PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 1, 1972

FILTRATIONS AND VALUATIONS ON RINGS

HELEN E. ADAMS

The concept of a multiplicative filtration on a ring is generalized so as to include among filtered rings, rings with valuation, pseudovaluation and semivaluation. The generalized filtration induces a topology on the ring, and it is shown that the Hausdorff completion of the resulting topological ring can be described by an inverse limit. The paper finishes with an example illustrating the theory.

1. Definitions and immediate consequences. In this section we define a generalized filtration and generalized pseudovaluation on a ring and show that a pseudovaluation induces a filtration on a ring.

If A and B are subsets of a ring we shall write AB to mean the set $\{xy: x \in A, y \in B\}$. By an *ordered semigroup* we mean a semigroup which is partially ordered as a set such that the ordering relation is compatible with the semigroup operation. A *directed semigroup* is an ordered semigroup which is directed above as an ordered set; and a *quasi-residuated semigroup* (Blyth and Janowitz [2]) is an ordered semigroup T with the property: given any $s, t \in T$, there exists $u \in T$ such that $ut \geqq s$ and $tu \geqq s$.

Let R be a ring and let S be a directed semigroup with the property:

(1.1) given any $s \in S$, there exists $t \in S$ such that $t^2 \geqq s$.

A *filtration* on R over S is a set of additive subgroups $\{P_s\}_{s \in S}$ of R, indexed by S, with the following properties:

(1.2) if $s, t \in S$ such that $s \geqq t$, then $P_s \subseteqq P_t$;

(1.3) for any $s, t \in S$, $P_s P_t \subseteqq P_{st}$;

(1.4) given $x \in R$, $s \in S$, there exists $t \in S$ such that $xP_t \subseteqq P_s$ and $P_t x \subseteqq P_s$.

Note that $\bigcap_{s \in S} P_s$ is a two-sided ideal of R. For a treatment of the classical multiplicative filtration on a ring, see Atiyah and Macdonald [1] and Northcott [6].

The following lemma gives a less general form of a filtration which will be shown to arise from a pseudovaluation on a ring. The proof of the lemma is straightforward.

LEMMA 1.1. *Let S be a quasi-residuated, directed semigroup. Let $\{P_s\}_{s \in S}$ be a set of additive subgroups of a ring R such that* (1.2), (1.3) *hold, and* (1.4′) $\bigcup_{s \in S} P_s = R$.

Then $\{P_s\}_{s \in S}$ is a filtration on R.

The following definition of a pseudovaluation on a ring allows us to treat at the same time Manis [5] valuations and pseudovaluations (Mahler [4]) on commutative rings, and semivaluations (Zelinsky [7]) on fields.

Let S be a quasi-residuated, directed semigroup, and let S_0 be the disjoint union of S and a zero element O_S with the properties: $O_S O_S = O_S$; and, for any $s \in S$, $O_S > s$ and $sO_S = O_S = O_S s$. A *pseudovaluation* on a ring R into S_0 is a map φ of R into S_0 such that: for all $a, b \in R$,

(1.5) $\varphi(ab) \geqq \varphi(a)\varphi(b)$;

(1.6) if $s \in S$ such that $s \leqq \varphi(a), \varphi(b)$, then $\varphi(a-b) \geqq s$;

(1.7) $\varphi(0) = O_S$;

(1.8) the set $\varphi(R) \backslash \{O_S\}$ is nonempty.

Let $\varphi\colon R \to S_0$ be a pseudovaluation on a ring R. Define, for any $s \in S$,

(1.9) $P_s = \{x\colon x \in R, \varphi(x) \geqq s\}$.

Then, from Lemma 1.1:

PROPOSITION 1.1. *The family of subsets* $\{P_s\}_{s \in S}$ *of* R, *defined in* (1.9), *is a filtration on* R.

2. The completion of a ring with respect to a filtration. Throughout this section, R is a ring with filtration $\{P_s\}_{s\in S}$. It will be shown that the filtration $\{P_s\}_{s\in S}$ induces a topology $\mathscr{T}$ on R compatible with the ring structure of R, and the completion of $(R, \mathscr{T})$ will be explicitly defined both algebraically and topologically.

From Bourbaki [3, III §1.2, example], the set $\{P_s\}_{s\in S}$ is the fundamental system of neighbourhoods of the zero for a uniquely determined topology $\mathscr{T}$ on R, addition in $(R, \mathscr{T})$ is continuous, and $\mathscr{T}$ is Hausdorff if and only if $\bigcap_{s\in S} P_s = \{0\}$. Further, multiplication in $(R, \mathscr{T})$ is continuous by the definition of a filtration and [3, III §6.3, $(\mathrm{AV_I})$ and $(\mathrm{AV_{II}})$]. Hence $(R, \mathscr{T})$ is a topological ring and, as such, admits a Hausdorff completion.

Now the Hausdorff completion of a topological ring is just the Hausdorff completion of the ring considered as an additive topological group [3, III §6.5]. Multiplication is then defined on the completion by a continuous extension of multiplication on the associated Hausdorff ring, in this case the factor ring $R/\bigcap_{s\in S} P_s$.

But in this case we already have, from [3, III §7.3, Proposition 2, Corollary 2], that the Hausdorff completion of the additive topological group $(R, \mathscr{T})$ is isomorphic, both algebraically and topologically, to the

Hausdorff group $(\tilde{R}, \tilde{\mathscr{T}})$ where $\tilde{R} = \varprojlim R/P_s$ and $\tilde{\mathscr{T}}$ is the usual topology induced on $\tilde{R}$ by the topology $\mathscr{T}$ on R. Hence the Hausdorff completion of the topological ring $(R, \mathscr{T})$ is isomorphic to the Hausdorff ring $(\tilde{R}, \tilde{\mathscr{T}}, \times)$ where $\times$ denotes the multiplication constructed on $\tilde{R}$ by means of a continuous extension of multiplication in $R/\bigcap_{s \in S} P_s$. The main aim of this section is to define explicitly the multiplication $\times$. This is not a straightforward task since each factor group R/P_s, $s \in S$, in the direct product $\prod_{s \in S} R/P_s$, is not a ring.

For reference we define the topological group $(\tilde{R}, \tilde{\mathscr{T}})$ explicitly [3, III §7]. Now $\tilde{R} = \{\{\xi_s\}_{s \in S} \in \prod_{s \in S} R/P_s$: for all $s, t \in S$ such that $s \leqq t$, $\xi_t \subseteqq \xi_s\}$. That is, the elements of $\tilde{R}$ are sets of subsets of R, indexed by S, and written $\{\xi_s\}_{s \in S}$ where: for each $s \in S$, $\xi_s \in R/P_s$; and, for any $s, t \in S$ such that $s \leqq t$, $\xi_t \subseteqq \xi_s$. Note that, for each $x \in R$, $\{X + P_s\}_{s \in S} \in \tilde{R}$. Equality and addition in $\tilde{R}$ are defined as follows: Let $\{\xi_s\}_{s \in S}$, $\{\eta\}_{s \in S} \in \tilde{R}$. Then $\{\xi_s\}_{s \in S} = \{\eta_s\}_{s \in S}$ if and only if, for each $s \in S$, $\xi_s = \eta_s$; and $\{\xi_s\}_{s \in S} + \{\eta_s\}_{s \in S} = \{\xi_s + \eta_s\}_{s \in S}$. When there is no risk of ambiguity, $\{\xi_s\}_{s \in S}$ will be written as $\{\xi_s\}$.

The topology $\tilde{\mathscr{T}}$ is defined on $\tilde{R}$ by inducing the usual quotient topology on each R/P_s, $s \in S$, then inducing the usual product topology on $\prod_{s \in S} R/P_s$, and finally restricting this topology to $\tilde{R}$, considered as a subspace of $\prod_{s \in S} R/P_s$.

Let $t \in S$ and let f_t: $\tilde{R} \to R/P_t$ be the canonical projection defined thus: For any $\{\xi_s\}_{s \in S} \in \tilde{R}$, $f_t(\{\xi_s\}_{s \in S}) = \xi_t$. Since R/P_t is discrete [3, III §7.3], the set $\tilde{P}_t = f_t^{-1}(P_t) = \{\{\xi_s\}_{s \in S} \in \tilde{R}$: $\xi_t = P_t\}$ is an open set in $(\tilde{R}, \tilde{\mathscr{T}})$, containing the zero $\{P_s\}_{s \in S}$ of $\tilde{R}$.

Further, it is easily checked that, for each $t \in S$, $\tilde{P}_t$ is a subgroup of $\tilde{R}$. Hence the set of subgroups $\{\tilde{P}_t\}_{t \in S}$ of $\tilde{R}$ forms a fundamental system of neighbourhoods of the zero of $(\tilde{R}, \tilde{\mathscr{T}})$ and thus, by [3, I §2.3, Example 3], defines the topology $\tilde{\mathscr{T}}$ on $\tilde{R}$.

Next we define a multiplication "$*$" in $\tilde{R}$, and show that $*$ is in fact the required multiplication $\times$. When there is no risk of ambiguity, we shall omit the multiplication sign $*$. Note that if each of the subgroups P_s, $s \in S$, were a two-sided ideal of R, then multiplication in $\tilde{R}$ would be as simple to define as addition: but this is not the case.

Let $\{\xi_s\}_{s \in S}$, $\{\eta_s\}_{s \in S} \in \tilde{R}$. Let $\{\xi_s\}_{s \in S} * \{\eta_s\}_{s \in S} = \{\Omega_s\}_{s \in S}$ where $\{\Omega_s\}_{s \in S}$ is defined as follows: Let $s \in S$. Then by (1.1) there exists $t \in S$ such that $t^2 \geqq s$. Choose $x_1 \in \xi_t$, $y_1 \in \eta_t$. From (1.4) there exist $u, v \in S$ such that $x_1P_u \subseteqq P_s$, $P_vy_1 \subseteqq P_s$. Let $w \in S$ be such that $w \geqq t, u, v$. Define $\Omega_s = xy + P_s$ where $x \in \xi_w$, $y \in \eta_w$. The following two lemmas show that Ω_s is well-defined and independent of the particular choice of w.

LEMMA 2.1. *With w chosen, the coset Ω_s does not depend upon the choice of x and y.*

Proof. Let $x, x' \in \xi_w$; $y, y' \in \eta_w$. Now

$$\begin{aligned} xy - x'y' &= (x - x')y_1 + x_1(y - y') \\ &\quad + (x - x')(y - y_1) + (x' - x_1)(y - y') . \end{aligned} \tag{2.2}$$

It is easily checked that each of the summands of (2.2) belongs to P_s. Hence $xy - x'y' \in P_s$ and the lemma follows.

LEMMA 2.2. *Let the notation be as above. Let $f, g \in S$ such that, for all $a', a'' \in \xi_f$ and for all $b', b'' \in \eta_g$, $a'b' - a''b'' \in P_s$. Then $\Omega_s = ab + P_s$ for any $a \in \xi_f$, $b \in \eta_g$.*

Proof. Let $a \in \xi_f$, $b \in \eta_g$. Let $h \in S$ such that $h \geqq w, f, g$. Let $c \in \xi_h$, $d \in \eta_h$. Then, by Lemma 2.1, $\Omega_s = cd + P_s$ since $c \in \xi_w$, $d \in \eta_w$. But $ab - cd \in P_s$ since $a, c \in \xi_f$ and $b, d \in \eta_g$. Hence $\Omega_s = ab + P_s$.

COROLLARY. *The definition of Ω_s is independent of the particular choice of w.*

Proof. Let $w' \in S$ be another possible choice for w (with possibly different t, u, v, x_1, y_1). Then, by Lemma 2.1, Lemma 2.2 holds for $f = g = w'$, and the corollary follows.

LEMMA 2.3. *In the above notation, $\{\Omega_s\}_{s \in S} \in \tilde{R}$.*

Proof. By the definition, for each $s \in S$, $\Omega_s \in R/P_s$. Let $\lambda, \mu \in S$ such that $\lambda \geqq \mu$. Then, by Lemma 2.1, there exist $m, n \in S$ such that $\Omega_\lambda = x'y' + P_\lambda$ for any $x' \in \xi_m$, $y' \in \eta_m$; and $\Omega_\mu = x''y'' + P_\mu$ for any $x'' \in \xi_n$, $y'' \in \eta_n$. Let $q \in S$ such that $q \geqq m, n$; and let $x \in \xi_q$, $y \in \eta_q$. Then $\Omega_\lambda = xy + P_\lambda$ and $\Omega_\mu = xy + P_\mu$. Hence $\Omega_\lambda \subseteqq \Omega_\mu$ since $P_\lambda \subseteqq P_\mu$. Therefore $\{\Omega_s\}_{s \in S} \in \tilde{R}$.

PROPOSITION 2.1. *With the multiplication defined above, $\tilde{R}$ is a ring which is commutative if R is commutative and has identity $\{1 + P_s\}_{s \in S}$ if R has identity 1.*

Proof. We already have that $\tilde{R}$ is an additive Abelian group.
(i) Using the definition of multiplication in $\tilde{R}$ and the directed

property of S, it is a straightforward task to show that multiplication in $\tilde{R}$ is associative and that both distributive laws hold. Hence $\tilde{R}$ is a ring which, by the definition of multiplication, is commutative if R is commutative.

(ii) Let R have identity 1. As noted before, $\{1 + P_s\}_{s \in S} \in \tilde{R}$. Again, using the directed property of S and the fact that, for each $s \in S$, $1 \in 1 + P_s$, it is a straightforward task to show that $\{1 + P_s\}_{s \in S}$ is the identity of $\tilde{R}$.

Next we show that $\{\tilde{P}_s\}_{s \in S}$, the fundamental system of neighbourhoods of the zero of $(\tilde{R}, \tilde{\mathscr{T}})$, is in fact a filtration on $(\tilde{R}, *)$ which defines the topology $\tilde{\mathscr{T}}$ as at the beginning of §2; and hence the multiplication $*$ is continuous in $(\tilde{R}, \tilde{\mathscr{T}}, *)$. We need the following preliminary result.

LEMMA 2.4. *Let $x \in R$, $t \in S$. Then there exists $u \in S$ such that $\{x + P_s\}_{s \in S} * \tilde{P}_u \subseteqq \tilde{P}_t$ and $\tilde{P}_u * \{x + P_s\}_{s \in S} \subseteqq \tilde{P}_t$.*

Proof. By (1.4) there exists $v \in S$ such that $xP_v \subseteqq P_t$; and by (1.1) there exists $w \in S$ such that $w^2 \geqq t$. Let $u \in S$ such that $u \geqq v, w$. Let $\{\eta_s\}_{s \in S} \in \tilde{P}_u$; that is, $\eta_u = P_u$. Let $x_1 \in x + P_u$, $y_1 \in P_u$. Then $x_1 y_1 \in P_t$ since $P_u \subseteqq P_v \cap P_w$, and so $xP_u \subseteqq P_t$, $P_u P_u \subseteqq P_t$. Therefore, for all $x', x'' \in x + P_u$ and for all $y', y'' \in \eta_u$, $x'y' - x''y'' \in P_t$. Hence, by Lemma 2.2, with $f = g = u$, $s = t$, $a = x_1$ and $b = y_1$, $\{x + P_s\}\{\eta_s\} = \{\Omega_s\}$ where $\Omega_t = P_t$: that is, $\{x + P_s\}\{\eta_s\} \in \tilde{P}_t$. Similarly $\tilde{P}_u\{x + P_s\} \subseteqq \tilde{P}_t$.

PROPOSITION 2.2. *$\{\tilde{P}_s\}_{s \in S}$ is a filtration on $\tilde{R}$ which defines the topology $\tilde{\mathscr{T}}$.*

Proof. (i) S is a directed semigroup with property (1.1) and, as noted, each $\tilde{P}_s$, $s \in S$, is an additive subgroup of $\tilde{R}$.

(ii) Let $t, u \in S$ such that $u \geqq t$. It is easily checked that $\tilde{P}_u \subseteqq \tilde{P}_t$.

(iii) Let $t, u \in S$. Again, it is easily checked that $\tilde{P}_t \tilde{P}_u \subseteqq \tilde{P}_{tu}$.

(iv) Let $\{\xi_s\} \in \tilde{R}$, $t \in S$. We must show that there exists $r \in S$ such that $\{\xi_s\}\tilde{P}_r \subseteqq \tilde{P}_t$ and $\tilde{P}_r\{\xi_s\} \subseteqq \tilde{P}_t$. Let $w \in S$ such that $w^2 \geqq t$ and let $x \in \xi_w$. Then $\{\xi_s\} - \{x + P_s\} \in \tilde{P}_w$. By Lemma 2.4 there exists $u \in S$ such that $\{x + P_s\}\tilde{P}_u \subseteqq \tilde{P}_t$. Let $r \in S$ such that $r \geqq u, w$; and let $\{\zeta_s\} \in \tilde{P}_r$. Now $\{\xi_s\}\{\zeta_s\} = (\{\xi_s\} - \{x + P_s\})\{\zeta_s\} + \{x + P_s\}\{\zeta_s\}$; $(\{\xi_s\} - \{x + P_s\})\{\zeta_s\} \in \tilde{P}_w \tilde{P}_r \subseteqq \tilde{P}_{wr} \subseteqq \tilde{P}_t$ by (ii) and (iii); and $\{x + P_s\}\{\zeta_s\} \in \{x + P_s\}\tilde{P}_u \subseteqq \tilde{P}_t$ since $r \geqq u$. Hence, by (i), $\{\xi_s\}\{\zeta_s\} \in \tilde{P}_t$. Similarly $\tilde{P}_r\{\xi_s\} \subseteqq \tilde{P}_t$. This completes the proof.

THEOREM 2.1. *The Hausdorff completion of* $(R, \mathscr{T})$ *is isomorphic to* $(\tilde{R}, \tilde{\mathscr{T}}, *)$.

Proof. By [3, III §7.3, Proposition 2], the mapping $i: R \to \tilde{R}$ given by: for all $x \in R$, $i(x) = \{x + P_s\}_{s \in S}$, has an image which is dense in $(\tilde{R}, \tilde{\mathscr{T}})$. From [3, III §6.5 and III §7.3, Proposition 2, Corollary 1], the mapping $i: R \to (\tilde{R}, \times)$ is a ring homomorphism. Hence $i(xy) = i(x) \times i(y)$. But

$$i(xy) = \{xy + P_s\}_{s \in S} = \{x + P_s\}_{s \in S} * \{y + P_s\}_{s \in S} = i(x) * i(y) .$$

Thus the multiplications $*$ and $\times$, which are continuous in $\tilde{\mathscr{T}}$, agree on the dense subset $i(R)$ of $(\tilde{R}, \tilde{\mathscr{T}})$. Therefore, by the principle of extension of identities [3, I §8.1], $*$ and $\times$ agree on $\tilde{R}$. Thus $(\tilde{R}, \tilde{\mathscr{T}}, *)$ is the Hausdorff completion of $(R, \mathscr{T})$.

3. Example. In this section we illustrate our theory with a semivaluation on the field Q of rational numbers (Zelinsky [7]).

We shall reserve the sign "$\geqq$" for the usual ordering on Q and shall denote the usual absolute value of the rational number x by $|x|$. Define $S = \{x: x \in Q, x > 0\}$. Order S as follows: For all $a, b \in S$, $a \geqq b$ if and only if $ab^{-1} \in I$ (the set of natural numbers). Then $(S, \geqq)$ is a quasi-residuated, directed semigroup under multiplication. Define a mapping $\varphi: Q \to S_0$ as follows: For all $x \in Q \backslash \{0\}$, $\varphi(x) = |x|$; and $\varphi(0) = O_S$. Then it can easily be checked that $\varphi: Q \to S_0$ is a pseudovaluation on Q. (In fact, φ is a semivaluation on Q, from Zelinsky [7]).

PROPOSITION 3.1. *The completion of* Q *with respect to* φ *is isomorphic to the ring of formal series* $\sum_{i=1}^{\infty} i!\, a_i$ *where* $a_i \in Q$, $0 \leqq a_1 < 2$, *and, for each* $i \in I \backslash \{1\}$, $a_i \in \{0, 1, \cdots, i\}$.

Proof. We shall use the notation of §§1 and 2 throughout. Now, for each $s \in S$,

$$P_s = \{x: x \in Q, \varphi(x) \geqq s\} = \{ms: m \in Z\} .$$

We shall use the fact that, for all $p, q \in I$, $p! \geqq p \geqq p/q$ and $p! \geqq (p-1)!$: that is, for all $\{\xi_s\}_{s \in S} \in \tilde{Q}$, $\xi_{p!} \subseteqq \xi_p \subseteqq \xi_{p/q}$ and $\xi_{p!} \subseteqq \xi_{(p-1)!}$.

(i) Let $\{\xi_s\}_{s \in S} \in \tilde{Q}$.

Let $x_1 \in \xi_2$. Then there exists a unique $a_1 \in Q$ such that $0 \leqq a_1 < 2$ and $x_1 - a_1 \in P_2$. Suppose that $x_1' \in \xi_2$ and $a_1' \in Q$ such that $0 \leqq a_1' < 2$ and $x_1' - a_1' \in P_2$. Then

$$a_1 - a_1' = (x_1' - a_1') - (x_1 - a_1) + (x_1 - x_1') \in P_2 .$$

Hence $a_1 = a_1'$, and so a_1 is independent of x_1. Then $\xi_2 = a_1 + P_2$.

Let $x_2 \in \xi_{3!} - (a_1 + P_{3!})$. Since $\xi_{3!} \subseteqq \xi_2$ and $P_{3!} \subseteqq P_2$, we have $x_2 \in \xi_2 - (a_1 + P_2) = P_2$. Hence $x_2/2$ is an integer. Let $a_2 \in \{0, 1, 2\}$ such that $a_2 \equiv x_2/2 \bmod 3$. Then $\xi_{3!} = a_1 + 2a_2 + P_{3!}$.

Next, suppose $k \in I\backslash\{1, 2\}$ such that $\xi_{k!} = a_1 + \sum_{i=2}^{k-1} i!\, a_i + P_{k!}$ where $a_i \in \{0, 1, \cdots, i\}$ for each $i \in \{2, 3, \cdots, k-1\}$. As before, we can show that there exists $a_k \in \{0, 1 \cdots, k\}$ such that $\xi_{(k+1)!} = a_1 + \sum_{i=2}^{k} i!\, a_i + P_{(k+1)!}$. Further, each a_i is unique.

Let $s \in S$. Then there exist unique $p, q \in I$ such that $s = p/q$ and $(p, q) = 1$. Now $\xi_{p!} \subseteqq \xi_{p/q}$. Hence $\xi_s = \sum_{i=1}^{p-1} i!\, a_i + P_s$.

Suppose that $\{\xi_s\}_{s \in S}$ and $\{\eta_s\}_{s \in S} \in S$ define the same set of a_i, $i \in I$. Then, for each $s \in S$, $\xi_s = \eta_s$. Hence $\{\xi_s\}_{s \in S}$ defines a unique set of a_i, $i \in I$.

(ii) Let $\{a_i\}_{i \in I}$ be given such that $a_1 \in Q$, $0 \leqq a_1 < 2$ and, for each $i \in I\backslash\{1\}$, $a_i \in \{0, 1, \cdots, i\}$. Let $s \in S$. Then, as before, there exists a unique $p \in I$ such that $p \geqq s$. Define $\xi_s = \sum_{i=1}^{p-1} i!\, a_i + P_s$. It is a straightforward task to show that $\{\xi_s\}_{s \in S} \in \tilde{Q}$.

Thus far we have established a one-one correspondence between the elements of $\tilde{Q}$ and formal power series $\sum_{i=1}^{\infty} a_i\, i!$ where $a_1 \in Q$, $0 \leqq a_1 < 2$, and, for each $i \in I\backslash\{1\}$, $a_i \in \{0, 1, \cdots, i\}$.

(iii) Let $\{\xi_s\}_{s \in S}, \{\eta_s\}_{s \in S} \in \tilde{Q}$ correspond to the series $\sum_{i=1}^{\infty} i!\, a_i, \sum_{i=1}^{\infty} i!\, b_i$ respectively. Now $\{\xi_s\}_{s \in S} + \{\eta_s\}_{s \in S} = \{\xi_s + \eta_s\}_{s \in S}$. Hence we can define addition of the series as would be hoped: $\sum_{i=1}^{\infty} i!\, a_i + \sum_{i=1}^{\infty} i!\, b_i = \sum_{i=1}^{\infty} i!\, (a_i + b_i)$ where at the ith stage $a_i + b_i$ is reduced modulo $(i+1)$ and the integral part of $(a_i + b_i)/(i + 1)$ carried on.

Let $\{\Omega_s\} = \{\xi_s\}\{\eta_s\}$. Let $s \in S$. Then there exists $t \in S$ such that, for all $x \in \xi_t$, $y \in \eta_t$, $\Omega_s = xy + P_s = \sum_{i=1}^{k-1} i!\, a_i \sum_{i=1}^{k-1} i!\, b_i + P_s$ for some $k \in I$. Hence we can define multiplication of the series in the usual way, taking care to correct each term as described for the addition. This proves the proposition.

Remark. The above example illustrates that the definition of multiplication $*$ in $\tilde{R}$ in §2 cannot be obviously simplified. For example, if $\{\xi_s\}_{s \in S} = \{5 + P_s\}_{s \in S}$ and $\{\eta_s\}_{s \in S} = \{3 + P_s\}_{s \in S}$, then $\{\Omega_s\}_{s \in S} = \{\xi_s\}_{s \in S}\{\eta_s\}_{s \in S} = \{15 + P_s\}_{s \in S}$. Now $\xi_2 = 1 + P_2 = \eta_2$, but $\Omega_4 = 3 + P_4$:

that is, it would not have been sufficient to choose the w of §2 such that $w^2 \geqq s$.

I would like to thank my supervisor, Dr C. F. Moppert, for his many valuable suggestions.

REFERENCES

1. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley (1969).
2. T. S. Blyth and M. F. Janowitz, *Residuation Theory*, Pergamon Press (1972).
3. N. Bourbaki, *General Topology*, Chapters 1 and 3, Hermann, Addison-Wesley (1966).
4. K. Mahler, *Über pseudobewertungen I*, Acta Math. **66** (1935) 79-119.
5. M. E. Manis, *Valuations on a commutative ring*, Proc. Amer. Math. Soc., **20** (1969), 193-198.
6. D. G. Northcott, *Lessons on Rings, Modules and Multiplicities*, Cambridge University Press, Cambridge (1968).
7. D. Zelinsky, *Topological characterization of fields with valuations*, Bull. Amer. Math. Soc., **54** (1948), 1145-1150.

Received April 2, 1971 and in revised form September 30, 1971.

MONASH UNIVERSITY, CLAYTON, VICTORIA, 3168, AUSTRALIA
AND
MOUNT HOLYOKE COLLEGE