup of M, then $M = \text{lex}\ K$ if and only if K is comparable with every convex ℓ-submodule of M.*

THEOREM 2.2. *Let g be a nonzero element of the f-module M over the directed po-ring R. If N is a value of g, then the largest convex ℓ-submodule of M contained in N is an R-value of g. This induces a natural one-to-one correspondence between the value set of g and the R-value set of g.*

Proof. Let N_1 be an R-value of g. Since $g \notin N_1$ and N_1 is a prime submodule, g has a unique value N containing N_1. Therefore, the correspondence $N_1 \rightarrow N$ is a well-defined mapping from the R-value set of g into its value set. Clearly N_1 is the largest convex ℓ-submodule of M contained in N, since N_1 is an R-value of g. Thus the mapping $N_1 \rightarrow N$ is one-to-one.

Now suppose that N is a value of g. Then N is a prime subgroup, and hence contains a minimal prime subgroup (the intersection of a maximal chain of prime subgroups contained in N). By Theorem 1.1 every minimal prime subgroup is a submodule, and hence N_1, the largest convex ℓ-submodule of M contained in N, is prime. If K is any convex ℓ-submodule of M properly containing N_1, then K properly contains N. For K and N are comparable, be primeness of N_1, and $K \nsubseteq N$, by maximality of N_1. Thus $g \in K$, and N_1 is an R-value of g.

Notice that, in general, there is no one-to-one correspondence between Γ and Γ_R. For let R be an ℓ-simple nonarchimedean f-ring, and let $M_R = R_R$. Then $\Gamma_R = \{0\}$, whereas Γ can be infinite.

We next show that, just as for ℓ-groups [6], there is a one-to-one correspondence between the R-values of g in M and the maximal convex ℓ-submodules of $C_R(g)$.

THEOREM 2.3. *For $0 \neq g \in M$ the map σ: $A \rightarrow A \cap C_R(g)$ is a one-to-one correspondence between the set of R-values of g in M and the set of maximal convex ℓ-submodules of $C_R(g)$. If N is a maximal convex ℓ-submodule of $C_R(g)$, then*

$$\sigma^{-1}(N) = \{x \in M: |xr| \wedge |g| \in N \textit{ for all } r \in R_*\} .$$

Proof. First note that N is a maximal convex ℓ-submodule of $C_R(g)$ if and only if N is an R-value of g in $C_R(g)$. Let $\{K_\alpha: \alpha \in A\}$ be the value set of g in M, and let $\{L_\alpha: \alpha \in A\}$ be the R-value set of g in M (where, of course, L_α is the largest convex ℓ-submodule of M contained in K_α). Now we have the theorem when $R = Z$ [6, Theorem 3.5]. Thus, $\{K_\alpha \cap C(g): \alpha \in A\}$ is the set of maximal convex ℓ-subgroups of $C(g)$, and $K_\alpha = \{x \in M: |x| \wedge |g| \in K_\alpha \cap C(g)\}$. Again, using this theorem with $M = C_R(g)$, $\{K'_\alpha: \alpha \in A\}$ is the set of values of g in $C_R(g)$, where $K'_\alpha = \{x \in C_R(g): |x| \wedge |g| \in K_\alpha \cap C(g)\}$. Thus, $\{K_\alpha \cap C_R(g): \alpha \in A\}$ is the set of values of g in $C_R(g)$. Clearly, $L_\alpha \cap C_R(g)$ is the largest convex ℓ-submodule of $C_R(g)$ contained in $K_\alpha \cap C_R(g)$, and so σ is one-to-one and onto by 2.2.

Finally, $L'_\alpha = \{x \in M: |xr| \wedge |g| \in L_\alpha \cap C_R(g)$ for all $r \in R_*\}$ is a convex ℓ-submodule of M since R is directed. Since $1 \in R_*$, $L'_\alpha \subseteq K_\alpha$. But if $x \in L_\alpha$ and $r \in R_*$, then $|xr| \wedge |g| \in L_\alpha \cap C_R(g)$. Thus $L_\alpha \subseteq L'_\alpha$, so $L_\alpha = L'_\alpha$ by maximality of L_α.

A nonzero element g of M is called *R-special* (*special*) if it has exactly one R-value (value) in M. Theorem 2.2 says that g is R-special exactly when it is special. If $N \in \Gamma_R (N \in \Gamma)$ is the unique R-value (value) of g in M, then N is called *R-special* (*special*) also.

The nonzero element g is called *basic* if $C_R(g)$ is totally ordered. Note that g is basic in M_R exactly when g is basic in M_Z.

THEOREM 2.4. *Let g be a nonzero element of the f-module M. The following statements are equivalent.*

(a) *g is R-special in M.*

(b) *g is R-special in $C_R(g)$.*

(c) *$C_R(g)$ is a lexicographic extension of a proper convex ℓ-submodule.*

(d) *g is special in M.*

(e) *g is special in $C(g)$.*

(f) *g is special in $C_R(g)$.*

(g) *$C(g)$ is a lexicographic extension of a proper convex ℓ-subgroup.*

If this is the case and if K (respectively N) is the unique R-value of g in M (respectively $C_R(g)$), then $N = K \cap C_R(g)$, $C_R(g) = \text{lex } N$, and $K = N \oplus g^{\perp}$.

Proof. The equivalence of (a) and (b) comes from 2.3. In [6] Conrad has proven that (d), (e), and (g) are equivalent, and thus that (e) and (f) are equivalent. The equivalence of (a) and (d) follows from 2.2.

If (b) is true, then $C_R(g)$ has a unique maximal convex ℓ-submodule N. Hence $C_R(g) = \text{lex } N$, since N is comparable to every convex ℓ-submodule of $C_R(g)$. Thus (b) implies (c). Conversely, if $C_R(g) = \text{lex } I$, then I is comparable to every convex ℓ-submodule of $C_R(g)$, and hence $C_R(g)$ has only one maximal convex ℓ-submodule.

Finally, Conrad has proven [5, Lemma 6.1] that if B is a convex ℓ-subgroup of M which is a lexicographic extension of a proper convex ℓ-subgroup, then $(B \oplus B^{\perp})^+ = \{x \in M^+ \colon x \text{ does not exceed every element of } B\}$. Since M is an f-module, $C_R(g)^{\perp} = g^{\perp}$. Thus, $(C_R(g) \oplus g^{\perp})^+ = \{x \in M^+ \colon x \text{ does not exceed every element of } C_R(g)\}$. Hence $K \subseteqq C_R(g) \oplus g^{\perp}$. But $g^{\perp} \subseteqq K$, so, by the modularity of the lattice of convex ℓ-submodules, $K = (K \cap C_R(g)) \oplus g^{\perp} = N \oplus g^{\perp}$.

COROLLARY 2.5. *Let g be a special element of the f-module M, and let N be the unique value of g in $C_R(g)$. Then N is the largest convex ℓ-subgroup of $C(g)$. Thus g has the same value in $C(g)$ as it has in $C_R(g)$.*

Proof. By 2.4 $N \cap C(g)$ is the largest convex ℓ-subgroup of $C(g)$, and $N = [N \cap C(g)] \oplus [g^{\perp} \cap C_R(g)]$. But $g^{\perp} \cap C_R(g) = 0$ since M is an f-module, so $N = N \cap C(g)$.

COROLLARY 2.6. *Suppose that g is special in M. Let K and N be the values of g in M and $C_R(g)$, respectively, and let K_1 and N_1 be the R-values of g in M and $C_R(g)$, respectively. Then K/K_1 and N/N_1 are isomorphic ℓ-groups, and K/N and K_1/N_1 are isomorphic ℓ-groups.*

Proof. $K = N \oplus g^{\perp}$ and $K_1 = N_1 \oplus g^{\perp}$ by 2.5 and 2.4.

COROLLARY 2.7. *If g is special but $C(g)$ is not totally ordered, then $C(g)$ contains a nonzero convex ℓ-submodule of M.*

Proof. If N is the value of g in $C_R(g)$ and N_1 is the R-value of g in $C_R(g)$, then $N_1 \subseteqq N \subseteqq C(g)$ by 2.5. Since N_1 is prime in $C_R(g)$ and thus in $C(g)$, $N_1 \neq 0$.

COROLLARY 2.8. *If g is special and N* (respectively N_1) *is the value* (respectively R-value) *of g in $C_R(g)$, then $C_R(g) = lex\ N$, $C_R(g) = lex\ C(g)$, and $C(g) = lex\ N_1$.*

Proof. $N_1 \subseteqq N \subseteqq C(g) \subseteqq C_R(g)$ by 2.5, and $C_R(g) = lex\ N_1$. So $C_R(g) = lex\ N$ and $C(g) = lex\ N_1$.

If g is special, then $C(g)/N$ is isomorphic to an ℓ-subgroup of the reals. In general, $C_R(g)/N_1$ and $C_R(g)/C(g)$ could be large. For instance, let R be an ℓ-simple f-ring, $M = R_R$, and $0 \neq g \in M$. Then $N_1 = 0$, $C_R(g) = M$, and so $C_R(g)/N_1 = M$, $C_R(g)/C(g) = M/C(g)$. In particular, if $R = Q[x]$ is ordered lexicographically with the highest term dominating, then R is ℓ-simple. If

$$0 \neq g = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

with $\alpha_n \neq 0$, then $C(g) = C(x^n)$. Thus, $C_R(g)/C(g) = R/C(x^n) \cong M$ as an ℓ-group.

The following result follows from Theorem 2.2 and from the known case $R = Z$ ([6, Theorem 3.7] and [7, p. 90]) (for a special case of the definition of *lex-sum*, see below; for the general definition see [7, p. 95]).

THEOREM 2.9. *A nonzero element g of the f-module M has only a finite number of R-values if and only if it is the finite sum of pairwise disjoint R-special elements. If $K_1, \cdots, K_n$ are the R-values of g in M, then $g = g_1 + \cdots + g_n$ where g_i is special with R-value K_i, and $|g_i| \wedge |g_j| = 0$ for $i \neq j$. This decomposition of g into disjoint special elements is unique. Finally,*

$$C_R(g) = C_R(g_1) \oplus \cdots \oplus C_R(g_n) ,$$

and $g^{\perp\perp}$ *is the lex-sum of* $g_1^{\perp\perp}, \cdots, g_n^{\perp\perp}$.

Note that if g has only a finite number of R-values but is not R-special, then the values of g in $C(g)$ and in $C_R(g)$ need not be the same. Thus, 2.5 does not generalize. Theorem 2.9 implies that if g has only a finite number of R-values, then each R-value of g is special. Just as for ℓ-groups, the converse of this statement is true.

THEOREM 2.10. *A nonzero element g of the f-module M has only a finite number of R-values if and only if each R-value of g is R-special.*

Proof. For a family $\{K_\alpha: \alpha \in A\}$ of convex ℓ-submodules of M let $V_R\{K_\alpha\} = \{x \in M$: each R-value of x is a submodule of K_α for some $\alpha \in A\} \cup \{0\}$. Then $V_R\{K_\alpha\}$ is a convex ℓ-submodule of M. Also, $K \in \Gamma_R$ is R-special if and only if $V_R(K) \nsubseteq K$. For if K is the unique R-value of x, then clearly $x \in V_R(K)\backslash K$. Conversely, if $x \in V_R(K)\backslash K$, then x has an R-value containing K, which must be K, since every R-value of x is contained in K. Thus, K is the only R-value of x.

Now suppose that each element of $\Gamma_R(g) = \{K_\alpha: \alpha \in A\}$ is R-special. Let $L = \sum_{\alpha \in A} V_R(K_\alpha)$. If $g \notin L$, then g has an R-value containing L, i.e., $L \subseteqq K_\alpha$ for some $\alpha \in A$. Thus $V_R(K_\alpha) \subseteqq K_\alpha$, which contradicts the hypothesis that K_α is R-special. So $g \in L$, and

$$g \in V_R(K_{\alpha_1}) + \cdots + V_R(K_{\alpha_n}) \subseteqq V_R(K_{\alpha_1}, \cdots, K_{\alpha_n}) .$$

Thus, every R-value of g is contained in one of the K_{α_i}, so $K_{\alpha_1}, \cdots, K_{\alpha_n}$ are the only R-values of g.

Following Conrad [6], we say that a lattice L is *generated by its set of meet irreducible elements* S if every element of L is the greatest lower bound of a dual ideal of S. L is *freely generated* by S if every element of L is the greatest lower bound of a unique dual ideal of S. Conrad has shown that the lattice of convex ℓ-subgroups of an ℓ-group M is freely generated by Γ if and only if M is finitely-valued, i.e., each element of M has at most a finite number of values. It is, of couse, no surprise that this result holds for f-modules.

A lattice L is *completely distributive* if the following equation and its dual hold in L, provided the indicated joins and meets exist:

$$\bigwedge_{i \in I} \bigvee_{j \in J} x_{ij} = \bigvee_{\phi \in J^I} \bigwedge_{i \in I} x_{i\phi(i)} .$$

THEOREM 2.11. *For an f-module M with lattice of convex ℓ-submodules $\mathscr{L}(M_R)$ the following statements are equivalent.*

(a) *Γ_R freely generates $\mathscr{L}(M_R)$.*

(a′) *Γ freely generates $\mathscr{L}(M_Z)$.*

(b) *$\mathscr{L}(M_R)$ is completely distributive.*

(b′) *$\mathscr{L}(M_Z)$ is completely distributive.*

(c) *$(B \vee (\bigwedge_\alpha A_\alpha) = \bigwedge_\alpha(B \vee A_\alpha)$ for every subset $\{A_\alpha, B\}$ of Γ_R.*

(c′) *$B \vee (\bigwedge_\alpha A_\alpha) = \bigwedge_\alpha(B \vee A_\alpha)$ for every subset $\{A_\alpha, B\}$ of Γ.*

(d) *Each element of Γ_R is special.*

(d′) *Each element of Γ is special.*

(e) *Each element of M has at most a finite number of R-values.*

(e′) *Each element of M has at most a finite number of values.*

(f) *Each element of M has a unique representation as the sum of a finite number of pairwise disjoint R-special elements.*

(f′) *The same as* (f) *with R replaced by Z.*

Proof. The equivalence of (a), (b), and (c), and that of (a′), (b′), (c′), (d′), (e′), and (f′) is proven in [6]. The equivalence of (d), (e), and (f) follows from 2.10 and 2.9. Theorem 2.2 implies that (e) and (e′) are equivalent. Since $\mathscr{L}(M_R)$ is a complete sublattice of $\mathscr{L}(M_Z)$, (b′) implies (b).

Now suppose that (a) is true and let $K \in \Gamma_R$. Then $\Delta_1 = \{N \in \Gamma_R: N \not\subseteq K\}$ and $\Delta_2 = \Delta_1 \cup \{K\}$ are distinct dual ideals of Γ_R. If $\cap \{N: N \in \Delta\} \subseteq K$, then $\cap \{N: N \in \Delta_1\} = \cap \{N: N \in \Delta_2\}$, contradicting (a). Thus there exists $g \in \cap \{N: N \in \Delta_1\} \backslash K$. Let L be an R-value of g containing K. If $L \supsetneqq K$, then $L \in \Delta_1$ and $g \in L$. Thus $L = K$. If P is any other R-value of g, then $P \in \Delta_1$, so $g \in P$. Thus K is R-special, and (a) implies (d).

The concept of finitely-valued is strongly related to that of direct sum.

THEOREM 2.12. *The f-module M_R is a direct sum of totally ordered R-modules if and only if it is finitely-valued and each special element is basic.*

Proof. Suppose that M is finitely-valued and each special element is basic. If g is a basic element, then $g^{\perp\perp}$ is totally ordered ([7, p. 88] or [1, Lemma 1]). Thus M is the sum of its totally ordered convex ℓ-submodules. Since two totally ordered convex ℓ-submodules are disjoint or comparable [7, 3.1], M is the direct sum of totally ordered R-modules.

The converse is trivial.

The following theorem shows that finitely-rooted f-modules can be built up from a finite family of totally ordered modules by means of direct sums and lexicographic extensions. When $R = Z$ and when the lattice of convex ℓ-subgroups of M has finite length the theorem is due to Birkhoff. It has been generalized to non-abelian ℓ-groups (without the assumption of finite length) by Conrad. Our proof is modelled after a proof that Weinberg [18, p. II 75] has given for the non-abelian case.

Observe that the f-module M_R has at most a finite number of roots exactly when M_Z has at most a finite number of roots. More generally, there is a one-to-one correspondence between the roots of Γ and the roots of Γ_R. For there is a one-to-one correspondence between the roots of Γ_R and the minimal prime submodules of M_R given by $M_\alpha \to \cap \{C: C \in M_\alpha\}$. Since each minimal prime subgroup of M is a submodule (1.1), the above correspondence establishes a bijection between the roots of Γ and those of Γ_R.

An f-module M_R is a *lexicosum* of the family of totally ordered R-modules $\{M_\alpha: \alpha \in A\}$ if it is in the smallest class $\mathscr{L}$ of f-modules containing $\{M_\alpha: \alpha \in A\}$ and satisfying

(1) If $A, B \in \mathscr{L}$, then $A \oplus B \in \mathscr{L}$.

(2) If $K = \text{lex } L$, where L and $K/L \in \mathscr{L}$, then $K \in \mathscr{L}$.

LEMMA 2.13. (Weinberg). *Let Γ be a rooted po-set with only a finite number of roots. If the trunk of Γ is empty, then Γ is the cardinal sum of two nonempty subsets.*

THEOREM 2.14. *Let M be an f-module over the directed po-ring R, and suppose that Γ_R has only a finite number of roots. Then M is a lexicosum of a finite number of totally ordered modules, and only a finite number of extensions are needed to get to M.*

Proof. Suppose that Γ_R has n roots. If $n = 1$, then M is totally ordered. Suppose that $n > 1$ and the theorem is true for M with less than n roots. If the trunk of Γ_R is empty, then Γ_R is the cardinal sum of non-empty subsets A and B, by 2.13. Let $M_1 = \cap \{C_\alpha: \alpha \in A\}$, $M_2 = \cap \{C_\beta: \beta \in B\}$. Then $M_1 + M_2 = \bigcap_\alpha C_\alpha + \bigcap_\beta C_\beta = \bigcap_\alpha \bigcap_\beta (C_\alpha + C_\beta)$, by 2.11. If $C_\alpha + C_\beta \subsetneqq M$, then $C_\alpha + C_\beta \subseteqq D$ for some $D \in \Gamma_R = A \cup B$. This clearly cannot happen, so $M = C_\alpha + C_\beta$. Thus $M = M_1 \oplus M_2$. Since $\Gamma_R(M)$ is the cardinal sum of $\Gamma_R(M_1)$ and $\Gamma_R(M_2)$, $\Gamma_R(M_1)$ and $\Gamma_R(M_2)$ each has less than n roots. So we are done, by induction.

If the trunk of Γ_R is not empty, then $M = lex\, M_0$ and M_0 is trunkless by 2.1. Since every minimal prime submodule of M is contained in M_0, M_0 has n roots, also. Thus the previous case applies to M_0.

3. Applications to f-rings.

An f-ring is a lattice-ordered ring S such that S_S and ${}_S S$ are both f-modules. (If S_S is an f-module, then ${}_SS$ need not be an f-module.) This two-sided condition may be reduced to a one-sided one. For $a \in S$ let L_a be the map defined by $(s)L_a = as$, and let T_a be the map defined by $(s)T_a = sa$. Then the subring R of $\mathrm{Hom}_Z(S, S)$ generated by the set $\{T_a, L_a \colon a \in S\}$ is a directed po-ring if its positive cone is defined by

$$R^+ = \{f \in R \colon S^+f \subseteqq S^+\} .$$

Now S is an f-ring if and only if S_R is an f-module. The convex ℓ-submodules of S_R are, of course, the ℓ-ideals of S. All of the preceding theory now applies to S_R.

Using Theorem 2.12 and Proposition 1.7 we can obtain generalizations of the results in [1]. In that paper Anderson studied f-rings S that satisfy the ascending chain condition for polars. Since Polar (S) is a Boolean algebra, this condition is equivalent to Polar (S) being finite, which is a special case of Polar (S) being atomic. In particular, we have that the f-ring S is an irredundant subdirect product of totally ordered rings exactly when Polar (S) is atomic. If S is semiprime, then it is an irredundant subdirect product of totally ordered domains exactly when Polar (S) is atomic. (This also follows from [13, Theorem 3.2].) Also, a finitely-valued semiprime f-ring is a direct sum of totally ordered domains if and only if each special element is basic. An example due to Anderson [1, p. 718] shows that a semiprime finitely-rooted f-ring need not have this latter property.

A more interesting situation arises if we assume that S is Johnson semisimple, i.e., that $J(S)$, the intersection of the regular maximal ℓ-ideals of S, is zero (see [13] for the theory of the Johnson radical for f-rings). The following theorem, and also Proposition 3.3, have analogues in the theory of archimedean ℓ-groups.

THEOREM 3.1. *The following statements are equivalent for an f-ring S.*

(a) *S is isomorphic to an f-subring of a direct product of a family of ℓ-simple unital f-rings that contains their direct sum.*

(b) *S is Johnson semisimple and its Boolean algebra of polars is atomic.*

Proof. Suppose that (b) holds, and let $\{P_\alpha \colon \alpha \in A\}$ be the set of maximal polars of S. By 1.7 $\bigcap \{P_\alpha \colon \alpha \in A\} = 0$. By [1, Lemma 6], $S = P_\alpha \oplus P_\alpha^\perp$ and $P_\alpha^\perp$ is an ℓ-simple unital f-ring. We have the isomorphism $\phi \colon S \to \prod_{\alpha \in A} P_\alpha^\perp$ which is induced by the projections $p_\alpha \colon S \to P_\alpha^\perp$. Clearly, $\sum_{\alpha \in A} \oplus P_\alpha^\perp \subseteqq \phi(S)$. Thus (b) implies (a).

That (a) implies (b) follows from 1.7 and from the fact that the Johnson radical is a radical.

Note that in an f-ring satisfying the conditions of 3.1 a regular maximal ℓ-ideal need not be a polar. For an example, let R_α be an ℓ-simple unital f-ring for each α in the infinite set A. Then $\sum_{\alpha \in A} \oplus R_\alpha$ is a regular maximal ℓ-ideal of the f-ring

$$\sum_{\alpha \in A} \oplus R_\alpha + Q1 \subseteqq \prod_{\alpha \in A} R_\alpha ,$$

but it is not a polar.

LEMMA 3.2. *If S is a unitable f-ring and e is an idempotent of S, then Se and $S(1-e)$ are ℓ-ideals of S.*

Proof. Since e is central [11, 2.1], Se is an ideal. Since $(re)^+ = r^+e$, Se is a sublattice. Suppose that $0 \leqq x \leqq re$, and let $\phi(S)$ be a totally ordered image of S. Then [11, 2.1] $\phi(e) = 0$ or 1, *so* $\phi(x) = \phi(x)\phi(e)$. Thus $x = xe \in Se$, and Se is an ℓ-ideal. Since $S(1-e)$ is the annihilator of e, it is an ℓ-ideal, also.

PROPOSITION 3.3. *If $J(S) = 0$, then every special element of S is basic.*

Proof. Let $g \in S$ *be* special. Then, by 2.4, $C_R(g) = \operatorname{lex} N$, where N is the maximal ℓ-ideal of S contained in $C_R(g)$. If K is a regular maximal ℓ-ideal of $C_R(g)$, then K is an ℓ-ideal of S, since $C_R(g)/K$ is semiprime [9, Lemma 61]. Thus $K = N$, and $J(C_R(g)) = N$. But $J(C_R(g)) = C_R(g) \cap J(S) = 0$ [13, Theorem 4.16]. Thus $N = 0$, and g is basic.

COROLLARY 3.4. *If $J(S) = 0$ and g is special in S, then there is an idempotent $e \in S$ such that $C_R(g) = Se$. Also, Se is an ℓ-simple f-ring.*

Proof. By 3.3 $C_R(g)$ is a totally ordered Johnson semisimple f-ring, hence a unital ℓ-simple f-ring. Let e be the identity of $C_R(g)$. Since S is unitable [13, Theorem 3.6], Se is an ℓ-ideal. Thus $Se = C_R(g)$.

The following corollary follows immediately from 2.12 and 3.3.

COROLLARY 3.5. *An f-ring is the direct sum of unital ℓ-simple f-rings if and only if it is finitely-valued and Johnson semisimple.*

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