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ON GROUPS OF EXPONENT FOUR SATISFYING AN ENGEL CONDITION

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Let $B(n)$ be the Burnside (i.e., freest) group of exponent 4 on n generators. It is known that $B(n)$ is nilpotent of class at most $3n-1$. This paper exhibits a commutator of length $3n-1$ in $B(n)$ which must be nontrivial if the class is exactly $3n-1$. The methods also yield an easy proof of the following.

THEOREM. **Let $E(n)$ be $B(n)$ reduced modulo the identical commutator relation**

$$(a_1, \cdots, a_{2n-4}, x, x, (y, z, z, z)) = 1.$$

Then $E(n)$ is nilpotent of class at most $2n+3$.

As an immediate corollary, every n-generator group of exponent 4 satisfying the Engel condition $(x, y, y, y) = 1$ identically is of class at most $2n+3$.

The theorem follows from Proposition 1 together with an elementary commutator calculation. The main point of the Proposition, however, is that it exhibits the stumbling block to a reduction in the class of $B(n)$ below $3n-1$ and at the same time suggests that perhaps if for some n the class is less than $3n-1$ then the class in general is at most $2n+k$ for some fixed k. Recent work of Gupta and others ([1], [2], [3]) has renewed interest in precise determination of the class and also in groups of exponent 4 satisfying Engel conditions. This paper updates the techniques of [4] as they appear to apply to these problems.

PRELIMINARIES. This paper may be viewed as a continuation of [4]. Notation is the same, and for $i = 1, \cdots, 9, A$ we denote formula (i) of [4] by (i) here, too. The symbol (i) in the margin at the right of an equation or congruence indicates that identity (i) justifies it. The notation $\langle x, \cdots, y\rangle$ stands for the group genarated by $\{x, \cdots, y\}$.

LEMMA. *The following commutator identities hold in a group of exponent* 4.

(B). $(x, (u, v, w)) \equiv (x, u, w, v,)(x, v, w, u) \bmod \langle x, u, w, v\rangle_5$.

(C). $(x, y, y, z, z, z) \equiv 1 \bmod \langle x, y, z\rangle_7$.

(D). $(x, y, y, y, (z, w)) \equiv 1 \bmod \langle x, y, z, w\rangle_7$.

Proof. Since

$$\begin{aligned}
&(x, (u, v, w)) \\
&\equiv (x, (u, v), w)(x, w, (u, v)) && (3) \\
&\equiv (x, u, v, w)(x, v, u, w)(x, u, (v, w))(x, v, (u, w)) && (3), (4) \\
&\equiv (x, u, w, v)(x, v, w, u) , && (3)
\end{aligned}$$

(B) holds.

Since

$$(x, y, y, z, z, z) \equiv (x, y, y, z)^2 \equiv ((x, y, y)^2, z) \equiv 1$$

by (2) and Theorem 2 of [4], (C) holds.

Finally, since

$$\begin{aligned}
(z, w, (x, y, y, y)) &\equiv (z, w, (x, y), y, y)(z, w, y, y, (x, y)) \\
&\equiv (z, w, x, y, y, y)(z, w, y, x, y, y) \\
&\quad \times (z, w, y, y, x, y)(z, w, y, y, y, x) \equiv 1 \qquad (3)
\end{aligned}$$

by (7) and (8), (D) holds.

LEMMA. *Let G be a group of exponent 4 with $G_{r+1} = 1$, and let a and x be in G. Then every commutator in G of length r of form*

$$(\cdots, x, x, a, x)$$

is a product of commutators of forms

$$(a, \cdots, x, x, x)$$

and

$$(a, \cdots, x, x, b, x)$$

each with the same entries as the given commutator.

Proof. By induction on r. Since $(x, x, a, x) = 1$, and

$$\begin{aligned}
(b, x, x, a, x) &\equiv (b, x^2, a, x) \\
&\equiv (a, x^2, b, x)(a, b, x^2, x) && (3) \\
&\equiv (a, x, x, b, x)(a, b, x, x, x) ,
\end{aligned}$$

the result is true for $r \leqq 5$. Now by (B),

$$\begin{aligned}
(c, \ldots, d, e, x, x, a, x) &\equiv (c, \ldots, d, e, x^2, a, x) \\
&\equiv (c, \ldots, d, a, x^2, e, x)(c, \ldots, d, (a, e, x^2), x) && (B) \\
&\equiv (c, \ldots, d, a, x^2, e, x)(c, \ldots, d, (a, e), x, x, x) \\
&\quad \times (c, \ldots, d, x, x, (a, e), x) . && (3)
\end{aligned}$$

The first two factors are products of commutators of the required forms by (A). The last factor is a product of commutators of forms

$$(a, e, \ldots, x, x, x)$$

and

$$(a, e, \ldots, x, x, b, x)$$

by the inductive assumption.

A consequence of this result is that Lemma 2 of [4] can be strengthened by the additional conclusion that $y_1 = x_1$, i.e., that the first entry in $(x_1, \ldots, x_n)$ can be held fixed. It is clear from the proof of Lemma 2 that each commutator which arises has $x_1, \ldots, x_n$ in some order as its entries.

The main results.

PROPOSITION 1. *Let G be a group of exponent 4, and let $r \geqq 3n \geqq 6$. Modulo G_{r+1}, every commutator $(a_1, \ldots, a_r)$ in which some n entries each appear three or more times is a product of commutators of form*

$$(a, b, \ldots, x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}, c, x_{n-1}, \ldots, x_1)$$

with entries some permutation of $a_1, \ldots, a_r$.

Proof. We may assume that $G_{r+1} = 1$, that $r > 3n$, by Theorem 3 of [4], and that no entry in $(a_1, \ldots, a_r)$ occurs more than three times, by Theorem 1 of [4]. Say each of $x_1, \ldots, x_n$ appears three times among $a_1, \ldots, a_r$. Since $r > 3_n$, we may suppose that $a_1 = a \notin \{x_1, \ldots, x_n\}$, by (A) of [4]. Since $n \geqq 2$, some x_i (say x_1) appears three times among $a_3, \ldots, a_r$. By Lemma 2 of [4] as just strengthened, we need only consider the forms

$$\text{(I)} \qquad (a, \ldots, x_1, x_1, x_1)$$

and

$$\text{(II)} \qquad (a, \ldots, x_1, x_1, b, x_1)\ .$$

Case (I). By (7), (I) is equivalent to

$$(a, b, x_1, x_1, x_1, \ldots)\ .$$

At least two of the last $r - 5$ entries here are the same, say x_2, since $n \geqq 2$ and $a \neq x_2$. By repeated use of (D) and (3) these entries can be brought forward to give

$$(a, b, x_1, x_1, x_1, x_2, x_2, \ldots)\ .$$

By (7), $(a, b, x, x, x, y, y) \equiv (a, b, y, y, x, x, x,)$, and now (C) applies. So $(a_1, \ldots, a_r)$ is trivial in this case.

Case (II). We have

$$
\begin{aligned}
&(a, c, \ldots, x_1, x_1, b, x_1) \\
&= (a, c, x_1, x_1, d, x_1, \ldots, b) && (9) \\
&= (a, c, x_1, d, x_1^2, \ldots, b) && (8) \\
&\equiv (a, c, (x_1, d), x_1^2, \ldots, b)(a, c, d, x_1, x_1, x_1, \ldots, b) && (3) \\
&= (a, x_1^2, (x_1, c), d, \ldots, b) && (5) \text{ and Case (I)}. \\
&= (a, x_1^2, c, x_1, d, \ldots, b)(a, x_1, x_1, x_1, c, d, \ldots, b) && (3) \\
&= (a, x_1^2, c, x_1, d, \ldots, b),
\end{aligned}
$$

the last step by the argument of Case (I).

Suppose inductively that we have reached the form

$$(a, x_1^2, \ldots, x_i^2, c, x_i, \ldots, x_1, \ldots)$$

with $1 \leqq i < n$. Some three of the last $r - 3i - 2$ entries are the same, say x_{i+2}, and the argument just given yields the form

$$(a, x_1^2, \ldots, x_i^2, x_{i+1}^2, c, x_{i+1}, x_i, \ldots, x_1, \ldots) ,$$

where the improved Lemma 2 is used to keep the starting block of length $3i - 2$ at the front. The proposition follows by finite induction, using (9).

Together with (D), Proposition 1 shows in particular that $B(n)_{3n-1} = 1$ precisely if all commutators of form

$$(a^2, x_1^2, x_2^2, x_3^2, \ldots, x_{n-1}^2, x_1, x_{n-1}, \ldots, x_3, x_2)$$

are trivial.

PROPOSITION 2. *Let G be a group of exponent 4. Let $m \geqq 9$. If every commutator of length $m - 1$ in G of form*

$$(\ldots, x, x, (w, y, y, y))$$

is in G_{m+1}, then every commutator of length m in G of form

$$(\ldots, x, x, y, y, z, y, x)$$

is in G_{m+1}.

Proof. We may assume that $G_{m+1} = 1$. Now for $a \in G_{m-7}$

$$
\begin{aligned}
&(a, x, x, y, y, z, y, x) \\
&= (a, x, x, z, y, y, x, y) && (9) \\
&= (a, x, x, z, y^2, (x, y))(a, x, x, z, y, y, y, x) && (3) \\
&= (a, x, x, (x, y), y^2, z)(a, x, x, (x, y, z, y^2)) \\
&\quad \times (a, x, x, y, y, y, z, x) && (\text{B}), (7)
\end{aligned}
$$

$$= (a, x, x, x, y, y, y, z)(a, x, x, y, x, y, y, z)$$
$$\times (a, x, x, (x, y, z, y^2) \qquad (3), (C)$$
$$= (a, x, x, x, y, z, y, y)(a, x, x, (x, y, z, y^2)) \qquad (C), (8), (9)$$
$$= (a, x, x, (y, x, z, y^2)) \qquad (C)$$
$$= (a, x, x, (y, x, y^2)(y, z, y^2)(y, xz, y^2)) = 1$$

by hypothesis.

Now let $n \geqq 3$ and let $E(n)$ be $B(n)$ reduced modulo the identical relation

$$(a_1, \ldots, a_{2n-4}, x, x, (y, z, z, z)) = 1 .$$

By Proposition 2 with $m = 2n + 3$, every commutator of length $2n + 3$ in $E(n)$ of form $(\ldots, x, x, y, y, z, y, x)$ is in $E(n)_{2n+4}$. Hence, by Proposition 1, every commutator of length 2_{n+4} in $E(n)$ in which three or more entries each appear three times is in $E(n)_{2n+4}$. Finally, by Theorem 1 of [4], every commutator of length $2n + 3$ in $E(n)$ in which some entry appears four or more times is in $E(n)_{2n+4}$. The theorem stated in the introduction now follows.

Added in proof. By substituting uv for y in (C) and linearizing, one obtains $(u, v, x, z, z, z) \equiv 1 \bmod \langle u, v, x, z\rangle_7$, which shortens some of the arguments given above.

Ī. D. Īvanjuta [Certain groups of exponent four, Dopovīdī Akad. Nauk Ukrain RSR Ser. A (1969), 787-790)] has shown that every n-generator group of exponent 4 satisfying $(x, y, y, y) = 1$ identically has class at most $2n$. His methods are specific to such groups, however, and do not apply readily to $B(n)$ or $E(n)$.

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