PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 2, 1973

ABEL SUMMABILITY OF CONJUGATE INTEGRALS

STANTON PHILIPP

It is proved here that the conjugate Fourier-Stieltjes integral of a finite-valued Borel measure μ on Euclidean k-space, $k \geqq 1$, taken with respect to a Calderon-Zygmund kernel in Lip $\alpha, 0 < \alpha < 1$, is almost everywhere (with respect to Lebesgue measure) Abel summable to the conjugate function of μ taken with respect to the above mentioned kernel. This has been already established for $k \leqq 3$ and for k even.

We make the following assumptions: k is a positive integer; $0 < \alpha < 1$; $\Omega \in \mathrm{Lip}\,\alpha(S)$, where S denotes the $(k-1)$-sphere in k-dimensional Euclidean space E_k; $\int_S \Omega(y)dS(y) = 0$, where dS refers to the natural measure on S; and μ is a real Borel measure on E_k as defined in [3].

Let $K(x) = \Omega(x/|x|)|x|^{-k}$ for each nonzero x in E_k (we use $|x|$ for the usual norm and $x \cdot y$ for the usual dot product and dx for Lebesgue measure, all in E_k). For y in E_k, set $\hat{\mu}(y) = (2\pi)^{-k} \int_{E_k} e^{-ix\cdot y}d\mu(x)$ and

$$\hat{K}(y) = (2\pi)^{-k} \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon \leqq |x| \leqq R} e^{-ix\cdot y}K(x)dx .$$

It is known [5, p. 69] that $\hat{K}$ is bounded on E_k. We define, for $t > 0$ and x in E_k,

$$I_t(x) = (2\pi)^k \int_{E_k} e^{-t|y|}\hat{k}(y)\hat{\mu}(y)e^{ix\cdot y}dy .$$

We shall prove the

THEOREM.
$$\lim_{t \to 0}\left[I_t(x) - \int_{|y-x|>t} K(x-y)d\mu(y)\right] = 0$$
except on a set of Lebesgue measure zero in E_k.

If $k = 1$, the theorem is classical (see [8, p. 103] for the essence of the matter). The case $k = 2$ and $1/2 < \alpha < 1$ was treated in [4]. The cases in which $0 < \alpha < 1$ and $k = 3$ or k is even were handled in [2]. Further references and motivation for the theorem are given in [2] and [4]. The proof given in the present paper covers all cases with $0 < \alpha < 1$ and $k \geqq 3$; modifications could be made in the proof to cover the cases $k = 2$, $0 < \alpha < 1$ and $k = 1$, but this seems pointless.

Proof of the theorem. Assume $k \geqq 3$. Define, for $t > 0$ and $n = 1, 2, \cdots$,

$$H_n^k(t) = \frac{\Gamma\left(\frac{n}{2}\right)}{2^{k/2}\Gamma\left(\frac{n+k}{2}\right)} \int_0^\infty e^{-ts} s^{k/2} J_{n-1+k/2}(s) ds , \tag{1}$$

where $J_{n-1+k/2}(s)$ denotes a standard Bessel function of first kind. We assume throughout that $t > 0$. It follows from (1) and [6, p. 385] and [7, p. 282] that for $n = 1, 2, \cdots$,

$$\begin{aligned} 0 &\leqq H_n^k(t) \\ &= \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+k+1}{2}\right)\pi^{-1/2}}{\Gamma\left(n+\frac{k}{2}\right)(t^2+1)^{(n+k)/2}} F\left(\frac{n+k}{2}, \frac{n-1}{2}; n+\frac{k}{2}; \frac{1}{1+t^2}\right) \\ &\leqq \left(\frac{1}{1+t^2}\right)^{(k+n)/2} . \end{aligned} \tag{2}$$

Here and throughout $F(a, b; c; x)$ refers to Gauss' hypergeometric series. Let the series of spherical harmonics associated with $\Omega(\xi)$ be $\sum_{n=1}^\infty Y_n(\xi)$. Let ω denote the $(k-1)$-dimensional volume of S and let ξ be a point of S. We shall use the symbol A generically to denote any positive constant which depends only on Ω and k. Let $\lambda = (k-2)/2$. The Gegenbauer polynomials $P_n^\lambda(\cos\theta)$ are defined in [5]. By (2) and the boundedness of Ω and the fact [1, p. 245] that

$$P_n^\lambda(\cos\theta) \,| \leqq P_n^\lambda(1) \leqq A n^{2\lambda-1} \tag{3}$$

for $n = 0, 1, 2, \cdots$ and $0 \leqq \theta \leqq \pi$, we may write

$$\begin{aligned} \sum_{n=1}^\infty H_n^k(t) Y_n(\xi) &= \sum_{n=1}^\infty H_n^k(t) \frac{n+\lambda}{\lambda\omega} \int_S \Omega(y) P_n^\lambda(\xi\cdot y) dS(y) \\ &= \frac{1}{\omega}\int_S \Omega(y)\left[\sum_{n=1}^\infty \frac{n+\lambda}{\lambda} H_n^k(t) P_n^\lambda(\xi\cdot y)\right] dS(y) . \end{aligned} \tag{4}$$

By the concluding argument given in [2], we see that in order to prove the theorem it suffices to prove that

$$\left|\sum_{n=1}^\infty H_n^k(t) Y_n(\xi) - \Omega(\xi)\right| \leqq A t^\alpha . \tag{5}$$

So we define a kernel $K_t(\theta)$ by

$$K_t(\theta) = 1 + \sum_{n=1}^\infty \frac{n+\lambda}{\lambda} H_n^k(t) P_n^\lambda(\cos\theta) \tag{6}$$

for $0 \leqq \theta \leqq \pi$. Now

$$\frac{1}{\omega}\int_S K_t(\xi\cdot y)dS(y) = 1\ , \tag{7}$$

as may be seen from (4). It follows from (4), (6), and (7) that

$$\sum_{n=1}^{\infty} H_n^k(t)\, Y_n(\xi) - \Omega(\xi) = \frac{1}{\omega}\int_S [\Omega(y) - \Omega(\xi)]K_t(\xi\cdot y)dS(y)\ . \tag{8}$$

Notice that if we can establish that

$$|K_t(\theta)| \leqq \frac{At}{\left(t^2 + \sin^2\dfrac{\theta}{2}\right)^{k/2}} \tag{9}$$

for $0 \leqq \theta \leqq \pi$, then it will follow from (8), (9), and the Lipschitz condition on Ω that

$$\begin{aligned}\left|\sum_{n=1}^{\infty} H_n^k(t)\, Y^u(\xi) - \Omega(\xi)\right| &\leqq A\int_S |K_t(\xi\cdot y)|\cdot|y - \xi|_\alpha dS(y)\\ &\leqq At\int_0^\pi \frac{(1-\cos\theta)^{\alpha/2}\sin^{k-2}\theta}{\left(t^2 + \sin^2\dfrac{\theta}{2}\right)^{k/2}}d\theta \leqq A\int_0^\pi \frac{t\theta^{k-2+\alpha}}{({}^2t+\theta^2)^{k/2}}d\theta \leqq At^\alpha\ .\end{aligned}$$

That is, once we prove (9), then (5) follows and we are done.

Now define

$$P_\nu(s,\, t) = \int_0^\infty e^{-ty} y s^{1/2} J_\nu(sy) J_\nu(y) dy \tag{10}$$

for $\nu > -(1/2)$ and $s > 0$. The relations

$$J_\nu(x) = 0(x^{-1/2}) \quad \text{as} \quad x \to \infty,\ J_\nu(x) = 0(x^\nu) \quad \text{as} \quad x \to 0+\ , \tag{11}$$

which are valid for each $\nu > -(1/2)$ and are proved in [6], will be useful to us. Using (10), (11), Fubini's theorem, the change of variable $sy = r$ and [6, p. 391], we obtain that

$$\begin{aligned}&\int_0^\infty s^{-\lambda-3/2} P_{n+\lambda}(s,\, t) ds\\ &\qquad = \int_0^\infty e^{-ty} y^{\lambda+1} J_{n+\lambda}(y) dy \int_0^\infty r^{-\lambda-1} J_{n+\lambda}(r) dr = H_n^k(t)\end{aligned} \tag{12}$$

for $n \geqq 1$. It follows from (6) and (12) that

$$K_t(\theta) = 1 + \sum_{n=1}^{\infty} \frac{n+\lambda}{\lambda} P_n^\lambda(\cos\theta) \int_0^\infty s^{-\lambda-3/2} P_{n+\lambda}(s,\, t) ds \tag{13}$$

for $0 \leqq \theta \leqq \pi$. By Gegenbauer's addition theorem [6, p. 363] we may write

$$\frac{J_\lambda(y\sqrt{s^2-2s\cos\theta+1})\,s^{\lambda+1/2}y^{\lambda+1}e^{-ty}}{(s^2-2s\cos\theta+1)^{\lambda/2}2^\lambda\Gamma(\lambda+1)} \tag{14}$$
$$=\sum_{n=0}^{\infty}\frac{n+\lambda}{\lambda}ys^{1/2}J_{n+\lambda}(y)J_{n+\lambda}(sy)P_n^\lambda(\cos\theta)e^{-ty}$$

for $y>0$, $s>0$, $0\leqq\theta\leqq\pi$. Because of (3) and the inequality [6, p. 49]

$$|J_{n+\lambda}(x)|\leqq\frac{Ax^{n+\lambda}}{\Gamma(n+\lambda)},$$

valid for $x>0$ and $n\geqq 0$, we may integrate the right member of (14) term by term with respect to y over $(0,\infty)$. So, by integrating both sides of (14) with respect to y over $(0,\infty)$ and then using (10) in the right member and [6, p. 391] in the left member, we obtain that

$$\sum_{n=1}^{\infty}\frac{n+\lambda}{\lambda}P_{n+\lambda}(s,t)P_n^\lambda(\cos\theta) \tag{15}$$
$$=\frac{\Gamma\left(\lambda+\frac{3}{3}\right)t(z-\cos\theta)^{-\lambda-3/2}}{2^{\lambda+1/2}\Gamma(\lambda+1)\pi^{1/2}s}-P_\lambda(s,t)$$

for $0\leqq\theta\leqq\pi$, $s>0$, where

$$z=\frac{s^2+t^2+1}{2s} \tag{16}$$

for $s>0$. We shall adhere to the notation (16). It follows from [6, p. 389] and [7, pp. 317 and 281] that

$$P_\nu(s,t) \tag{17}$$
$$=\frac{\Gamma\left(\nu+\frac{3}{2}\right)ts^{-1}}{2^{\nu+1/2}\pi^{1/2}\Gamma(\nu+1)}z^{-\nu-3/2}F\left(\frac{\nu+\frac{1}{2}}{2},\frac{\nu+\frac{3}{2}}{2}:\nu+1;z^{-2}\right)$$
$$+\frac{\Gamma\left(\nu+\frac{5}{2}\right)ts^{-1}}{2^{\nu+3/2}\pi^{1/2}\Gamma(\nu+2)}z^{-\nu-7/2}F\left(\frac{\nu+\frac{5}{2}}{2},\frac{\nu+\frac{7}{2}}{2}:\nu+2;z^{-2}\right)$$

for $\nu>1/2$ and $s>0$. Proceeding formally, we multiply both sides of (15) by $s^{-\lambda-3/2}$ and integrate with respect to s over $(0,\infty)$ term by term in the left member; using (13), the result is that

$$K_t(\theta) \tag{18}$$
$$=1+\int_0^\infty s^{-\lambda-3/2}\left[\frac{\Gamma\left(\lambda+\frac{3}{2}\right)t(z-\cos\theta)^{-\lambda-3/2}}{2^{\lambda+1/2}\Gamma(\lambda+1)\pi^{1/2}s}-P_\lambda(s,t)\right]ds$$

for $0 \leqq \theta \leqq \pi$. To justify this formal procedure, we observe that, by (17), $P_{n+\lambda}(s, t) \geqq 0$ for $n \geqq 1$, $s > 0$; therefore, by (3), (15) and Fubini's theorem, we can be sure that (18) holds if the integral in the right member of (18) is a finite Lebesgue integral when $\theta = 0$. But it follows from (17) that for fixed t and $\theta = 0$, the integrand in the right member of (18) is $O(1)$ as $s \rightarrow 0$; and by (10) and (11), this same integrand is $O(s^{-\lambda-3/2})$ as $s \rightarrow \infty$. Therefore (18) holds.

We proceed to estimate the right member of (18) by using the expression for $P_\lambda(s, t)$ given by (17). Observe that if $s>0$ and $|s-1| \geqq 1/2$, then $z \geqq 13/12$; and recall that the radius of convergence of the hypergeometric series is at least one. Therefore, using [7, p. 299, Ex. 18], we may write

$$F\left(\frac{\lambda + \frac{1}{2}}{2}, \frac{\lambda + \frac{3}{2}}{2}; \lambda + 1; z^{-2}\right) = 1 + \varphi(z) , \tag{19}$$

where

$$|\varphi(z)| \leqq Az^{-2} \qquad \text{for} \quad |s - 1| \geqq \frac{1}{2}, \quad s > 0 , \tag{20}$$

and

$$|\varphi(z)| \leqq A \log (1 - z^{-2})^{-1} \qquad \text{for} \quad |s - 1| < \frac{1}{2} . \tag{21}$$

It follows from (20) and (21) that

$$\int_0^\infty s^{-\lambda-3/2} t z^{-\lambda-3/2} s^{-1} \varphi(z) ds = 0(t) \quad \text{as} \quad t \rightarrow 0 . \tag{22}$$

We conclude from (17), (18), (19), and (22) that, as $t \rightarrow 0$,

$$K_t(\theta) = O(t) + B(t, \theta) + \psi(t) \tag{3}$$

for $0 \leqq \theta \leqq \pi$, where we have set

$$B(t, \theta) = \frac{\Gamma\left(\lambda + \frac{3}{2}\right)t}{2^{\lambda+1/2}\pi^{1/2}\Gamma(\lambda + 1)} \int_0^\infty s^{-\lambda-5/2}[(z - \cos \theta)^{-\lambda-3/2} - z^{-\lambda-3/2}]ds \tag{24}$$

for $0 \leqq \theta \leqq \pi$ and

$$\begin{aligned} \psi(t) = 1 - &\frac{\Gamma\left(\lambda + \frac{5}{2}\right)t}{2^{\lambda+3/2}\pi^{1/2}\Gamma(\lambda + 2)} \\ &\times \int_0^\infty s^{-\lambda-5/2} z^{-\lambda-7/2} F\left(\frac{\lambda + \frac{5}{2}}{2}, \frac{\lambda + \frac{7}{2}}{2}, \lambda + 2; z^{-2}\right) ds . \end{aligned} \tag{25}$$

By the mean value theorem we may write, for $0 \leqq \theta \leqq \pi$,

$$
\begin{aligned}
B(t, \theta) &= At\int_0^\infty s^{-\lambda-5/2}\left(\frac{1}{z-\cos\theta} - \frac{1}{z}\right)w^{\lambda+1/2}ds \\
&= At\int_0^\infty \frac{s^{-\lambda-5/2}(\cos\theta)w^{\lambda+1/2}}{z(z-\cos\theta)}ds \,,
\end{aligned} \tag{26}
$$

where w lies between z^{-1} and $(z - \cos\theta)^{-1}$. If $\pi/2 \leqq \theta \leqq \pi$, we obtain from (26) that

$$
|B(t, \theta)| \leqq At\int_0^\infty \frac{s^{-\lambda-5/2}}{z^2}\left(\frac{1}{z}\right)^{\lambda+1/2} ds \leqq At \,.
$$

If $0 \leqq \theta \leqq \pi/2$, we obtain from (26) that

$$
\begin{aligned}
|B(t, \theta)| &\leqq At\int_0^\infty \frac{s^{-\lambda-5/2}}{z(z-\cos\theta)^{\lambda+3/1}}ds \\
&\leqq At\int_0^\infty \frac{ds}{[(s-\cos\theta)^2 + t^2 + \sin^2\theta]^{\lambda+3/2}} \\
&\leqq At\int_{-\infty}^\infty \frac{ds}{[(s-\cos\theta)^2 + t^2 + \sin^2\theta]^{\lambda+3/2}} \\
&= \frac{At}{(t^2 + \sin^2\theta)^{k/2}} \,.
\end{aligned}
$$

It follows from these estimates that

$$
|B(t, \theta)| \leqq \frac{At}{\left(t^2 + \sin^2\dfrac{\theta}{2}\right)^{k/2}} \tag{27}
$$

for $0 \leqq \theta \leqq \pi$. Now we wish to estimate $\psi(t)$. It is shown in [7, pp. 286 and 282] that

$$
\begin{aligned}
&F\left(\frac{\lambda + \frac{5}{2}}{2}, \frac{\lambda + \frac{7}{2}}{2}; \lambda + 2; z^{-2}\right) \\
&\quad = (1 - z^{-2})^{-1} F\left(\frac{\lambda + \frac{1}{2}}{2}, \frac{\lambda + \frac{3}{2}}{2}; \lambda + 2; z^{-2}\right)
\end{aligned} \tag{28}
$$

and

$$
\lim_{x \to -0} F\left(\frac{\lambda + \frac{1}{2}}{2}, \frac{\lambda + \frac{3}{2}}{2}; \lambda + 2; x\right) = L \,, \tag{29}
$$

where

$$L = \frac{\Gamma(\lambda + 2)}{\Gamma\left(\frac{\lambda + 5/2}{2}\right)\Gamma\left(\frac{\lambda + 1/2}{2}\right)}.$$

So, by (25) and (28), we have that

$$\begin{aligned}\psi(t) &= \left[1 - \frac{L\Gamma\left(\lambda + \frac{5}{2}\right)t}{2^{\lambda+3/2}\pi^{1/2}\Gamma(\lambda + 2)}\int_0^\infty s^{-\lambda-5/2}z^{-\lambda-7/2}(1 - z^{-2})^{-1}ds\right] \\ &\quad + \frac{\Gamma\left(\lambda + \frac{5}{2}\right)t}{2^{\lambda+3/2}\pi^{1/2}\Gamma(\lambda + 2)}\int_0^\infty s^{-\lambda-5/2}z^{-\lambda-7/2} \\ &\quad \times \left[\frac{L - F\left(\frac{\lambda + 1/2}{2}, \frac{\lambda + 3/2}{2}; \lambda + 2; z^{-2}\right)}{1 - z^{-2}}\right]ds.\end{aligned} \tag{30}$$

Using (29), [7, p. 281], the mean value theorem, and then estimates of the type (20) and (21), all in the second term of the right member of (30), we obtain after simplifying the gamma functions which occur in the first term of the right member of (30) that

$$\psi(t) = \left[1 - \frac{t}{\pi}\int_0^\infty s^{-\lambda-5/2}z^{-\lambda-7/2}(1 - z^{-2})^{-1}ds\right] + O(t) \tag{31}$$

as $t \to 0$. It follows from (31) and (16) and the change of variable $s = 1 + xt$ that

$$\begin{aligned}\psi(t) &= O(t) + 1 - 2^{\lambda+3/2}\frac{t}{\pi}\int_0^\infty(s^2 + t^2 + 1)^{-\lambda-3/2} \\ &\quad \times \left[\frac{1}{(s - 1)^2 + t^2} - \frac{1}{(s + 1)^2 + t^2}\right]ds \\ &= O(t) + 1 - 2^{\lambda+3/2}\frac{t}{\pi}\int_0^\infty(s^2 + t^2 + 1)^{-\lambda-3/2}[(s - 1)^2 + t^2]^{-1}ds \\ &= O(t) + 1 - 2^{\lambda+3/2}\frac{t}{\pi}\int_{1/2}^{3/2}(s^2 + t^2 + 1)^{-\lambda-3/2}[(s - 1)^2 + t^2]^{-1}ds \\ &= O(t) + 1 - \frac{1}{\pi}\int_{-1/2t}^{1/2t}\left(1 + \frac{t^2 + x^2t^2 + 2xt}{2}\right)^{-\lambda-3/2}\frac{dx}{1 + x^2}\end{aligned} \tag{32}$$

as $t \to 0$. So, assuming as we may that $0 < t < 1/4$ and using the binomial theorem, we obtain from (32) that

$$\begin{aligned}\psi(t) &= O(t) + \left[1 - \frac{1}{\pi}\int_{-1/2t}^{1/2t}\frac{dx}{1 + x^2}\right] \\ &\quad + \frac{(\lambda + 3/2)}{2\pi}\int_{-1/2t}^{1/2t}\frac{t^2 + x^2t^2 + 2xt}{1 + x^2}dx\end{aligned} \tag{33}$$

$$+ \frac{1}{\pi}\int_{-1/2t}^{1/2t} \frac{O((t^2 + x^2t^2 + 2xt)^2)}{1 + x^2}dx$$
$$= O(t) + O(t) + O(t) + O(t) = O(t)$$

as $t \to 0$. Finally, (9) follows from (23), (27), and (33).

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, **2**, New York, 1953.
2. S. Philipp, V. L. Shapiro, and W. H. Sills, *On the Conjugate Multiple Fourier-Stieltjes Integral*, to appear.
3. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1966.
4. V. L. Shapiro, *The conjugate Fourier-Stieltjes integral in the plane*, Bull. Amer. Math. Soc., **65** (1959), 12-15.
5. ———, *Topics in Fourier and geometric analysis*, Amer. Math. Soc. Memoir, Number 39, 1961.
6. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge, 1922.
7. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge, 1927.
8. A. Zygmud, *Trigonometric Series*, **1**, Cambridge, 1959.
9. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math., **88** (1952), 25-139.

Received March 3, 1971.

UNIVERSITY OF CALIFORNIA, SANTA CRUZ