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THE UNIFORMIZING FUNCTION FOR A CLASS OF RIEMANN SURFACES

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This paper considers a class of simply connected Riemann surfaces which are shown to be of parabolic type. Infinite product representations are obtained for both the uniformizing function and its derivative.

The class of surfaces. For each integer $n \geqq 1$ let $[a_{2n-1}, b_{2n-1}]$ and $[b_{2n}, a_{2n}]$ denote closed intervals of the real line satisfying $0 < a_{2n-1} < b_{2n-1} < b_{2n}$ and $b_{2n+1} < b_{2n} < a_{2n}$. Let S_n denote a copy of the w-sphere. Slit S_1 along $[a_1, b_1]$, slit S_{2n} along both $[a_{2n-1}, b_{2n-1}]$ and $[b_{2n}, a_{2n}]$, and slit S_{2n+1} along $[a_{2n+1}, b_{2n+1}]$ and $[b_{2n}, a_{2n}]$. A surface F belonging to the class is constructed by joining S_{2n-1} to S_{2n} along $[a_{2n-1}, b_{2n-1}]$ and S_{2n} to S_{2n+1} along $[b_{2n}, a_{2n}]$ with the intervals forming first order branch lines.

The uniformizing function. F is a simply connected, open Riemann surface and is thus either parabolic or hyperbolic. There is a unique analytic one-to-one mapping $f(z)$ which maps $\{|z| < r \leqq \infty\}$ onto F and satisfies $f(0) = 0 \in S_1$ and $f'(0) = 1$. An argument similar to that in [2, p. 1137] shows that $f(z)$ is real if z is real. For notation let $f(\delta_k) = 0 \in S_k$, $f(\gamma_k) = \infty \in S_k$, $f(\alpha_k) = a_k$ and $f(\beta_k) = b_k$. The image of S_1 under $f^{-1}(z)$ is a region containing the origin and bounded by a Jordan curve C_1 which is symmetric about the real axis. For $n > 1$ the image of S_n is an annular region about the origin bounded by two Jordan curves, C_{n-1} and C_n, each symmetric about the real axis. For $n \geqq 1$, C_n intersects the real axis at α_n and β_n only. Furthermore,

$$\beta_{n+1} < \beta_n < \gamma_1 < 0 < \alpha_{2n-1} < \delta_{2n} < \gamma_{2n} < \alpha_{2n} < \gamma_{2n+1} < \delta_{2n+1} < \alpha_{2n+1} .$$

The closed surfaces and rational functions. Let F_n denote the surface formed from the first $2n$ sheets of F with the cut along $[b_{2n}, a_{2n}]$ on S_{2n} deleted. F_n is an elliptic surface so there is a unique rational function $R_n(z)$ mapping the z-sphere one-to-one and onto F_n which satisfies $R_n(0) = 0 \in S_1$, $R_n(\infty) = \infty \in S_{2n}$ and $R'_n(0) = 1$. For notation let $R_n(\delta_{k,n}) = 0 \in S_k$, $R_n(\gamma_{k,n}) = \infty \in S_k$, $R_n(\alpha_{k,n}) = a_k$ and $R_n(\beta_{k,n}) = b_k$. Also, throughout the following the notation $1 - z/\alpha_\phi = \alpha^*_\phi$, $1 - z/\beta_\phi = \beta^*_\phi$, $1 - z/\gamma_\phi = \gamma^*_\phi$ and $1 - z/\delta_\phi = \delta^*_\phi$ is used. Then

$$R_n(z) = [z/\gamma^*_{1,n}] \prod_{k=2}^{2n-1} [\delta^*_{k,n}/\gamma^*_{k,n}]\, \delta^*_{2n,n}$$

and

$$R_n'(z) = \prod_{k=1}^{2n-1} [\alpha_{k,n}^* \beta_{k,n}^* / (\gamma_{k,n}^*)^2]$$

since $R_n(z)$ and $R_n'(z)$ must contain exactly these factors. The zeros and poles of $R_n(z)$ and the points corresponding to the branch points of F_n are real and their ordering is similar to that for $f(z)$.

LEMMA 1. *F is parabolic.*

Proof. Let D_n be the plane with $(-\infty, \beta_{2n-1\ n}]$ on the real axis deleted. Let Δ_n be the domain in the plane which is the interior of the curve C_{2n} excluding the segments $[\beta_{2n}, \beta_{2n-1}]$ and $[\gamma_{2n}, \alpha_{2n}]$. Then $\psi_n(z) = f^{-1}[R_n(z)]$ maps D_n onto Δ_n. An argument similar to that in [2, p. 1138] shows that F cannot be hyperbolic so that F is parabolic. Thus $f(z)$ maps the plane onto F. Furthermore, the sequence $\{D_n\}$ converges to its kernel which is the plane.

LEMMA 2. *$R_n(z) \rightarrow f(z)$ subuniformly (uniformly on compact subsets) in the plane as $n \rightarrow \infty$. Furthermore, $\delta_{k,n} \rightarrow \delta_k$, $\gamma_{k,n} \rightarrow \gamma_k$, $\alpha_{k,n} \rightarrow \alpha_k$ and $\beta_{k,n} \rightarrow \beta_k$ as $n \rightarrow \infty$.*

Proof. Since the sequences of domains $\{D_n\}$ and $\{\Delta_n\}$ converge to their kernels which in both cases is the plane then the sequence $\{f^{-1}[R_n(z)]\}$ converges subuniformly in the plane [3, p. 18] to the identity. Hence $R_n(z) \rightarrow f(z)$ subuniformly in the plane. It follows from Hurwitz's theorem that $\delta_{k,n} \rightarrow \delta_k$, $\gamma_{k,n} \rightarrow \gamma_k$, $\alpha_{k\ n} \rightarrow \alpha_k$ and $\beta_{k,n} \rightarrow \beta_k$ as $n \rightarrow \infty$.

LEMMA 3. *The infinite product*

$$\Pi(z) = (z/\gamma_1^*) \prod_{k=2}^{\infty} (\delta_k^*/\gamma_k^*)$$

converges subuniformly in the plane.

Proof. Since $\delta_k \rightarrow \infty$ and $\gamma_k \rightarrow \infty$ as $k \rightarrow \infty$ then if $R > 0$ there is an integer $n_0 = n_0(R) > 1$ such that for $k \geqq n_0 - 1$ both $\delta_k > R$ and $\gamma_k > R$. Thus, $\log[\delta_k^*/\gamma_k^*]$ is defined for $|z| \leqq R$. Since for $k \geqq 1$, $0 < \delta_{2k} < \gamma_{2k} < \gamma_{2k+1} < \delta_{2k+1}$, then for $n \geqq 0$, $p \geqq 0$ and $|z| \leqq R$,

$$\begin{aligned} \left| \sum_{k=n_0+n}^{n_0+n+p} \log(\delta_k^*/\gamma_k^*) \right| &= \left| \sum_{m=1}^{\infty} (z^m/m) \sum_{k=n_0+n}^{n_0+n+p} (1/\gamma_k^m - 1/\delta_k^m) \right| \\ &\leqq \sum_{m=1}^{\infty} (R/\delta_{n_0+n-1})^m = R/(\delta_{n_0+n-1} - R) . \end{aligned}$$

Because $R/(\delta_{n_0+n-1} - R) \to 0$ as $n \to \infty$ then the uniform Cauchy criterion is satisfied in $|z| \leqq R$ by the infinite series $\sum_{k=n_0}^{\infty} \log [\delta_k^*/\gamma_k^*]$. This is sufficient for $\Pi(z)$ to converge subuniformly in the plane.

LEMMA 4. $\Pi(z) = f(z)$.

Proof. Because $\gamma_{k,n} \to \gamma_k$ and $\delta_{k,n} \to \delta_k$ as $n \to \infty$ there exists $R > 0$ and $N > 0$ such that if $n > N$ and $|z| \leqq R$ the quotient $R_n(z)/\Pi(z)$ is nonzero and analytic with value 1 at $z = 0$. Thus, using the principal value of the logarithm,

$$\log [R_n(z)/\Pi(z)] = \log (\gamma_1^*/\gamma_{1,n}^*) + \sum_{m=1}^{\infty} c_m z^m \text{ where for } 2 < p \leqq 2n - 1\,,$$

$$c_m = c_m(n) = \frac{1}{m}\Big[1/\delta_{2n,n}^m + \sum_{k=2}^{p-1} (1/\gamma_k^m - 1/\delta_k^m - 1/\gamma_{k,n}^m + 1/\delta_{k,n}^m)$$
$$+ \sum_{k=p}^{\infty} (1/\gamma_k^m - 1/\delta_k^m) - \sum_{k=p}^{2n-1} (1/\gamma_{k,n}^m - 1/\delta_{k,n}^m)\Big].$$

Because

$$\Big|\sum_{k=p}^{\infty} (1/\gamma_k^m - 1/\delta_k^m)\Big| < 1/\delta_{p-1}^m$$

and

$$\Big|\sum_{k=p}^{2n-1} (1/\gamma_{k,n}^m - 1/\delta_{k,n}^m)\Big| < 1/\delta_{p-1,n}^m$$

then $|c_m(n)| \leqq |1/\delta_{2n,n}^m|$

$$+ \Big|\sum_{k=2}^{p-1} (1/\gamma_k^m - 1/\delta_k^m - 1/\gamma_{k,n}^m + 1/\delta_{k,n}^m)\Big| + 1/\delta_{p-1}^m + 1/\delta_{p-1,n}^m\,.$$

This bound for $c_m(n)$ has limit $2/\delta_{p-1}^m$ as $n \to \infty$ and $2/\delta_{p-1}^m \to 0$ as $p \to \infty$. Hence $c_m(n) \to 0$ as $n \to \infty$. The convergence of $\{\log [R_n(z)/\Pi(z)]\}$ is subuniform in the plane and $\gamma_{1,n} \to \gamma_1$ as $n \to \infty$. Thus, as $n \to \infty$, $\lim \log [R_n(z)/\Pi(z)] = \log [f(z)/\Pi(z)] = 0$ so that $f(z) = \Pi(z)$.

LEMMA 5. *The sequences* $A_n = \sum_{k=1}^{2n-1} 1/\alpha_{k,n}$, $B_n = \sum_{k=1}^{2n-1} 1/\beta_{k,n}$ *and* $C_n = \sum_{k=1}^{2n-1} 1/\gamma_{k,n}$ *are bounded.*

Proof. There is some $R > 0$ such that $R_n'(z) \neq 0$ if $|z| \leqq R$ and thus,

$$\log R_n'(z) = \sum_{m=1}^{\infty} (z^m/m) \sum_{k=1}^{2n-1} (2/\gamma_{k,n}^m - 1/\alpha_{k,n}^m - 1/\beta_{k,n}^m)\,.$$

Let $\nu = \nu_n$ denote the coefficient of z in this series expansion. For $n \geqq 1$ and $k > 1$, $0 < \gamma_{k,n} < \alpha_{k,n}$ and for $k \geqq 1$, $\beta_{k,n} < 0$ so that, $2/\gamma_{1,n} -$

$1/\alpha_{1,n} - \nu_n = B_n + \sum_{k=2}^{2n-1}(1/\alpha_{k,n} - 1/\gamma_{k,n}) - \sum_{k=2}^{2n-1} 1/\gamma_{k,n} < B_n < 0$. As $n \to \infty$, $\log R_n'(z) \to \log f'(z)$ subuniformly in the plane and thus,

$$-\infty < \lim(2/\gamma_{1,n} - 1/\alpha_{1,n} - \nu_n) \leqq \lim\inf B_n \leqq 0.$$

Hence the sequence $\{B_n\}$ is bounded. The remaining two sequences are bounded below and the inequalities $C_n < \nu_n + 1/\alpha_{1,n} - 1/\gamma_{1,n}$ and $A_n < C_n + 1/\alpha_{1,n} - 1/\gamma_{1,n}$ show they are bounded above.

LEMMA 6. *The series* $\sum_{k=1}^{\infty} 1/\beta_k$, $\sum_{k=1}^{\infty} 1/\gamma_k$, $\sum_{k=1}^{\infty} 1/\alpha_k$ *and* $\sum_{k=2}^{\infty} 1/\delta_k$ *are convergent.*

Proof. Each of these series is monotone. Using Lemma 2 and the notation and results of Lemma 5 it follows for $p \geqq 1$ that as $n \to \infty$,

$$-\infty < \lim\inf B_n \leqq \lim \sum_{k=1}^{p} 1/\beta_{k,n} = \sum_{k=1}^{p} 1/B_k < 0 .$$

Thus the first series converges. $\sum_{k=1}^{\infty} 1/\gamma_k$ converges since it is monotone increasing and for $p \geqq 1$ and $n \to \infty$,

$$\sum_{k=1}^{p} 1/\gamma_k = \lim \sum_{k=1}^{p} 1/\gamma_{k,n} \leqq \lim\sup C_n < \infty .$$

The remaining two series have positive terms and are dominated by convergent series since for $k > 1$, $0 < 1/\delta_{k+1} < 1/\alpha_k < 1/\gamma_k$. Thus, they also converge.

LEMMA 7. *The infinite product*

$$Q(z) = \prod_{k=1}^{\infty} [\alpha_k^* \beta_k^* / (\gamma_k^*)^2]$$

converges subuniformly in the plane.

Proof. This follows from Lemma 6.

As a further consequence of Lemma 6 both $Q(z)$ and $\Pi(z)$ may also be expressed as a quotient of products.

LEMMA 8. $f'(z) = Q(z)\exp(\delta z)$ *with* δ *real.*

Proof. For some $R>0$ both $Q(z)$ and $R_n'(z)$ are analytic and non-zero in $|z| < R$. Hence, for $|z| < R$,

$$\log [R_n'(z)/Q(z)] = \sum_{m=1}^{\infty} (z^m/m) \Big[\sum_{k=1}^{\infty} 1/\alpha_k - \sum_{k=1}^{2n-1} 1/\alpha_{k,n}^m + \sum_{k=1}^{\infty} 1/\beta_k - \sum_{k=1}^{2n-1} 1/\beta_{k,n}^m - \sum_{k=1}^{\infty} 2/\gamma_k^m + \sum_{k=1}^{2n-1} 2/\gamma_{k,n}^m \Big] .$$

From Lemmas 5 and 6 there exists $M > 0$ such that for $n \geqq 1$

$$\sum_{k=1}^{2n-1} 1/\alpha_{k,n} < M \text{ and } \sum_{k=1}^{\infty} 1/\alpha_k < M .$$

Also, for $k \geqq 1$ and $n \geqq 1$, $\alpha_k < \alpha_{k+1}$ and $\alpha_{k,n} < \alpha_{k+1,n}$ so that

$$k/\alpha_k < \sum_{p=1}^{\infty} 1/\alpha_p < M$$

and

$$k/\alpha_{k,n} < \sum_{p=1}^{2n-1} 1/\alpha_{p,n} < M .$$

Thus, for $p \geqq 1$, $|\sum_{k=p}^{\infty} 1/\alpha_k^m - \sum_{k=p}^{2n-1} 1/\alpha_{k,n}^m|$

$$< \sum_{k=p}^{\infty} (M/k)^m + \sum_{k=p}^{\infty} (M/k)^m = 2M^m \sum_{k=p}^{\infty} (1/k)^m .$$

This last expression has limit zero as $p \to \infty$ provided $m \geqq 2$. Thus, for $m \geqq 2$, it follows that as $n \to \infty$,

$$\lim \left[\sum_{k=1}^{\infty} 1/\alpha_k^m - \sum_{k=1}^{2n-1} 1/\alpha_{k,n}^m \right]$$
$$= \lim \left[\sum_{k=p}^{\infty} 1/\alpha_k^m - \sum_{k=p}^{2n-1} 1/\alpha_{k,n}^m \right] = 0 .$$

Similar arguments show that as $n \to \infty$ and provided $m \geqq 2$,

$$\lim \left[\sum_{k=1}^{\infty} 1/\beta_k^m - \sum_{k=1}^{2n-1} 1/\beta_{k,n}^m \right] = \lim \left[\sum_{k=1}^{\infty} 1/\gamma_k^m - \sum_{k=1}^{2n-1} 1/\gamma_{k,n}^m \right] = 0 .$$

Hence, if δ denotes the limit as $n \to \infty$ of the coefficient of z in the expansion of $\log [R_n'(z)/Q(z)]$ then as $n \to \infty$, $\delta z = \lim \log [R_n'(z)/\Pi(z)] = \log [f'(z)/Q(z)]$ so that $f'(z) = Q(z) \exp (\delta z)$.

LEMMA 9. $\delta = 0$.

Proof. Since $Q(z)$ is composed of cannonical products of genus zero then for $\varepsilon > 0$ there exists $R > 0$ such that if $|z| > R$ and $0 < \rho < |\arg z| < \pi - \rho$ then $|Q(z)| \leqq \exp (\varepsilon |z|)$ and $1/|Q(z)| \leqq \exp (\varepsilon |z|)$. Thus,

$$\exp (\delta \mathscr{R}(z) - \varepsilon |z|) \leqq |f'(z)| \leqq \exp (\delta \mathscr{R}(z) + \varepsilon |z|) .$$

Let V_1 and V_2 denote open sectors in the first and second quadrants, respectively, with vertex at the origin and sides contained in the open quadrant. If $\delta < 0$ and $z \in V_1$ then $\mathscr{R}(z) > 0$ and there exists $\phi_1 > 0$ such that for $|z| > R$, $|f'(z)| \geqq \exp (\phi_1 |z|)$. Let r_n denote the distance from the origin to the portion of the curve C_n in V_1 and let z_n and ζ_n denote the intersection of C_n with the sides of V_1 where $0 < \theta =$

$\arg \zeta_n - \arg z_n$. For n sufficiently large,

$$b_{2n+1} - a_{2n+1} > f(\zeta_{2n+1}) - f(z_{2n+1})$$
$$= \int_{z_{2n+1}}^{\zeta_{2n+1}} f'(z)\,|dz| \geqq \theta r_{2n} \exp(\phi_1 r_{2n})$$

where the integral is along C_{2n+1}. If $n \to \infty$ then $\theta r_{2n} \exp(\phi_1 r_{2n}) \to \infty$ and since $a_{2n+1} > 0$ then $b_{2n+1} \to \infty$.

However, if $z \in V_2$ then $\mathscr{R}(z) < 0$ and there exists $\phi_2 > 0$ such that for $|z| > R$, $|f'(z)| \leqq \exp(-\phi_2 |z|)$. It follows that $f(z)$ is bounded in V_2. If $z \in V_2$ and $z \in C_{2n}$ then $0 < b_{2n} < f(z)$ so that $\{b_{2n}\}$ is bounded. This is a contradiction since $b_{2n+1} < b_{2n}$. Thus $\delta \leqq 0$. A similar argument shows $\delta \geqq 0$ so that $\delta = 0$.

THEOREM. *A Riemann surface belonging to the class described is parabolic and a uniformizing function $f(z)$ for a member of the class has the representation*

$$f(z) = (z/\gamma_1^*) \prod_{k=2}^{\infty} (\delta_k^*/\gamma_k^*) .$$

The derivative has the representation

$$f'(z) = \prod_{k=1}^{\infty} [\alpha_k^* \beta_k^* / (\gamma_k^*)^2] .$$

For $k \geqq 1$,

$$\beta_{k+1} < \beta_k < \gamma_1 < 0 < \alpha_{2k-1} < \delta_{2k} < \gamma_{2k} < \alpha_{2k} < \gamma_{2k+1} < \delta_{2k+1} < \alpha_{2k+1} .$$

Furthermore, $\sum_{k=1}^{\infty} 1/\alpha_k$, $\sum_{k=1}^{\infty} 1/\beta_k$, $\sum_{k=1}^{\infty} 1/\gamma_k$ and $\sum_{k=2}^{\infty} 1/\delta_k$ converge.

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REFERENCES

1. H. B. Curtis, Jr., *Some properties of functions which uniformize a class of simply connected Riemann surfaces*, Proc. Amer. Math. Soc., **10** (1959), 525-530.

2. ———, *The uniformizing function for certain simply connected Riemann surfaces*, Pacific J. Math., **15** (1965), 1137-1144.

3. G. R. MacLane, *Riemann surfaces and asymptotic values associated with real entire functions*, The Rice Institute Pamphlet (1952).

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