PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

DEGREES OF MEMBERS OF Π_1^0 CLASSES

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This paper deals with the degrees of members of Π_1^0 classes of sets or function which lack recursive members. For instance, it is shown that if a is a degree and $0 < a \leqq 0'$, then there exists a Π_1^0 class of sets which has a member of degree a but none of degree 0. By way of contrast there is no Π_1^0 class of functions which has members of all nonzero r.e. degrees but no recursive members. Furthermore, each nonempty Π_1^0 class of sets has a member of r.e. degree but not necessarily of r.e. degree less than $0'$. As corollaries results are derived about degrees of theories and degrees of models such as: (1) There is an axiomatizable, essentially undecidable theory with a complete extension of minimal degree; (2) (Scott-Tennenbaum): There is a complete extension of Peano arithmetic of degree $0'$ but none of r.e. degree $< 0'$; (3) There is no nonstandard model of Peano arithmetic of r.e. degree $< 0'$. The recursion theorem is applied to yield new information about standard constructions such as Yates' simple non-hypersimple set of given nonzero r.e. degree.

We now turn to definitions and notation. Let N be the set of natural numbers. A *function* is a mapping from a subset of N into N, and a *total function* is a mapping from N into N. A *class* is any collection of total functions. A class is called *special* in case it is nonempty and has no recursive member. Two classes are called *degree-isomorphic* if there is a one-to-one correspondence between them which preserves (Turing) degree.

A *theory* is any deductively closed set of sentences in a countable language. We often view theories as sets of numbers via Gödel-numbering.

A *string* is a function whose domain is a finite initial segment of N. If f is a function and $j \in N$, we define $[f]_j$ to be the restriction of f to $\{k: k < j\}$. Thus if f is total, $[f]_j$ is a string for all j. If S is a set of strings, we define S^* to be the class of total functions f such that $[f]_j \in S$ for all $j \in N$. If S is a recursive set of strings (under some Gödel numbering), S^* is called a *Π_1^0 — class*. Any Π_1^0 class may be put in the form T^*, where T is a recursive set of strings closed under initial substring, i.e., a *recursive tree*. Then T^* may be viewed intuitively as the class of all infinite paths through the tree T.

A class is called *recursively bounded* if there exists a recursive function which majorizes all of its members. (We say *g majorizes f* if $g(n) \geqq f(n)$ for all $n \in N$.) We identify sets with their characteristic functions so that any collection of sets is a recursively bounded class. In the converse direction, every recursively bounded Π_1^0 class (of functions) is degree-isomorphic to a Π_1^0 class of sets. This may be proved by a simple coding argument or by appeal to [4, Proposition 6.5] where it is shown that every recursively bounded Π_1^0 class is degree-isomorphic to the class of complete extensions of some axiomatizable theory. Thus, for our purposes, there is no essential distinction between recursively bounded Π_1^0 classes and Π_1^0 classes of sets.

If A is an infinite set, the principal function of A (denoted p_A) is the total function which enumerates A in increasing order. For sets A, B the notation $A \leqq_T B$ means that A is recursive in B, and similarly for functions. We assume a standard indexing and enumeration of all r.e. sets. The e^{th} r.e. set is denoted W_e and the finite subset of W_e enumerated after s steps is denoted W_e^s.

In general, there is a strong distinction between recursively bounded Π_1^0 classes and arbitrary Π_1^0 classes, but the first theorem indicates a point of similarity.

THEOREM 1. *For any special Π_1^0 class $\mathscr{S}$ there is a special recursively bounded Π_1^0 class $\mathscr{T}$ such that $\{\mathbf{f}: f \in \mathscr{S}\} \subseteqq \{\mathbf{f}: f \in \mathscr{T}\}$, where $\mathbf{f}$ denotes the degree of f.*

Proof. Let us say that a string has *type* 1 if its range is contained in $\{0, 1\}$ and *type* 2 if its range is disjoint from $\{0, 1\}$. Clearly there is a recursive set T_2 of strings of type 2 such that T_2^* is degree-isomorphic to $\mathscr{S}$. Let T_1 be any recursive set of strings of type 1 such that T_1^* is a special Π_1^0 class of sets. (For instance, T_1^* could be the class of complete extensions of some axiomatizable essentially undecidable theory [14].)

If σ, τ are strings, we write $\sigma * \tau$ for the string obtained by viewing σ and τ as finite sequences and then concatenating them. If $\sigma * \tau$ is a string, σ may be "factored" as follows:

$$\sigma = \sigma_1 * \tau_1 * \sigma_2 * \tau_2 * \cdots * \tau_n$$

where for $1 \leqq i \leqq n$, σ_i is of type 2 and τ_i is of type 1. To make the factorization unique we also require that at most σ_1, τ_n be empty and $n > 0$. Now define $t(\sigma)$ (the 1-tail of σ) to be τ_n and $s(\sigma)$ (the 2-skeleton of σ) to be $\sigma_1 * \sigma_2 * \cdots * \sigma_n$.

We define T to be the set of strings σ such that

(i) $t(\sigma) \in T_1$ and $s(\sigma) \in T_2$

(ii) $\sigma(n) \leqq n + 1$ for all n in domain of σ.

We claim that if $\mathscr{T} = T^*$, then $\mathscr{T}$ satisfies the requirements of the theorem. T^* is clearly a recursively bounded Π_1^0 class. To show that T^* has no recursive member it suffices to show that if $f \in T^*$ then there exists $g \in T_1^* \cup T_2^*$ such that $g \leqq_T f$. (Recall that neither T_1^* nor T_2^* has any recursive members.) Assume $f \in T^*$.

Case 1. $\{n: f(n) > 1\}$ is infinite. Let g be the "2-skeleton" of f. More precisely, let $g = f \circ h$, where h is the principal function of $\{n: f(n) > 1\}$. Then $g \leqq_T f$ and $g \in T_2^*$ because for each n there exists k such that $[g]_n = s([f]_k)$. Since $[f]_k \in T$ for all k, $[g]_n \in T_2$ for all n.

Case 2. $\{n: f(n) > 1\}$ is finite. Let g be the "1-tail" of f. More precisely, let i_0 be the least number j such that $f(i) \in \{0, 1\}$ for all $i \geqq j$ and define $g(n) = f(i_0 + n)$ for all n. Then $g \leqq_T f$ and $g \in T_1^*$ because for each n there exists k such $[g]_n = t([f]_k)$. This completes the proof that T^* has no recursive member. It remains to show that if $g \in T_2^*$, then T^* has a member f of the same degree as g. Given $g \in T_2^*$, let f be a function in T^* whose 2-skeleton (as defined in Case 1) is g. The function f is constructed by viewing g as an infinite sequence and inserting between each pair of consecutive terms a sufficiently long string from T_1 to insure $f(n) \leqq n + 1$. This is possible because T_1 is an infinite set of strings of type 1. Since the 2-skeleton of f is g, $g \leqq_T f$. Also $f \leqq_T g$ assuming the strings from T_1 used to construct f are chosen in an effective manner. Thus g is the desired function in T^* of the same degree as f, so the proof is complete.

COROLLARY 1.1 *If $\boldsymbol{a}$ is a degree and $0 < \boldsymbol{a} \leqq 0'$, then $\boldsymbol{a}$ is the degree of a member of some special recursively bounded Π_1^0 class.*

Proof. It is shown in [3, Theorems 4.10 and 4.13], that if $0 \leqq \boldsymbol{a} \leqq 0'$ then $\boldsymbol{a}$ contains a function f such that $\{f\}$ is a Π_1^0 class. The Corollary follows from this and the Theorem.

COROLLARY 1.2 *There is an axiomatizable, essentially undecidable theory which has a complete extension of minimal degree.*

Proof. Since by [4, Proposition 6.5] each recursively bounded Π_1^0 class is degree-isomorphic to the class of complete extensions of some axiomatizable theory, the Corollary follows from Corollary 1.1 and the existence of a minimal degree below $0'$ [10, Theorem 1].

According to a result of Scott and Tennenbaum [13] the theory in Corollary 1.2 cannot be Peano arithmetic. In fact it is shown in

[4, Corollary 4.4] that if $\boldsymbol{a}$ is the degree of any complete extension of Peano arithmetic, then any countable partially ordered set can be embedded in the degrees $< \boldsymbol{a}$.

The following Corollary extends Theorem 1.

COROLLARY 1.3 *For any special Π_2^0 class $\mathscr{S}$ there is a special recursively bounded Π_1^0 class $\mathscr{T}$ such that $\{\boldsymbol{f}: f \in \mathscr{S}\} \subseteq \{\boldsymbol{f}: f \in \mathscr{T}\}$ where $\boldsymbol{f}$ denotes the degree of f.*

Proof. By [3, Theorem 3.1] each Π_2^0 class is degree-isomorphic to some Π_1^0 class.

It is not possible to extend Corollary 1.3 by replacing Π_2^0 by Π_3^0. This is because by [16, p. 691] the class of *all* nonrecursive functions is Π_3^0 while the class of all functions Turing equivalent to any member of a special Π_1^0 class is meager, i.e. of first category [4, Theorem 5.1]. In fact by using the effectiveness of the proof of [4, Theorem 5.1] one may show that there is a degree $\boldsymbol{a}$ such that $\boldsymbol{0} < \boldsymbol{a} \leqq \boldsymbol{0}''$ and no member of any special Π_1^0 class has degree $\boldsymbol{a}$. Therefore $\boldsymbol{0}'$ cannot be replaced by $\boldsymbol{0}''$ in Corollary 1.1.

We now develop notation for functionals which will be useful in the proof of the next theorem. In this we are essentially following Lachlan [5, pp. 537–538], and we refer the reader there for the definitions omitted here. A *functional* is a mapping from the set of all functions into itself. If Φ is a functional and f is a function, we write $\Phi(f; n)$ for the value (if any) of the function $\Phi(f)$ at argument n. We assume that $\{\Phi_e^s\}$ is an s.r.e. double sequence of finite functionals such that for each e, Φ_e^s is increasing in s with limit denoted Φ_e, and every partially recursive functional is Φ_e for some e.

In a sense, the next theorem contrasts with Corollary 1.1, because it shows that there is no single special Π_1^0 class of functions sufficiently large to have members of all degrees $\boldsymbol{a}$, $\boldsymbol{0} < \boldsymbol{a} \leqq \boldsymbol{0}'$.

THEOREM 2. *If $\mathscr{P}$ is a special Π_1^0 class of functions, then there exists a nonzero r.e. degree $\boldsymbol{a}$ such that $\mathscr{P}$ has no member of degree $\leqq \boldsymbol{a}$.*

Proof. Let $\mathscr{P}$ be T^*, where T is a recursive tree. We must construct a nonrecursive r.e. set A such that for no e is $\Phi_e(A) \in T^*$. Let A^s be the finite subset of A enumerated by the end of stage s of the construction. In order to arrange $\Phi_e(A) \notin T^*$ we follow the method of Sacks [11, §5, Theorem 1]. That is, we attempt to preserve information about A^s which forces a long initial subfunction of $\Phi_e^s(A^s)$ to be *in* T. Although at first glance this appears to be just the opposite of what one should do, it has the desired effect because if $\Phi_e(A)$ were in T^*, the values of $\Phi_e^s(A^s)$ would be so well preserved that $\Phi_e(A)$ would be recursive, in contradiction to the assumption that T^* has no recursive member. To insure that A is nonrecursive, we make A simple as in [8, §5].

We now define three binary recursive functions p, q, r.

$$p(e, s) = \max\{n\colon n \leqq s \ \& \ [\Phi_e^s(A^s)]_n \in T\}$$
$$q(e, s) = \min\{j\colon \Phi_e^s([A^s]_j;\ n) \text{ is defined for all } n < p(e, s)\}$$
$$r(e, s) = \max\{q(e, s')\colon s' \leqq s\}.$$

To see that $p(e, s)$ is defined for all e and s, observe that we may assume without loss of generality that the empty string is in T. It then follows easily that $q(e, s)$ and $r(e, s)$ are also defined. In the construction all numbers $< r(e, s)$ will be prevented from entering A^{s+1} (in order to insure $\Phi_e(A) \notin T^*$ as described previously) unless such a number can be used to make $W_{e'}^{s+1} \cap A^{s+1} \neq \emptyset$ for some $e' < e$. More precisely, the construction is as follows:

Stage 0. Let $A^0 = \emptyset$.

Stage $s + 1$. Let e_{s+1} be the least number $e \leqq s$ such that $W_e^s \cap A^s = \emptyset$ and W_e^s contains a number $u \geqq \max(\{r(e', s)\colon e' \leqq e\} \cup \{2e\})$. Let $A^{s+1} = A^s \cup \{u\}$, where u is the least such number for $e = e_{s+1}$. (If e_{s+1} does not exist, let $A^{s+1} = A^s$.)

The construction is effective, so A is r.e.

LEMMA 2.1. *No function in T^* is recursive in A.*

Proof. Assume, for a contradiction, that $\Phi_e(A) \in T^*$ (and thus $\Phi_e(A)$ is a total function.)

Then it follows from the continuity of Φ_e that $\lim_s p(e, s) = \infty$. We claim now that $\Phi_e(A)$ must be recursive. Let t be a stage such that for all $e' < e$, if $W_{e'} \cap A \neq \emptyset$, then $W_{e'}^t \cap A^t \neq \emptyset$. Thus for all $s > t$, either e_s is undefined or $e_s \geqq e$. To compute $\Phi_e(A;\ n)$ effectively, find a number $s_n \geqq t$ such that $n < p(e, s_n)$. Note that $\Phi_e(A^{s_n};\ n)$ is defined because $n < p(e, s_n)$. We claim in fact that $\Phi_e(A^{s_n};\ n) = \Phi_e(A;\ n)$. To show this it suffices to prove that no number $u < q(e, s_n)$ enters A after stage s_n. But if $u \in A^{s+1} - A^s$ and $s + 1 > s_n$, then $e_{s+1} \geqq e$ and so $u \geqq r(e, s) \geqq q(e, s_n)$ by the construction. Since $\Phi_e(A^{s_n};\ n)$ is effectively computable from n, it follows that $\Phi_e(A)$ is recursive.

LEMMA 2.2. *For fixed e, $r(e, s)$ is bounded over all s.*

Proof. Let n be the least number such that $[\Phi_e(A)]_n \notin T$. (Such a number exists by Lemma 2.1). Let t be as in the proof of Lemma 2.1. By almost exactly the same argument as for Lemma 2.1 one may show, using the assumption that $[\Phi_e(A)]_n \notin T$, that $p(e, s) \leqq n$ for all $s \geqq t$. By choice of n, $\Phi_e(A; n')$ is defined for all $n' < n$. Choose a number j such that $\Phi_e([A]_j; n') = \Phi_e(A; n')$ for all $n' < n$. Thus if s is sufficiently large $\Phi_e([A^s]_j; n')$ is defined for all $n' < n$ and hence for all $n' < p(e, s)$. Then it follows from the definition of q that $q(e, s) \leqq j$ for all sufficiently large s and in turn from the definition of r that $r(e, s)$ is bounded over all s.

LEMMA 2.3. *A is simple.*

Proof. Since each r.e. set W_e "contributes" at most one member $u \geqq 2e$, A contains at most e elements $\leqq 2e$ and the complement of A is infinite, just as for Post's simple set [8, § 5]. Fix a number e. Let j be the largest of the numbers $r(e', s)$ over all $e' \leqq e$ and all s. The bound j exists by Lemma 2.2. Then if $s \geqq t$ (where t is as in Lemma 2.1) and W_e^s contains a member $\geqq \max\{j, 2e\}$, then $W_e^{s+1} \cap A \neq \varnothing$ by the construction. Hence if W_e is infinite then $W_e \cap A \neq \varnothing$. This completes the proof of the theorem.

In Theorem 2 we proved in particular that no special Π_1^0 class has members of all nonzero r.e. degrees. On the other hand, we now show that if such a class is recursively bounded, it must have a member of *some* r.e. degree.

THEOREM 3. *If $\mathscr{P}$ is a nonempty recursively bounded Π_1^0 class of functions, $\mathscr{P}$ has a member of r.e. degree.*

Proof. Let f_0 be the least member of $\mathscr{P}$ in the lexicographic ordering of functions by first differences. (Since $\mathscr{P}$ is closed and nonempty, f_0 exists.) Let $\mathscr{P} = T^*$, where T is a recursively bounded recursive tree. Clearly $f_0 \equiv_T B$, where B is the set of nodes on T which strictly precede f_0 in the lexicographic ordering. To see that B is r.e. note that the set C of nodes of T not extendible to functions of T^* is r.e. because T is recursively bounded. If $lh(\sigma) = lh(\tau)$, let $\tau \prec \sigma$ denote that τ lexicographically precedes σ. Since

$$B = \{\sigma : \sigma \in C \ \& \ (\forall \tau)[\tau \in T \ \& \ lh(\sigma) = lh(\tau) \ \& \ \tau \prec \sigma \Rightarrow \tau \in C]\}$$

and since T is recursively bounded, B is r.e.

Conversely to Theorem 3, for every nonrecursive r.e. set B there is a special Π_1^0 class of sets whose lexicographically least element f_0 has the same degree as B.

In [14, Theorem 2] Shoenfield proved that every nonempty recursively bounded Π_1^0 class has a member of degree $\boldsymbol{a} < \boldsymbol{0}'$. In view of this and Theorem 3 it is natural to conjecture that every nonempty recursively bounded Π_1^0 class has a member of r.e. degree $< \boldsymbol{0}'$. We now refute this conjecture.

THEOREM 4. *There is a nonempty recursively bounded Π_1^0 class $\mathscr{P}$ such that if $\boldsymbol{a}$ is the degree of any member of $\mathscr{P}$ and $\boldsymbol{b}$ is an r.e. degree and $\boldsymbol{a} \leqq \boldsymbol{b}$, then $\boldsymbol{b} = \boldsymbol{0}'$.*

Before proving Theorem 4, we give a definition and a lemma. A set A is called *effectively immune* in case A is infinite and there is a recursive function g such that for all e, if $W_e \subseteqq A$ then $|W_e| < g(e)$, where $|W_e|$ is the cardinality of W_e. (This definition is essentially due to Smullyan [17].) In [6], Martin proved that every r.e. set with effectively immune complement has degree $\boldsymbol{0}'$. The following lemma, which is an easy extension of his result, will immediately yield Theorem 4.

LEMMA 4.1. *If A is effectively immune, B is r.e., and $A \leqq_T B$, then B has degree $\boldsymbol{0}'$.*

Proof. We only outline the proof, since it is mostly a repetition of Martin's argument in altered format. From the assumptions that B is r.e., A is infinite, and $A \leqq_T B$ it is easy to show as in [15, Theorem 2] that there is a recursive sequence of recursive functions $\{\alpha_n\}$ such that $\lim_n \alpha_n = p_A$ (where p_A is the principal function of A) and $\{\alpha_n\}$ has a modulus recursive in B. (cf [15, Theorem 2].) Assume that g is a recursive function such that if $W_e \subset A$, then $|W_e| < g(e)$. Let C be any r.e. set. We shall prove $C \leqq_T B$. Fix an enumeration of C and let C^s be the finite subset of C obtained after s steps. Given a number e, define an r.e. set $W_{h(e)}$ as follows. If $e \in C$, let $s(e) = \mu s(e \in C^s)$. Then let $W_{h(e)}$ be the range of $[\alpha_{s(e)}]_{gh(e)}$. If $e \notin C$, let $W_{h(e)} = \varnothing$. Note that $h(e)$ enters (circularly) into the definition of $W_{h(e)}$, but this may be justified by the recursion theorem. Also we may assume that h is recursive.

Let $r(e)$ be a number such that for $i \geqq r(e)$, $[\alpha_i]_{gh(e)} = [p_A]_{gh(e)}$. In particular, if $e \in C$ and $s(e) \geqq r(e)$, then $W_{h(e)} \subset A$ and $|W_{h(e)}| = gh(e)$, which is impossible. It then follows that

$$e \in C \longleftrightarrow e \in C^{r(e)}$$

for all e.

However $r(e)$ may be found recursively in B because g and h are recursive and $\{\alpha_i\}$ has a modulus recursive in B. Thus the above equivalence shows that $C \leqq_T B$, and the proof of Lemma 4.1 is complete.

Proof of Theorem 4. Let S be Post's simple, nonhypersimple set [8, §5]. As Smullyan [17, p. 893] observed, $\bar{S}$ is effectively immune. Let $\{D_{f(k)}\}$ be an s.r.e. sequence of pairwise disjoint sets all intersecting $\bar{S}$. Let

$$\mathscr{P} = \{E\colon E \subset \bar{S} \text{ and } (\forall k)[D_{f(k)} \cap E \neq \emptyset]\} .$$

Then $\mathscr{P}$ is easily seen to be a nonempty Π_1^0 class of sets containing only effectively immune sets. Theorem 4 now follows from Lemma 4.1.

COROLLARY 4.2. *There is a degree* $\boldsymbol{a}$ *such that* $\boldsymbol{a}' = \boldsymbol{0}'$ *and the only r.e. degree* $\geqq \boldsymbol{a}$ *is* $\boldsymbol{0}'$.

Proof. Let $\mathscr{P}$ be the nonempty recursively bounded Π_1^0 class of Theorem 4. Then by [4, Theorem 2.1], $\mathscr{P}$ has a member of degree $\boldsymbol{a}$ such that $\boldsymbol{a}' = \boldsymbol{0}'$.

In [19, p. 270] Yates proved that there is a degree $\boldsymbol{a} < \boldsymbol{0}'$ such that the only r.e. degrees comparable with $\boldsymbol{a}$ are $\boldsymbol{0}$ and $\boldsymbol{0}'$. It is easy to modify his proof so that $\boldsymbol{a}' = \boldsymbol{0}'$ also and thus obtain a different proof of Corollary 4.2. On the other hand, we have been unable to modify our proof of Corollary 4.2 to obtain Yates' result. What we lack is a proof of the following conjecture.

Conjecture. Every special recursively bounded Π_1^0 class $\mathscr{P}$ contains a member of some degree $\boldsymbol{a} < \boldsymbol{0}'$, such that the only r.e. degree $\leqq \boldsymbol{a}$ is $\boldsymbol{0}$.

(If $\boldsymbol{0}'$ is replaced by $\boldsymbol{0}''$, the conjecture follows from [4, Corollary 2.11]).

We now apply Theorem 4 to Peano arithmetic. Considerable further information along these lines may be found in [4].

By the *degree of a countable model* M we mean the degree of a set which encodes the elements and relations of the model. It is well-known that Peano arithmetic has no recursive nonstandard models. (The standard model is of course recursive.) The proof given in reference [1, p. 48] establishes more. Namely,

REMARK. If M is any nonstandard model of Peano arithmetic, and $\mathscr{P}$ is any nonempty recursively bounded Π_1^0 class, then $\mathscr{P}$ contains a member recursive in M.

Scott [12, p. 118] has observed that if T is any complete extension of Peano arithmetic, then the above remark holds with T in place of M. Finally, one can replace "complete extension" by "consistent extension" since it is easy to see that if T is any consistent extension of Peano arithmetic then there is a complete extension T' recursive in T [4, Proposition 6.1]. If we let $\mathscr{P}$ be the special Π_1^0 class of Theorem 4, these observations yield:

COROLLARY 4.3. *If* $\boldsymbol{a}$ *is the degree of any consistent extension or nonstandard model of Peano arithmetic, and* $\boldsymbol{b}$ *is an r.e. degree such that* $\boldsymbol{a} \leqq \boldsymbol{b}$, *then* $\boldsymbol{b} = \boldsymbol{0}'$.

In [13] Scott and Tennenbaum announced that no complete extension of Peano arithmetic can have r.e. degree $< \boldsymbol{0}'$. This result, which of course is a consequence of Corollary 4.3 can also be used to refute the conjecture made just before Theorem 4.

COROLLARY 4.4. (Scott-Tennenbaum [13]). *There is a complete extension of Peano arithmetic of degree* $\boldsymbol{0}'$.

Proof. Let $\mathscr{S}$ be the class of all complete extensions of Peano arithmetic. Then $\mathscr{S}$ is a nonempty Π_1^0 class of sets, so by Theorem 3, $\mathscr{S}$ has a member of r.e. degree $\boldsymbol{b}$. It follows from Corollary 4.3 that $\boldsymbol{b} = \boldsymbol{0}'$.

In [13], Scott and Tennenbaum actually announced the stronger result that if $\boldsymbol{a} \geqq \boldsymbol{0}'$, then there is a complete extension of Peano arithmetic of degree $\boldsymbol{a}$. This result can be obtained by first using the Friedberg completeness criterion [9, p. 265] to obtain a degree $\boldsymbol{b}$ such that $\boldsymbol{b}' = \boldsymbol{a}$ and then relativizing Theorem 3 and Corollary 4.3 to $\boldsymbol{b}$ with the aid of the two results of Scott stated on [12, p. 118].

All of the results we have obtained here for Peano arithmetic apply as well to any theory in which the provable and refutable formulas are effectively inseparable, such as ZF set theory (cf [4], Proposition 6.1 or 6.4).

We close with an extension of Theorem 4 in which $\boldsymbol{0}'$ is replaced by an arbitrary r.e. degree $\boldsymbol{c}$. The proof is similar to the former except that Post's simple set is replaced by Yates' simple, nonhypersimple set of degree $\boldsymbol{c}$ for $\boldsymbol{c} \neq 0$ [20, Theorem 2]. An additional difficulty arises in the proof, however, and to overcome it we introduce a method which yields new information about results of Yates and Martin as we discuss at the end of the proof.

THEOREM 5. *If* $\boldsymbol{c}$ *is any r.e. degree, there is a recursively bounded*

*Π_1^0 class $\mathscr{P}$ such that the r.e. degrees of members of $\mathscr{P}$ are precisely the r.e. degrees $\geqq$ **c**.*

Proof. Assume $\mathbf{c} \neq 0$ (else the theorem is obvious) and let A be an r.e. set of degree $\mathbf{c}$. Following Yates [20, Theorem 2], let $A =$ range f, where f is a $1-1$ recursive function. At stage s, for each $e \leqq s$, if $W_e^s \cap B^s = \emptyset$ and W_e^s contains a number $u_e \geqq \max\{2e, f(s)\}$ place the least such u_e into B^{s+1}. It follows that B is simple, nonhypersimple, and $B \leqq_T A$. Construct $\mathscr{P}$ as in the proof of Theorem 4 with B in place of S. Yates then constructs his simple set S as the join of B and a hypersimple set of the same degree as A to insure that $A \leqq_T S$. The following lemma shows that this is unnecessary since automatically $A \leqq_T B$.

LEMMA 5.1. *If C is an r.e. set and D is an infinite subset of $\bar{B}$ and $D \leqq_T C$, then $A \leqq_T C$.*

Proof. As in the proof of Lemma 4.1 there is a recursive sequence of recursive functions $\{\alpha_n\}$ such that $\lim_n \alpha_n = p_D$, the principal function of D, and $\{\alpha_n\}$ has a modulus recursive in C. Given e, let $s(e) = \mu s[e = f(s)]$ if $e \in A$. Define $W_{h(e)}$ to be the range of $[\alpha_{s(e)}]_{2h(e)}$. (If $e \notin A$, let $W_{h(e)} = \emptyset$.) Let $r(e)$ be so large that $[\alpha_i]_{2h(e)} = [P_D]_{2h(e)}$ for all $i \geqq r(e)$. Such r can be found recursively in C because $\{\alpha_n\}$ has a modulus recursive in C.

We cannot argue as in Lemma 4.1 that $e \in A$ iff $e \in A^{r(e)}$ since if $s(e) > r(e)$ we can only conclude that $W_{h(e)}$, not $W_{h(e)}^{s(e)}$, has a member $u \geqq 2h(e)$. Thus we must define,

$$\hat{A} = \{e\colon e \in A - A^{r(e)}\} .$$

Case 1. $\hat{A}$ is finite. Then clearly $A \leqq_T C$.

Case 2. $\hat{A}$ is infinite. Clearly $\hat{A}$ is r.e. in C. Let k be a given number. Find $e \in \hat{A}$ such that $2h(e) > k$. This e exists because $\hat{A}$ is infinite and we may assume that h is $1-1$. Also e may be found recursively in C because $\hat{A}$ is r.e. in C. Let s_k be a stage such that $s_k \geqq h(e)$ and $W_{h(e)}^{s_k}$ contains a number $u \geqq 2h(e)$. ($W_{h(e)}$ has such a u because $e \in \hat{A}$ implies that $s(e) \geqq r(e)$.) Then u (or some $v \leqq u$) will be used in the construction to make $W_{h(e)} \cap B \neq \emptyset$, unless $f(s) \geqq u (> k)$ for all $s \geqq s_k$. This must be the case because $W_{h(e)} \subseteqq D \subseteqq \bar{B}$ since $e \in \hat{A}$. Thus,

$$k \in A \longleftrightarrow k \in A^{s_k} .$$

Since s_k can be found recursively in C, uniformly in k, $A \leqq_T C$.

Remark. The division of the above proof into two cases makes the argument nonuniform. This lack of uniformity may be avoided by an application of the recursion theorem to slightly modify Yates' construction [20, Theorem 2] so that $\hat{A}$ must be empty.

By Lemma 5.1, the r.e. degrees of members of $\mathscr{P}$ are all above $\boldsymbol{c}$. We complete the proof of Theorem 5 by establishing the converse.

LEMMA 5.2. *Given any degree $\boldsymbol{a} \geqq \boldsymbol{c}$, there is a member $A \in \mathscr{P}$ of degree $\boldsymbol{a}$.*

Proof. Let B be as above, and assume that the strong array $\{D_{f(n)}\}_{n \in N}$, which witnesses the nonhypersimplicity of B, satisfies $|D_{f(n)} \cap \bar{B}| \geqq 2$ for all n. Choose a set A_1 of degree $\boldsymbol{a}$, and find $A \in \mathscr{P}$ such that $A \leqq_T A_1$ and $|D_{f(n)} \cap \bar{A}| = 1$ iff $n \in A_1$. Clearly A has degree $\boldsymbol{a}$, so the proof of Theorem 5 is complete.

Frequently in recursion theory, one constructs an r.e. set S having some property P and recursive in a given nonrecursive r.e. set A by enumerating elements in A for the sake of P only when A "permits", i.e., only when a sufficiently small element is enumerated in A. This clearly guarantees that $S \leqq_T A$, but the method of Lemma 5.1 sometimes insures that $A \leqq_T S$ also. For example, by the "permitting" method Yates [20, Theorems 1 and 2] constructs a simple set and a semicreative set each recursive in a given nonrecursive r.e. set A, while Martin [7, Theorems 2 and 4.1] constructs a maximal set and an r.e. set with no maximal superset each recursive in a given dense simple set A. Yates and Martin also code A into the constructed set S, so that $A \leqq_T S$. By applying the recursion theorem as in Lemma 5.1, one can show that the latter is unnecessary, since in each case automatically $A \leqq_T S$. In the case of Martin's maximal set construction this saves considerable work as is shown in [2].

These observations suggest a "maximum degree principle" which asserts that if an r.e. set S is constructed with weak "negative requirements" it automatically has degree $\mathbf{0}'$ and if S is constructed recursive in A by Yates' method then $A \leqq_T S$ automatically. Although most finite injury priority constructions of an r.e. set S can be combined with Yates' method to produce $S \leqq_T A$, a given nonrecursive r.e. set [18, Theorem 3], it rarely happens that $A \leqq_T S$ automatically unless the negative requirements of the original construction are extremely weak. For example in the original Friedberg-Muchnik construction of incomparable r.e. degrees $\boldsymbol{a}, \boldsymbol{b}$, one automatically gets $\boldsymbol{a} \cup \boldsymbol{b} = \mathbf{0}'$, but when the incomparable r.e. degrees $\boldsymbol{a}, \boldsymbol{b}$ are constructed below a fixed nonzero r.e. degree $\boldsymbol{c}$ by Yates' method, one does not automatically get $\boldsymbol{a} \cup \boldsymbol{b} = \boldsymbol{c}$ [18].

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Received October 27, 1970. This work was supported by National Science Foundation Grants GP-7421, GP-23707, GP-8866, and GP-19958. We are grateful to A. H. Lachlan for proposing this problem area.

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