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ON THE GENUS OF THE COMPOSITION OF TWO GRAPHS

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Given two graphs G and H, a new graph $G(H)$, called the composition (or lexicographic product) of G and H, can be formed. In this paper, a formula is developed to give the genus for a large class of lexicographic products. In the simplest special case, the genus of the product is given by the first Betti number of one of the factors.

In the present context, a *graph* is a finite 0- or 1- complex. For terms not defined below, see [2] and [6].

The *genus*, $\gamma(G)$, of a graph G is the minimum genus among the genera of all closed orientable 2-manifolds M in which G can be imbedded. An imbedding of G in M is said to be *minimal* if M has genus $\gamma(G)$. The *first Betti number*, $\beta(G)$, of a graph G is given by $\beta(G) = q - p + k$, where G has q edges, p vertices, and k components; $\beta(G)$ counts the number of independent cycles in G. Given two graphs G and H with disjoint vertex sets $V(G)$, $V(H)$ and edge sets $E(G)$, $E(H)$ respectively, the *composition* (or *lexicographic product*) $G(H)$ has vertex set given by the cartesian product $V(G) \times V(H)$, with two vertices (u_i, v_j) and (u_k, v_m) adjacent in $G(H)$ if and only if either: (i) $u_i = u_k$ and $v_j v_m$ is in $E(H)$, or (ii) $u_i u_k$ is in $E(G)$. For example, the regular complete m-partite graph on mn vertices is just the composition $K_m(\overline{K_n})$, where K_s denotes the complete graph on s vertices, and $\overline{K_s}$ denotes the complement of K_s (a 0-complex).

We will also employ the following notions. If G is imbedded in M, the components of $M - G$ are called *regions*. A region bounded by a circuit of length 3(4) in G is said to be *triangular* (*quadrilateral*). The number of triangular (quadrilateral) regions in a given imbedding is denoted by $r_3(r_4)$. In general, r_k designates the number of regions having a connected boundary consisting of k edges of G, and r denotes the total number of regions. It is well known (see, for example, [6]) that, for a minimal imbedding of a connected graph G having p vertices and q edges, the Euler formula $p - q + r = 2 - 2\gamma(G)$ applies. Also, it is easy to show that $2q = \sum_{i \geqq 3} i r_i$. We note that a 3-cycle in a graph G need not bound a triangular region in a given minimal imbedding of G. For example, there are 35 3-cycles in K_7; yet any minimal imbedding of K_7 has $r = r_3 = 14$.

The following result of Battle, Harary, Kodama, and Youngs [1] will be useful:

THEOREM. *The genus of a graph is the sum of the genera of its components.*

We are now prepared to state the main result.

THEOREM. *Let G have p vertices of positive degree, q edges, k nontrivial components, and no 3-cycles. Let H have $2n(n \geqq 1)$ vertices and maximum degree less than two. Then $\gamma(G(H)) = k + n(nq - p)$.*

Proof. Let G have nontrivial components C_i, $i = 1, \cdots, k$; then $G(H)$ has nontrivial components $C_i(H)$, $i = 1, \cdots, k$. It will suffice to prove the theorem for G connected, since then (by the result of Battle, Harary, Kodama and Youngs):

$$\begin{aligned}\gamma(G(H)) &= \sum_{i=1}^{k} \gamma(C_i(H)) \\ &= \sum_{i=1}^{k} (1 + n(nq_i - p_i)) \\ &= k + n(nq - p) .\end{aligned}$$

We therefore assume G to be connected. Let $V(G) = \{u_1, \cdots, u_p\}$, and $V(H) = \{v_1, \cdots, v_{2n}\}$.

Suppose the vertices (u_i, v_j), (u_k, v_m), and (u_r, v_s) form a 3-cycle in $G(H)$. Since there are no 3-cycles in G, the vertices u_i, u_k, and u_r cannot be distinct in $V(G)$. Hence every 3-cycle in $G(H)$ must contain an edge of the form $(u_i, v_j)(u_i, v_m)$. There are exactly pe such edges in $G(H)$, where e designates the number of edges in $H(0 \leqq e \leqq n)$. Since each one of these edges can appear in the boundary of at most 2 triangular regions, it follows that $r_3{}^* \leqq 2pe$ in any imbedding of $G^* = G(H)$. (A parameter with (without) an asterisk will apply to graph $G^*(G)$).

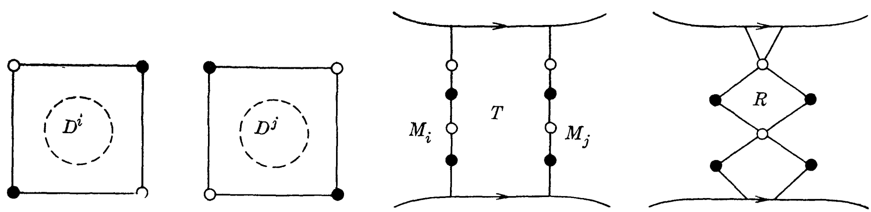
We will construct an imbedding of G^* so that $r_3^* = 2pe$ and $r_4^* = r^* - 2pe$; since $r^* = \sum_{i \geqq 3} r_i^*$ and $2q^* = \sum_{i \geqq 3} i r_i^*$, r^* will be maximal for such an imbedding. Then, by the Euler formula, the imbedding itself will be minimal. Now, for G^*, $p^* = 2np$, and $q^* = pe + 4n^2 q$. Also, if $r_3^* = 2pe = r^* - r_4^*$, then $r^* = pe + 2n^2 q$, since $2q^* = 2pe + 8n^2 q = 3(2pe) + 4(r^* - 2pe)$. Then, from the Euler formula,

$$\begin{aligned}\gamma(G^*) &= 1 + 1/2(q^* - p^* - r^*) \\ &= 1 + 1/2(pe + 4n^2 q - 2np - (pe + 2n^2 q)) \\ &= 1 + n(nq - p) .\end{aligned}$$

We now construct such an imbedding. Let the edges of G be designated by $x_1, \cdots, x_q$. For each edge there is a subgraph of $G(H)$ isomorphic to the complete bipartite graph $K_{2n\,2n}$. Imbed q copies of $K_{2n,2n}$, minimally, in q closed orientable 2-manifolds $M_1, \cdots, M_q$ of genus $(n-1)^2$ each, in the fashion described by Ringel [3]. Select these 2-manifolds so that each is exterior to any other. Each imbedding has $r' = r'_4 = 2n^2$, and it has been shown in [5] that the $2n^2$ quadrilateral regions can be partitioned into $2n$ mutually disjoint sets of n regions each, each set containing all $4n$ vertices of the graph $K_{2n,2n}$. Furthermore, in any region, diagonally opposite vertices are in the same part of the vertex set partition for $K_{2n,2n}$.

Suppose edges x_i and x_j are adjacent in G. We make $2n$ vertex identifications between M_i and M_j as follows. Select one set of n quadrilateral regions in M_i and the $2n$ vertices of one part of the vertex set partition for $K_{2n,2n}$ from the boundaries of these regions (two diagonally opposite vertices are selected from the boundary of each region). Make similar selections in M_j. Now attach n tubes between M_i and M_j, one tube for each pair of regions (one from each 2-manifold) that we have selected. The first such tube may be attached as follows. Let region R^i in M_i correspond to region R^j in M_j. Let C^h be a simple closed curve bounding the open disk D^h interior to R^h, $h = i, j$. Let T be a topological cylinder, with bases C_i and C_j, such that $(M_i \cup M_j) \cap T = C_i \cup C_j$. Form $(M_i - D^i) \cup (M_j - D^j) \cup T$. It is clear how to add the remaining tubes. The result is a closed orientable 2-manifold M (of genus $2(n-1)^2 + n - 1$).

We now make two vertex identifications per tube, as indicated by the sequence of operations in Figure 1.



1a 1b 1c

FIGURE 1

This process destroys two quadrilaterals and creates two new quadrilaterals for each tube. Furthermore, the two identifications for each tube yield two vertices diagonally opposite in a common region R. If edge x_k is also adjacent to x_i (and to x_j) in G, there are now n regions available on the 2-manifold M with which to make the appropriate $2n$ identifications with the 2-manifold M_k. From these n regions on M, we select the diagonally opposite vertices that resulted from the first identification. It is clear that this process may be continued until a quadrilateral imbedding of $G(\overline{K_{2n}})$ results. We need only insure that, for 2-manifold M_i corresponding to edge $x_i = [u_{i1}, u_{i2}]$ in G, if we selected the $2n$ vertices of one part of the vertex-set partition of $K_{2n,2n}$ with which to make the identifications at u_{i1} in G, then we must select the $2n$ vertices of the second part of the vertex-set partition of $K_{2n,2n}$ with which to make the identifications at u_{i2} in G.

Corresponding to each vertex of G, there is now a copy of $\overline{K}_{2n}$, within which the e edges of H may be added. Each such edge converts one quadrilateral region of the imbedding of $G(\overline{K}_{2n})$ into two triangular regions. The result is an imbedding of G^* having $r_3^* = 2pe$ and $r_4^* = r^* - 2pe$, as desired. This completes the proof.

We note that the value $r^* = 2qn^2 + pe$ may be verified by a direct count, since $r_4^* = q(2n^2) - pe$. Also, the genus of $G(H)$ may be computed directly, for this construction, without recourse to any Euler type formula. The contributions to the genus are of three types:

(i) $q(n-1)^2$, representing the collective genera of the q 2-manifolds with which we began our construction;

(ii) $(2q-p)(n-1)$, representing the contribution of the $2q-p$ sets of $2n$ vertex identifications each, each "bundle" of n tubes adding $n-1$ to the genus;

(iii) $\beta(G) = q - p + 1$, representing the contribution of the bundles of tubes taken collectively.

Adding, we find:

$$\begin{aligned}\gamma(G(H)) &= q(n-1)^2 + (2q-p)(n-1) + (q-p+1)\\ &= 1 + n(nq-p)\,.\end{aligned}$$

It is no surprise that the formula $\gamma(K_{2n,2n}) = (n-1)^2$ is included in the above theorem. For the case where G is the cycle C_s and $H = \overline{K}_{2n}$, we may combine the theorem with the result of Ringel and Youngs [4] that $\gamma(K_3(\overline{K}_m)) = ((m-1)(m-2))/2$ to establish the following:

COROLLARY 1.

$$\gamma(C_s(\overline{K}_{2n})) = \begin{cases}1 + n(2n-3), & \text{if } s = 3\\ 1 + ns(n-1), & \text{if } s \geqq 4\,.\end{cases}$$

In the situation where G is the complete bipartite graph $K_{r,s}$ and H is as in the statement of the theorem, we have:

COROLLARY 2. $\gamma(K_{r,s}(H)) = (nr-1)(ns-1)$.

We list here only one other result, for the special case $n = 1$ of the theorem:

COROLLARY 3. *Let G be a graph containing no 3-cycles. Then* $\gamma(G(K_2)) = \gamma(G(\overline{K}_2)) = \beta(G)$.

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