

IRREDUCIBLE SUBGROUPS OF ORTHOGONAL GROUPS GENERATED BY GROUPS OF ROOT TYPE I

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In this paper it is shown that the only irreducible subgroups of orthogonal groups over finite fields of odd characteristic generated by groups of root type 1 are (1) unitary groups, (2) groups of type G_2 , or (3) the commutator subgroups of the orthogonal groups in question. This is a generalization of a previous result by the author.

In a previous paper [6], the author proved that the only subgroups of orthogonal groups over finite fields of odd characteristic which are generated by groups of root type 1 (defined below) and transitive on one-dimensional singular subspaces are the groups (1), (2), and (3) above.

The result of this paper is related to two current lines of research. Firstly we note that the irreducible groups that we study, together with the vector spaces they act on, form a quadratic pair in the sense of Thompson [8] [see e.g., 4], since the elements of a group of root type 1 have a minimal polynomial $(x - 1)^2$.

Thus we have a classification of certain subgroups of orthogonal groups which form a quadratic pair. However, our methods depend on the orthogonal geometry and hence are unlike the methods of Thompson. In addition, we include the case $p = 3$, not considered by Thompson.

Secondly, the groups of root type 1 form a conjugacy class in the orthogonal groups as well as in the groups (1), (2), and (3) above. (This is not proved here, but is proved in [8] § 16, where Thompson shows Σ is a conjugacy class in G , and can also be proved directly.) Furthermore, any two groups of root type 1 generate either an abelian p -group or $SL(2, q)$ (where q is the number of elements in a group of root type 1) or a group of order q^3 which is isomorphic to the Sylow p -subgroup of $SL(3, q)$. Thus our result is analogous to results obtained by, for example, Aschbacher [2], or Fischer [3], who have investigated the classification of groups generated by a conjugacy class of p -elements such that any two p -elements generate a group from a given class of groups.

1. Terminology and restatement of theorem. Let V be a finite dimensional vector space over a finite field of characteristic not 2. Let B be a symmetric bilinear form on V . B determines a quadratic

form Q on V by $B(x, x) = 2Q(x)$. In addition, suppose there is no $x \neq 0$ in V such that $B(x, y) = 0$ for all y in V . Then we say B is nondegenerate on V . In this case, the group of linear transformations on V preserving B is called the orthogonal group (with respect to B), and is denoted $0(V)$. The commutator subgroup of $0(V)$ is denoted $\Omega(V)$.

If for all $x \in S \neq \{0\}$, $S \subseteq V$, $B(x, x) = 0$, we say S is singular. We remark that the condition $S \neq \{0\}$ is not standard. Our "singular vectors" and "singular subspaces" are always nonzero. If a vector u is nonzero and not singular, it is nonsingular. This is standard. Further, we use projective terminology. Thus, a one-dimensional subspace is a point and a two-dimensional subspace is a line. Let $\langle \rangle$ denote "subspace generated by". Thus $\langle x \rangle$ is the point generated by the vector x .

The set of vectors x such that $B(x, y) = 0$ for all $y \in Y \subseteq V$ is denoted Y^\perp . If $X \subseteq Y^\perp$ we say X is perpendicular to Y . Since B is bilinear, this is equivalent to saying $\langle X \rangle$ is perpendicular to $\langle Y \rangle$. Since B is symmetric, $X \subseteq Y^\perp$ implies $Y \subseteq X^\perp$.

Now let x be a singular vector (hence nonzero by definition), and let u be in x^\perp . Define a linear transformation $\rho_{x,u}$ by: For $z \in x^\perp$, $\rho_{x,u}$ sends z to $z + B(z, u)x$. This transformation preserves B on the $n - 1$ dimensional space x^\perp . (Note that in case $u \in \langle x \rangle$ $\rho_{x,u}$ acts as the identity on x^\perp .) By Witt's theorem (see, for example, Artin [1, p. 121]), every linear transformation which preserves B on a subspace of V can be extended to a member of $0(V)$. Tamagawa [7] shows that the extension ρ of $\rho_{x,u}$ to a member of $0(V)$ is unique. In fact, if y is a singular vector such that $B(y, u) = 0$ and $B(x, y) = 1$ (when $u \in \langle x \rangle$, such vectors always exist) then ρ sends y to $y - Q(u)x - u$. We abuse notation by allowing $\rho_{x,u}$ to stand for its extension to a member of $0(V)$; $\rho_{x,u}$ is called a Siegel transformation.

Since, as a direct consequence of its definition,

$$(1) \quad \rho_{x,u}\rho_{x,v} = \rho_{x,u+v}$$

we see that the set $\Sigma = \{\rho_{x,ku} \mid x \text{ singular, } u \in x^\perp, u \notin \langle x \rangle; x, u \text{ fixed, } k \in F\}$ is a group isomorphic to the additive group of F . If u is singular we say Σ is a group of root type 1. If u is nonsingular we say Σ is a group of root type 2.

Let $|F| = q = p^n$, where p is an odd prime. Let $\dim V = t$ and $\text{index } V = \nu$. We remark that if $\Omega(V)$ contains a group of root type 1, ν must be at least 2, and hence t is at least 4. We restate the main theorem of this paper:

MAIN THEOREM. *Let G be an irreducible subgroup of $\Omega(V)$ generated by groups of root type 1. Then*

- (i) $G \cong G_2(q)$
- (ii) $G \cong SU(t/2, q^2)$ and either

$$t = 4m + 2, \nu = 2m, m \geq 1 \quad \text{or} \quad t = 4m, \nu = 2m, m > 1$$

or

- (iii) $G = \Omega(V)$.

2. The use of standard basis notation. Artin [1] shows that V can always be represented in the following way:

$$V = \langle x_1, x_{-1} \rangle \perp \langle x_2, x_{-2} \rangle \perp \cdots \langle x_k, x_{-k} \rangle \perp w$$

where $\langle x_i, x_{-i} \rangle$ is a “hyperbolic line”,

i.e.,
$$B(x_i, x_{-i}) = 1 \quad \text{and} \quad Q(x_i) = Q(x_{-i}) = 0$$

and W is anisotropic (containing no singular points). The dimension of W is 1, 2 or 0. The index of V is k .

By Witt’s theorem, if v_1, \dots, v_t are any vectors in V with the same multiplication table $B(v_i, v_j)$ as some subset of a standard basis, they can be extended to a standard basis. Thus we use $\langle x_1, x_2 \rangle$, for example, to stand for any singular line in V .

3. Preliminary results. We restate for the reader some of the results from [6].

LEMMA 1. *The groups of root type 1 are in one-to-one correspondence with the singular lines of V .*

The follows from elementary properties of Siegel transformations, namely

$$(2) \quad \rho_{x, u+kx} = \rho_{x, u} \quad k \in F$$

$$(3) \quad \rho_{x, cu} = \rho_{cx, u} \quad c \in F^*$$

and for singular u

$$(4) \quad \rho_{x, u} = \rho_{-u, x}.$$

Thus we may make the following definition:

Let $\Sigma l = \{ \rho_{x, u} \mid l = \langle x, u \rangle \} \cup \{1\}$.

If $\Sigma l \subseteq G$, say l is an *axis line* for G .

For the following lemmas let Σl and Σm be groups of root type 1 and let H be the group generated by Σl and Σm . Let $W = \langle l, m \rangle$ and suppose $l \cap m = \{0\}$.

LEMMA 2.

(1) If $l \cap m^\perp = l$, H is abelian.

(2) If $l \cap m^\perp = P$, (a point),

H is isomorphic to the p -Sylow subgroup of $\mathrm{SL}_2(q)$ ($q = p^n$).

(3) If $l \cap m^\perp = \{0\}$, $H \cong \mathrm{SL}_2(q)$.

LEMMA 3. If $l \cap m^\perp = \{0\}$, H is reducible on W , every singular point in W is on exactly one axis line for H , and the orbits of singular vectors under H form singular lines.

To illustrate, if $l = \langle x_1, x_2 \rangle$, $m = \langle x_{-1}, x_{-2} \rangle$, then $\langle x_1, x_{-2} \rangle$ is an orbit of singular vectors under H , i.e., for any $c, d \in F$, not both zero there is a $\sigma \in H$ such that $\sigma(x_{-2}) = cx_1 + dx_{-2}$.

LEMMA 4. If $l \cap m^\perp = P$, P is on two distinct axis lines in H .

To illustrate Lemma 4, let $l = \langle x_1, x_2 \rangle$, $m = \langle x_{-1}, x_3 \rangle$. Then $P = \langle x_2 \rangle$, and $\langle x_2, x_3 \rangle$ as well as $\langle x_2, x_1 \rangle$ are axis lines for H .

LEMMA 5. If $l \cap m^\perp = \{0\}$, H is transitive on the nonsingular vectors of a given length in W .

4. **Baer's theorem.** In Lemma 2 we have remarked that two groups of root type 1 either generate a p group or $\mathrm{SL}_2(q)$. In the latter case, we say the two groups of root type 1 are opposites, following Thompson [8]. Baer's theorem allows us to assume that each group of root type 1 in an irreducible subgroup of $\mathcal{Q}(V)$ has an opposite. For the convenience of the reader we state the relevant theorems:

LEMMA 6 (Baer). (Theorem 3.8.2 [5]). Let K be a conjugacy class of p -elements of the group G . If every pair of elements of K generate a p -group then K lies in a normal p -subgroup of G .

LEMMA 7 (Theorem 3.1.3 [5]). If G possesses a faithful irreducible representation on a vector space over a field of characteristic p , then G has no nontrivial normal p -subgroups.

5. **No two axis lines intersect (Case 1).** In §§ 5 and 6 we assume that no two axis lines of G intersect. Suppose $l = \langle x_1, x_2 \rangle$ is an axis line for G . By Baer's theorem (Lemmas 6 and 7), we must have an opposite, " m ". Without loss of generality, we may use standard basis notation and let $m = \langle x_{-1}, x_{-2} \rangle$. Each singular point in $W = \langle l, m \rangle$ is on exactly one axis line for H , the group generated by Σl and Σm .

We retain these definitions for H, W, l , and m in §§5 and 6. Thus if G is to be irreducible and have no two axis lines intersect, the dimension of V must be greater than 4.

In order that $\langle x_1, x_{-2} \rangle = X$ not be fixed by G , some point in X must be moved by an element of G out of X . Without loss of generality (by Lemma 3) let us assume $\langle x_1 \rangle$ is moved out of X . Then we must have an axis line of form $n = \langle x_{-1} + y, z \rangle$. We may assume $z \in x_1^\perp$ since $n \cap x_1^\perp$ has dimension one. If $z \in \langle x_1, x_2 \rangle^\perp$, then $\langle z \rangle$ is on two axis lines, by Lemma 4. Thus

$$n = \langle x_{-1} + cx_1 + dx_2 + v, x_{-2} + ax_2 + bx_1 + u \rangle$$

where $v, u \in W^\perp, a, b, c, d \in F$.

Case 2. Suppose $a = 0$. Then since $y = x_{-2} + bx_1 + u$ is perpendicular to $\langle bx_2 - x_{-1}, x_{-2} + bx_1 \rangle = k$, an axis line for H , we must have $x = x_{-1} + cx_1 + dx_2 + v$ also in k^\perp , so that y is not on two axis lines (Lemma 4). Hence $c = 0$ and $d = -b$ and $n = \langle x_{-1} - bx_2 + v, x_{-2} + bx_1 + u \rangle$.

Thus u and v are both singular, $u \in v^\perp$, and $\langle u, v \rangle \in W^\perp$. Thus the dimension of V must be at least 8. We return to this case later.

Case 1. Suppose $a \neq 0$. By Lemma 5 we may send $x_{-2} + ax_2 + bx_1 + u$ to $z = x_{-2} + ax_2 + u$ by an element of H .

Hence we get an axis line for $G, n = \langle w, z \rangle$. Choose w on n in u^\perp . Then $w = ex_1 + fx_{-1} + gx_2 + hx_{-2} + x$ where $x \in \langle W, u \rangle^\perp$. Hence $g + ah = 0$ or $g = -ah$. At this point, we need more information. We thus note:

LEMMA 8 [6]. *Using standard basis notation, if $l = \langle x_1, x_2 \rangle, m = \langle x_{-1}, x_{-2} \rangle, n = \langle x_{-2} + ax_2 + u, hx_{-2} - ahx_2 + ex_1 + fx_{-1} + x \rangle$ where $\langle x, u \rangle \in \langle l, m \rangle^\perp$ and $\langle x, u \rangle$ is anisotropic, then $\Sigma l, \Sigma m$, and Σn generate a group G isomorphic to $SU_3(q)$. In addition, G is transitive on the singular vectors in $V = \langle l, m, n \rangle$ and is transitive on the non-singular vectors of a given length, and every singular point of V is on an axis line for G .*

By Lemma 8, if $\langle u, x \rangle$ is anisotropic, the group generated by H and Σn is isomorphic to $SU_3(q)$.

Now suppose $\langle u, x \rangle$ contains one singular point, $\langle y \rangle$. Then $\langle t + y \rangle \in n$ for some $t \in W$. But since y is singular, t is singular and hence t belongs to some axis line s of H . Thus $\langle t + y \rangle \in s^\perp$. But $n \cap s^\perp \neq n$ since $\langle n, W \rangle$ does not contain a 4-dimensional singular space. Thus by Lemma 4, $\langle t + y \rangle$ must be on two axis lines for G , contradicting the hypothesis of this section.

LEMMA 9. *Let m and l be axis lines for G such that $m \cap l^\perp = \phi$. Let Σn move some point of $\langle m, l \rangle$ outside of $\langle m, l \rangle$. If we assume that for the group T generated by Σm , Σl , and Σn , no two lines intersect, then either (a) $\langle m, l, n \rangle$ contains a 4-dimensional singular space (Case 2 above) or else (b) $T \cong SU_3(q)$ and $\langle m, l, n \rangle$ is non-degenerate with dimension 6 and index 2.*

As a consequence of Lemma 9, we now prove

LEMMA 10. *Suppose no two axis lines of G intersect. If $T \cong SU_k(q)$ is a subgroup of G fixing a $2k$ -dimensional subspace U of V , then either*

- (1) $U = V$,
- (2) *there is a Case 1 axis line moving U and $G \cong SU_{k+1}(q)$, or*
- (3) *there is no Case 1 axis line moving U ; axis lines moving U are Case 2 axis lines.*

Proof. If $U \neq V$, T is reducible on V and there must be another axis line h . Thus Σh is a subgroup of G which does not fix U and for some pair of opposites Σl , Σm in T , Σh does not fix $\langle l, m \rangle$. If h is a "Case 1" line, $\langle h, U \rangle \cap U^\perp$ has dimension 2 and is anisotropic. By work in [6], Σh corresponds to a unitary transvection with center $\langle x + y \rangle$ where x is in U and y is nonisotropic in U^\perp . (We have abused notation by identifying the orthogonal geometry of U and the corresponding unitary geometry of U .) One wishes to know that $SU_k(q)$ and such a unitary transvection generate $SU_{k+1}(q)$. But one need only remark that $SU_k(q)$ is transitive on nonisotropic vectors of a given length, to see that if $\langle x + y \rangle$ is a center of a transvection for G , then so is $\langle z + y \rangle$ for any z in U of the same length. Thus T and Σh generate a group isomorphic to SU_{k+1} , since such unitary groups are generated by the set of all unitary transvections.

6. No two axis lines of G intersect (Case 2). We now assume that Case 1 does not occur in G . We have in G , the subgroup H , generated by the opposites Σl and Σm . In addition, by Lemma 3, we have for each c and d not both zero the axis line

$$n(c, d) = \langle cx_{-1} + dx_2 + x_3, cx_{-2} - dx_1 + x_4 \rangle.$$

But H and $\Sigma n(c, d)$ fix $\langle x_3, x_4 \rangle$. In order that G be irreducible we need still another axis line k such that Σk moves $\langle x_3, x_4 \rangle$. (Hence Σk is opposite to $\Sigma n(c, d)$ for some c, d .)

First assume k is in W^\perp . Then let $\langle x_{-4} \rangle = k \cap \langle x_3 \rangle^\perp$ and let $\langle x_{-3} \rangle = k \cap \langle x_4 \rangle^\perp$. Thus $k = \langle x_{-3}, x_{-4} \rangle$.

If k is not in W^\perp , there are two possibilities

- (1) $k = l(c, d) = \langle cx_1 + dx_2 + x_{-3}, cx_2 - dx_{-1} - x_{-4} \rangle$
- (2) $k = p(c, d) = \langle cx_1 + dx_{-2} + x_{-4}, cx_2 - dx_1 + x_{-3} \rangle$.

Since the group generated by $\Sigma l(c, d)$, $\Sigma n(c, d)$, and H is isomorphic to the group generated by $\Sigma \langle x_{-3}, x_{-4} \rangle$, $\Sigma n(c, d)$, and H , we may identify possibility (1) with $k = \langle x_{-3}, x_{-4} \rangle$.

Further, a line of type (2) in addition to a line of type $n(c, d)$ produces a Case 1 configuration in G : $\Sigma \langle x_1 + x_{-4}, x_2 + x_{-3} \rangle$ sends $\langle x_{-1} + x_3, x_{-2} + x_4 \rangle$ to $\langle x_{-1} + x_2 + x_{-3} + x_3 - x_1 - x_{-4}, \dots \rangle$.

Thus, we may choose $k \in W^\perp$. We remark that the group T generated by Σk , $\Sigma n(c, d)$, and H fixes the space $\langle x_{-1}, x_2, x_3, x_{-4} \rangle$.

We generalize this process:

LEMMA 11. *Suppose G has no Case 1 configuration and G contains a subgroup $T(n) (n \geq 2)$ generated by groups of root type 1 with the following axis lines: $\langle x_1, x_2 \rangle, \langle x_{-1}, x_{-2} \rangle, \dots, \langle x_{-1} + x_{2n-1}, x_{-2} + x_{2n} \rangle, \langle x_{-(2n-1)}, x_{-2n} \rangle$. Then (1) $T(n)$ is transitive on the (singular) vectors in the space $X = \langle x_{-1}, x_2, x_3, x_{-4}, \dots, x_{2n-1}, x_{-2n} \rangle$ and $T(n)$ fixes X .*

(2) *If k is a singular line in U , the space spanned by axis lines of $T(n)$, and Σk is not an axis line for $T(n)$, then Σk is not an axis line for G .*

(3) *G contains $T(n + 1)$.*

We remark that as a consequence of this lemma, since V is finite dimensional, there can be no group G without a Case 1 configuration.

The proof of (1) is tedious but straightforward. One shows all vectors in $\langle x_{-1}, x_2, x_{2j-1}, x_{-2j} \rangle 2 \leq j \leq n$ are in the same orbit by applying in sequence various transformations in H , $\Sigma \langle x_{-(2j-1)}, x_{-2j} \rangle$ and $\Sigma \langle x_{-1} + x_{2j-1}, x_{-2} + x_{2j} \rangle$. Then one remarks that x_{-1} can be sent to an arbitrary vector of X , $a_1x_{-1} + a_2x_2 + \dots + a_{2n}x_{-2n}$ by sending x_{-1} to $x_{-1} + a_3x_3 + a_4x_{-4}$, then fixing $a_3x_3 + a_4x_{-4}$ and sending x_{-1} to $x_{-1} + a_5x_5 + a_6x_{-6}$, repeating until in the last place where a_{2j-1} and a_{2j} are not both zero, one sends x_{-1} to $a_1x_{-1} + a_2x_2 + a_{2j-1}x_{2j-1} + a_{2j}x_{-2j}$.

By part (1) every point in X is on (exactly) one axis line for $T(n)$. A new axis line k in U must have no point in X . Since $U \cap X^\perp = X$, k must not be on X^\perp . Without loss of generality, say $k = \langle x_1 + y, z \rangle$ and let $z = k \cap \langle x_{-1} \rangle^\perp$. Then $z = x_2 + w$, since Σk is an opposite to $\Sigma \langle x_{-1}, x_{-2} \rangle$. But $z \notin X$. Thus $w \in X$. Thus there is a component of z in $Y = \langle x_1, x_{-2}, x_{-3}, x_4, \dots, x_{2n} \rangle$. Say bx_{2j} is a component, then $\Sigma \langle x_{-1} + x_{-2j}, x_{-2} + x_{-(2j-1)} \rangle (\cong T(n))$ moves z to $z + (x_{-1} + x_{-2j}) - b(x_{-2} + x_{-(2j-1)})$. But then we have a Case 1 configuration (if $bx_{-(2j-1)}$ is a component, use $\Sigma \langle x_{-1} + x_{2j-1}, x_{-2} + x_{2j} \rangle$). This proves (2).

Thus we must have a line r moving X and lying outside of U .

Without loss of generality, say r moves x_{-1} , $r = \langle x + y, z + w \rangle$ where $x, z \in U$ and $y, w \notin U$. Say z has an x_1 component. Say $\langle x + y \rangle \subseteq \langle x_{-1} \rangle^\perp \cap r$. Then $x = x_2 + a_3x_3 + b_3x_{-3} + \dots + a_{2n}x_{2n} + b_{2n}x_{-2n}$. Since $T(n)$ is transitive on vectors in X , and fixes X , some element in $T(n)$ sends x to $y = x_2 + c_3x_{-3} + c_4x_4 + \dots + c_{2n-1}x_{-(2n-1)} + c_{2n}x_{2n}$. Some $\sigma \in \Sigma \langle x_{-1} + x_{2j-1}, x_{-2} + x_{2j} \rangle$ sends y to $y + c_{2j-1}(-x_{-2} - x_{2j}) + x_{-1} + x_{2j-1}$. But since there are no Case 1 configurations, $c_{2j-1} = 0$ for all j . Further, some $\tau \in \Sigma \langle x_{-2j}, x_{-(2j-1)} \rangle$ sends y to $y + c_{2j}(x_{-(2j-1)})$. Thus $c_{2j} = 0$ as well, since the coefficient of $x_{-(2j-1)}$ must be zero. Thus we may choose $x = x_2$, or without loss of generality x can be chosen to be any vector in X , say $x = x_{-1}$. The hypothesis of no two axis lines intersecting yields $r = \langle x_{-1} + y, x_{-2} + w \rangle$ with $y, w \in U^\perp$. Without loss of generality $r = \langle x_{-1} + x_{2n+1}, x_{-2} + x_{2n+2} \rangle$. In addition, we have shown that an axis line of form $\langle x + y, z + w \rangle$ with $x, z \in U$ and $y, w \in U^\perp$ can be written with $x \in X$. Hence, if we are to have a line which does not fix $\langle x_{2n+1}, x_{2n+2} \rangle$, we must have lines of form $\langle x_{-(2n+1)}, x_{-(2n+2)} \rangle$, or $\langle x_{-1} + x_{-(2n+1)}, x_{-2} + x_{-2n} \rangle$ as before. But the latter produces a Case 1 configuration. This proves part (3).

We now prove:

THEOREM 1. *If no two axis lines of G intersect, $G \cong SU_k(q)$, $k \geq 3$. (In this case $\dim V = 2k$. If k is even, index $V = k$. If k is odd, index $V = k - 1$.)*

Proof. We have shown that if G is a group satisfying the hypothesis of this paper, such that no two axis lines intersect, then G must contain $SU_3(q)$. Let n be the largest number such that $T \cong SU_n(q) \subseteq G$ and let U be the $2n$ -dimensional space fixed by T . If $U = V$, we are done. If not, there must be an axis line r for G such that Σr does not fix U . Since T is transitive on the singular vectors of U (for this we need $n \geq 3$) and since no two axis lines of G intersect, we may write $r = \langle x_{-1} + x_{n+1}, x_{-2} + x_{n+2} \rangle$. (If r were Case 1, we would contradict the maximality of n .) In addition we must have $\langle x_{-(n+1)}, x_{-(n+2)} \rangle$. But T is transitive on the singular vectors of U . So we then also have as axis line $\langle x_1 + x_{n+1}, x_2 + x_{n+2} \rangle$, and $\Sigma \langle x_{-(n+1)}, x_{-(n+2)} \rangle$ sends $\langle x_1 + x_{n+1}, x_2 + x_{n+2} \rangle$ to

$$k = \langle x_1 + x_{n+1} + x_{-(n+2)}, x_2 + x_{n+2} - x_{-(n+1)} \rangle$$

and Σk sends r to $\langle x_{-1} + x_2 + x_{n+2} - x_{-(n+1)} + x_{n+1} + x_1 + x_{n+1} + x_{-(n+2)}, \dots \rangle$, a Case 1 configuration.

7. Two axis lines contain the same point and are perpendicular to each other. If two axis lines l and m for G contain the same point P , then either l is perpendicular to m (e.g., $\langle x_1, x_2 \rangle$ and

$\langle x_1, x_3 \rangle$) or else $l \cap m^\perp = P$ (e.g., $\langle x_1, x_2 \rangle$ and $\langle x_1, x_{-2} \rangle$). We suppose in this section that the former case occurs, but the latter does not, and in addition for no point Q is the space of axis lines containing Q of dimension larger than 3.

Let $P = \langle x_1 \rangle$. Using standard basis notation we assume $\langle x_1, x_2 \rangle$ is an axis line containing P . Let the space spanned by the axis lines containing P be called S .

Not that the hypotheses of this section imply S is singular.

Let an opposite to $\langle x_1, x_2 \rangle$ be denoted $\langle x_{-1}, x_{-2} \rangle$ as usual. Then $l = S \cap \langle x_{-2} \rangle^\perp$ has dimension 2 and includes x_1 . By the following lemma, l is an axis line for G .

LEMMA 12. *If S is the space spanned by axis lines for G which contain the singular point P , and $P \subseteq m \subseteq S$, where m is a line, then m is an axis line for G .*

Proof. Say $l_i = \langle P, x_i \rangle$ $1 \leq i \leq k$ are the axis lines spanning S . Then $m = \langle P, \sum_{i=1}^k c_i x_i \rangle$. By the fact that $\rho_{x,u} \rho_{x,v} = \rho_{x,u+v}$, we see that m is an axis line for G . (If S is not singular, Σm may be a group of root type 2.)

Thus using standard notation, we may name our l above " $\langle x_1, x_3 \rangle$ " where $x_3 \in \langle x_{-2}, x_{-1}, x_2, x_1 \rangle^\perp$.

We remark that $\Sigma \langle x_{-1}, x_{-2} \rangle$ sends $\langle x_1, x_3 \rangle$ to $\langle x_1 + ax_{-2}, x_3 \rangle$. By Lemma 12, $\langle x_{-2}, x_3 \rangle$ is also an axis line for G . By Baer's theorem (Lemmas 6 and 7), we know that $\langle x_1, x_3 \rangle$ must have an opposite; call it m .

Let $m = \langle x + w, y \rangle$ where $w + x \in x_3^\perp$ and $y \in x_1^\perp$, and $w \in \langle x_1, x_2, x_{-1}, x_{-2} \rangle^\perp$. Thus $x = x_{-1} + ax_1 + bx_2 + cx_{-2}$ for some $a, b, c \in F$.

Case A. Assume w is singular or zero. Hence x is singular. Let H be the group generated by $\Sigma \langle x_1, x_2 \rangle$ and $\Sigma \langle x_{-1}, x_{-2} \rangle$. Since x is singular we can find a $\sigma \in H$ which sends m to $\sigma(m) = \langle x_2 + dx_1 + w, \sigma(y) \rangle$ where $d \in F$. (Note that since the coefficient of x_{-1} is nonzero in x , the coefficient of x_2 is nonzero in $\sigma(x)$. Further, note that $\sigma(y)$ must contain an " x_{-3} " component since y did, and $\sigma \in H$.)

If $\sigma(y)$ is perpendicular to $\langle x_1, x_2 \rangle$, then $\Sigma \sigma(m)$ sends $\langle x_1, x_3 \rangle$ to $\langle x_1, x_3 + x_2 + dx_1 + w \rangle$. Thus $w \in \langle x_3 \rangle$. However we show $w \in \langle x_3 \rangle$ even if $\sigma(y)$ is not perpendicular to $\langle x_1, x_2 \rangle$.

If $\sigma(y)$ is not perpendicular to $\langle x_1, x_2 \rangle$, then by Lemma 4, $\langle x_2 + dx_1 + w \rangle$ is on an axis line with some point P on $\langle x_1, x_2 \rangle$. Since $\sigma(y)$ is not perpendicular to x_3 , $\langle x_2 + dx_1 + w \rangle$ is on an axis line with some point Q in $\langle x_1, x_3 \rangle$. Thus $\langle x_2 + dx_1 + w \rangle$ is on axis lines with P, Q , and $\sigma(y)$. If $P \neq Q$, these three points are linearly independent and $\langle x_2 + dx_1 + w \rangle$ is on too many linearly independent axis lines for the hypothesis of this section. Thus $P = Q = \langle x_1 \rangle$. But then $\langle x_1 \rangle$ is on

too many axis lines, unless $w \in \langle x_3 \rangle$.

Therefore, we may suppose that $\sigma(m) = \langle x_2 + dx_1 + cx_3, x_{-3} + ax_{-2} + bx_1 + ex_2 + fx_3 + z \rangle$ where $z \in \langle x_1, x_2, x_3, x_{-1}, x_{-2}, x_{-3} \rangle^\perp$, and there is no x_{-1} component since $\langle x_2 + dx_1 + cx_3, x_1 \rangle$ is an axis line. In addition, by Lemma 12, we may assume $b = 0$.

There is an element $t \in \Sigma \langle x_{-2}, x_3 \rangle$ which sends $\sigma(m)$ to $t\sigma(m) = \langle x_2 + dx_1, x_{-3} + hx_{-2} + ex_2 + gx_3 + z \rangle$. Since $t\sigma(m)$ is singular, $h = 0$. By Lemma 12 we may also assume $e = 0$, and we obtain $t\sigma(m) = \langle x_2 + dx_1, x_{-3} + gx_3 + z \rangle$. Here d may be zero or not.

Let T be the group generated by $H, \Sigma \langle x_1, x_3 \rangle$, and Σm for Case A. But T fixes the space $U = \langle x_1, x_{-2}, x_{-3} + gx_3 + z \rangle$, so T is not irreducible. Further, T is transitive on the vectors of U . Thus we have an axis line for G which moves x_1 , say $l = \langle x_{-1} + y, w \rangle$ where $w \in U$. Let w be in x_1^\perp . If w is also in x_2^\perp , w is on an axis line with some point in $\langle x_1, x_2 \rangle$, not $\langle x_1 \rangle$. If w is also in x_3^\perp , w is on an axis line with still another point. Further, $\langle w, ax_1 + x_2, bx_1 + x_3 \rangle$ must be 3-dimensional since no point in $\langle x_1, x_2, x_3 \rangle$ can be on an axis line with $x_{-1} + y$, these all being on axis lines with $\langle x_1 \rangle$. Hence $\langle w, ax_1 + x_2, bx_1 + x_3, x_{-1} + y \rangle$ (in the space of axis lines containing w) is 4-dimensional. This implies w must either not be perpendicular to x_2 or else not to x_3 .

Suppose w is not perpendicular to x_2 , but is perpendicular to x_3 . Then $w = ax_1 + bx_2 + x_{-2} + cx_3 + x$ where $x \in \langle x_1, x_2, x_3, x_{-1}, x_{-2}, x_{-3} + gx_3 + z \rangle^\perp$ and w is on an axis line with $x_1 + dx_3$, $d \neq 0$. We may use Lemma 12 with $\langle x_1 + dx_3, x_3 \rangle$ and $\langle x_1 + dx_3, x_1 \rangle$ to get the axis line $m = \langle x_1 + dx_3, bx_2 + x_{-2} + x \rangle$. If $b = 0$, Σm sends $\langle x_{-2}, x_{-1} \rangle$ to $\langle x_{-2}, x_{-1} + x_{-2} + x \rangle$. But then x_{-2} is on too many axis lines (since x is linearly independent of x_{-1} and x_3). Thus $b \neq 0$. But by [6], this gives us a G_2 configuration in the 7-dimensional space $\langle x_1, x_2, x_{-1}, x_{-2}, x_3, x_{-3} + gx_3 + z, x \rangle$.

If w is not perpendicular to x_3 , we have another opposite to $\langle x_1, x_3 \rangle$. If we are in Case B (which follows) we end up with a line $m = \langle x_1 + dx_3, x_{-2} + bx_2 + x \rangle$ and G_2 , as above. So we assume Case A again occurs and we have a line of form $l = \langle x_{-1} + ax_{-2}, x_{-3} + fx_3 + x \rangle$, where $x_{-3} + fx_3 + x \in \langle x_{-3} + gx_3 + z \rangle$ since $w \in U$, and U is fixed by T . But $x_{-3} + fx_3 + x \in \langle x_1, x_2, x_{-1}, x_{-2} \rangle^\perp$. If $x_{-3} + fx_3 + x \in \langle x_{-3} + gx_3 + z \rangle^\perp$ then $x_{-3} + fx_3 + x$ is on axis lines with $x_{-1} - dx_{-2}$ and $x_2 + dx_1$, as is $x_{-3} + gx_3 + z$. Whether or not $d = a$, $x_2 + dx_1$ ends up on too many axis lines. Thus $\langle x_{-3} + fx_3 + x \rangle \notin \langle x_{-3} + gx_3 + z \rangle^\perp$ and again by [6], and Witt's theorem, we get G_2 .

Case B. We recall that $\langle x_1, x_2 \rangle$, $\langle x_{-1}, x_{-2} \rangle$, $\langle x_1, x_3 \rangle$, and $\langle x_{-2}, x_3 \rangle$ are axis lines for G . Further Σm is an opposite to $\Sigma \langle x_1, x_3 \rangle$, and $m = \langle x + w, y \rangle$, where $\langle y \rangle = m \cap \langle x_1 \rangle^\perp$, $\langle x + w \rangle = m \cap \langle x_3 \rangle^\perp$, and

$w \in \langle x_1, x_2, x_{-1}, x_{-2} \rangle^\perp$. In Case B, w is nonsingular.

By Lemma 5, H , the group generated by $\Sigma \langle x_1, x_2 \rangle$ and $\Sigma \langle x_{-1}, x_{-2} \rangle$ is transitive on nonsingular vectors of a given length. Thus we may assume for $a \neq 0$, $m = \langle x_{-2} + ax_2 + w, y \rangle$ where y has an x_{-3} component.

If $w = bx_3 + z$, where $z \in \langle x_3, x_{-3} \rangle^\perp$, then apply $\sigma \in \Sigma \langle x_{-2}, x_3 \rangle$ to obtain $\sigma(m) = \langle x_{-2} + ax_2 + z, x \rangle$. We may choose $x = \sigma(m) \cap z^\perp$. Again, $x \notin x_3^\perp$, so we may use Lemma 4 to see $x_{-2} + ax_2 + z$ is on an axis line with some point P in $\langle x_1, x_3 \rangle$. We obtain $l = \langle x_{-2} + ax_2 + z, x_1 + bx_3 \rangle$ where b is not zero, for otherwise x_1 is on too many axis lines. (Since x_3 is on an axis line with x_{-2} , $P \neq \langle x_3 \rangle$.)

We recall that $\sigma(m) (= \langle x_{-2} + ax_2 + z, x \rangle)$ is an axis line for G , and $x \notin x_3^\perp$, $x \in \langle x_1 + bx_3 \rangle^\perp \cap z^\perp$. Thus $x = x_{-3} - bx_{-1} + cx_2 + dx_3 + ex_1 + fx_{-2} + t$ where $t \in \langle x_1, x_2, x_3, x_{-1}, x_{-2}, x_{-3}, z \rangle^\perp$. By Lemma 12, there is an axis line $n = \langle x_{-2} + ax_2 + z, x_{-3} - bx_{-1} + cx_2 + gx_3 + fx_{-2} + t \rangle$. Since n is singular, $f = -a^{-1}c$. Look at the point $P = n \cap x_{-2}^\perp = n \cap \langle x_{-1}, x_{-2} \rangle^\perp$. $P = \langle x_{-3} - bx_{-1} + fx_{-2} + gx_3 + t - a^{-1}cx_{-2} - a^{-1}cz \rangle = \langle x_{-3} - bx_{-1} - 2a^{-1}cx_{-2} + gx_3 + t - a^{-1}cz \rangle$. But P is on an axis line with some point in $\langle x_{-1}, x_{-2} \rangle$, as well as with $x_{-2} + ax_2 + z$. Thus P is on an axis line with $\langle x_{-1} \rangle$. Apply Lemma 12 to see $r = \langle x_{-1}, x_{-3} + gx_3 + t - a^{-1}cz \rangle$ is an axis line. By [6], the group generated by $\Sigma r, \Sigma l, H$, and $\Sigma \langle x_1, x_3 \rangle$ is isomorphic to G_2 on the 7-dimensional space $\langle x_1, x_2, x_{-1}, x_{-2}, x_3, x_{-3} + gx_3 + t - a^{-1}cz, z \rangle$. We remark for the convenience of the reader that in [6], we showed that G_2 is generated by $\Sigma \langle x_1, x_2 \rangle, \Sigma \langle x_{-1}, x_{-2} \rangle, \Sigma \langle x_1, x_3 \rangle, \Sigma \langle x_{-1}, x_{-3} \rangle, \Sigma \langle x_2, x_{-3} \rangle, \Sigma \langle x_{-2}, x_3 \rangle$ and $\Sigma \langle x_1 + dx_3, bx_2 + x_{-2} + x \rangle$ where $x \in \langle x_1, x_2, x_{-1}, x_{-2}, x_3, x_{-3} \rangle^\perp$. In Case A, “ x_{-3} ” was replaced with “ $x_{-3} + gx_3 + z$ ”. (Witt’s theorem makes this substitution legal.) In Case B, “ x_{-3} ” was replaced by “ $x_{-3} + gx_3 + t - a^{-1}cz$ ” and “ x ” by “ $z + kx_3$ ” where k is chosen such that $z + kx_3 \in \langle x_{-3} + gx_3 + t - a^{-1}cz \rangle^\perp$. Then note we have the line $\langle x_{-2} + ax_2 + z + kx_3, x_1 + bx_3 \rangle$ as an axis line for G by Lemma 12.

Thus we have seen we have (under the hypotheses of this section) a G_2 formation, which by [6] is transitive on the singular points of the 7-dimensional space U spanned by the axis lines. Suppose then that G contains G_2 and acts irreducibly on a space of dimension larger than 7.

Thus, without loss of generality we have an axis line $l = \langle x_{-1} + w, \nu \rangle$ where no point on l is in U , since all the singular points in U are already on the required number of axis lines. Let ν be the point on l in x_1^\perp . But l must be opposite to all lines of form $\langle x_1, ax_2 + bx_3 \rangle$, a and b not both zero, or by Lemma 4 some point on one of these lines will be on an axis line with ν . This is impossible since ν must

be perpendicular to some point on $\langle x_2, x_3 \rangle$. This establishes:

THEOREM 2. *The irreducible subgroups G of $\Omega(V)$ generated by groups of root type 1 such that (1) two axis lines l and m contain the same point P and $l \subseteq m^\perp$, but no two axis lines intersect which are not perpendicular to each other and (2) for no point Q is the space of axis lines of dimension larger than 3, are the groups $G_2(q)$. The dimension of V is 7 in this case.*

8. The nondegenerate axis line space. Let S be a maximal dimension space spanned by axis lines containing any one singular point. If we name this point $\langle x_1 \rangle$, then let $l = \langle x_1, x_2 \rangle$ be one of the axis lines in $S (= S(x_1))$.

Suppose one of the opposites to l is $m = \langle x_{-1}, x_{-2} \rangle$. Then $T = S \cap \langle x_{-1} \rangle^\perp$. Suppose in this section that T is nondegenerate.

Let $W = T \cap \langle x_{-2} \rangle^\perp$. We are going to work with W and $\langle x_{-2}, x_{-1} \rangle$.

Let $w \in W$. Then $\rho_{x_1, w}$ sends $\langle x_{-2}, x_{-1} \rangle$ to $\langle x_{-2}, x_{-1} - Q(w)x_1 - w \rangle$. Let $z = Q(w)x_1 + w$. Then $\rho_{x_{-2}, z}$ sends $\langle x_1, x_2 \rangle$ to $\langle x_1, x_2 - Q(z)x_{-2} - z \rangle$. If z (and hence w) is nonsingular, then one obtains $\langle x_1, x_{-2} \rangle$ as an axis line. In both cases (w singular or nonsingular), applying Lemma 12 to $\langle x_{-2}, x_{-1} - Q(w)x_1 - w \rangle$ and $\langle x_{-2}, x_{-1} \rangle$ and (if w is nonsingular) $\langle x_{-2}, x_1 \rangle$, one obtains $\langle x_{-2}, w \rangle$ is an axis line for G .

If there are no nonsingular vectors in W , we may still obtain $\langle x_{-2}, x_1 \rangle$ as an axis line as follows: We know every point in W is on an axis line with $\langle x_{-2} \rangle$. In addition, $\langle x_{-1} \rangle$ is on an axis line with $\langle x_{-2} \rangle$, and $W \subseteq \langle x_1, x_{-1} \rangle^\perp$. If $x_{-2} \notin W$, then the geometry of the space of axis lines containing $\langle x_{-2} \rangle$ is different from the geometry of $S(x_1)$, (i.e., $S(x_{-2})$ has dimension equal to $\dim S(x_1)$ but $T(x_{-2})$ is degenerate, since $x_{-1} \subseteq S(x_{-2}) \subseteq x_{-1}^\perp$).

But the group generated by $\Sigma \langle x_1, x_2 \rangle$ and $\Sigma \langle x_{-1}, x_{-2} \rangle$ sends x_1 to x_{-2} , so this is impossible. Thus $\langle x_1, x_{-2} \rangle$ is an axis line.

We have thus obtained:

LEMMA 13. *If $S(x)$ is the space of axis lines containing x , and m is opposite to l , one of those axis lines (m and l both singular), then $S(m \cap x^\perp) \cong (S(x) \cap m^\perp) \oplus m$. If in addition $T(x)$ is nondegenerate, or if $S(x) \cap m^\perp$ contains a nonsingular point, then $m \cap x^\perp$ is on an axis line with x , and $\langle m, S(x) \rangle = \langle l, S(m \cap x^\perp) \rangle$.*

Let $n = \langle x_{-1} + u, x_2 + v \rangle$ be an opposite for $\langle x_1, x_{-2} \rangle$, where $v, u \in \langle x_1, x_{-2} \rangle^\perp$. Let $T(x_1) = \langle x_2, x_{-2}, U \rangle$ where $U \subseteq \langle x_1, x_{-1}, x_{-2}, x_2 \rangle^\perp$.

Using Lemma 13, we obtain that $\langle x_2 + v \rangle$ must be on an axis line with x_1 , so that $v = ax_1 + bx_{-2} + w, w \in U(x_1)$. A similar argument using $S(x_{-2})$ (instead of $S(x_1)$) shows that $x_{-1} + u$ must be on an

axis line with x_{-2} , so that in particular $u \in \langle S(x_{-2}) \rangle$, ($u = dx_1 + cx_{-2} + x$, $x \in U(x_1)$). Let $Z = \langle S(x_1), x_{-1} \rangle$. We have shown $Z = \langle S(x_{-1} + u), x_1 \rangle = \langle S(x_2 + v), x_{-2} \rangle = \langle S(x_{-2}), x_2 \rangle$. We show, as in [6], that $\Omega(Z) \subseteq G$.

Let $l = \langle x, y \rangle$ be a singular line in Z . If $\langle x_2 + v \rangle$ or $\langle x_{-2} \rangle$ are on l , then l is an axis line for G by Lemma 12. If not:

Case 1. $\langle x, y \rangle \in \langle x_2 + v, x_{-2} \rangle^\perp$. Here $\langle x_2 + v, x \rangle$ and $\langle x_{-2}, y \rangle$ are axis lines, so by Lemma 4, so is $\langle x, y \rangle$.

Case 2. $\langle x \rangle \in \langle x_2 + v, x_{-2} \rangle^\perp$, $\langle y \rangle \in \langle x_{-2} \rangle^\perp$, (by symmetry, $y \in \langle x_2 + v \rangle^\perp$ could be used instead). Then $\langle x, x_{-2} \rangle$ is an axis line and $\langle x_{-2}, y \rangle$ is an axis line where $kx_{-2} + y \in \langle x_2 + v \rangle^\perp$. By Lemma 4, $\langle x, kx_{-2} + y \rangle$ is an axis line and by Lemma 12, $\langle x, y \rangle$ is an axis line.

Case 3. $\langle x, y \rangle \in \langle x_{-2} \rangle^\perp$ (resp. $\langle x, y \rangle \in \langle x_2 + v \rangle^\perp$). Since some point on the line l will be perpendicular to $\langle x_2 + v \rangle$ (resp. $\langle x_{-2} \rangle^\perp$) this reduces to Case 2.

Case 4. $\langle x \rangle \in \langle x_{-2}, x_2 + v \rangle^\perp$ but $\langle y \rangle \notin \langle x_{-2} \rangle^\perp$, $y \notin \langle x_2 + v \rangle^\perp$ but $y = a(x_2 + v) + b(x_{-2}) + w$ where $w \in \langle x_{-2}, x_2 + v \rangle^\perp$. But by Case 1, $\langle x, w \rangle$ is an axis line. Thus since $\langle x, w \rangle$, $\langle x, x_{-2} \rangle$, and $\langle x, x_2 + v \rangle$ are axis lines, Lemma 12 implies $\langle x, y \rangle$ is an axis line.

There are no more cases, because Z is nondegenerate, so if l is a singular line in Z and m is a hyperbolic line in Z it cannot happen that $Z \cap m^\perp \cap l$ is $\{0\}$. (If this were true $Z = m^\perp \oplus l$, m^\perp nondegenerate and Z nondegenerate.) Thus $\Omega(Z) \subseteq G$. In addition, we have proved:

LEMMA 14. *Let Z be nondegenerate. Then the group generated by all groups of root type 1 $\{\Sigma \langle x, y \rangle \mid y \in x^\perp \cap Z\}$ and $\{\Sigma \langle z, w \rangle \mid w \in z^\perp \cap Z\}$ where $\langle x, z \rangle$ is some fixed hyperbolic line, is $\Omega(Z)$.*

Now we wish to show $\Omega(Z) = G$. Clearly, so far Z is fixed so that if $\Omega(Z) \neq G$, G is reducible. Since the axis lines of $\Omega(Z)$ containing one point P in Z span a space which by hypothesis has maximal dimension, there can be no axis lines with one point in Z and one point outside Z .

Without loss of generality, assume $l = \langle x_{-1} + u, v \rangle$ is an axis line, with no point on l in Z . Let v be the point on l in x_1^\perp . If l is opposite some line in $S(x_1)$, use Lemma 13. If not, $v \in S(x_1)^\perp$. Then v is on axis lines with every point in $S(x_1) \cap \langle x_{-1} + u \rangle^\perp$ yielding a contradiction. Thus we have:

THEOREM 3. *The irreducible groups generated by groups of root type 1 with nondegenerate axis line spaces are the groups $\Omega(V)$.*

9. The elimination of other possibilities: I. Mixed axis line spaces. We again suppose $S(x_1)$ is the space spanned by axis lines containing x_1 and that $S(x_1)$ is of maximal dimension. Let $S(x_1) = \langle x_1 \rangle \oplus T(x_1)$ indicating only that x_1 is linearly independent of $T(x_1)$. In the last section, we showed that if $T(x_1)$ were nondegenerate, $G = \Omega(V)$. In this section we assume T is neither singular nor nondegenerate. Thus $T = U \oplus W$, where U is a maximal nondegenerate subspace of T and W is singular and $W \subseteq U^\perp$, and U and W are both nonzero.

Some set of singular vectors spans T . They cannot all be pairwise perpendicular, or U would be empty. Thus there are singular vectors x, y such that $\langle x, y \rangle$ is hyperbolic and $\langle x, y \rangle \subseteq U$. Say $\langle x \rangle = x_2$. Then suppose $\langle x_{-1}, x_{-2} \rangle$ is an opposite to $\langle x_1, x_2 \rangle$. Choose $T(x_1)$ in $\langle x_{-1} \rangle^\perp$ as before. Then $y = x_{-2} + ax_2 + z$, where $z \in \langle x_1, x_2, x_{-1}, x_{-2} \rangle^\perp$.

As in the previous section, since the group generated by $\Sigma \langle x_{-1}, x_{-2} \rangle$ sends x_1 to x_{-2} , the geometries of their axis line spaces must be the same. But if we apply Lemma 13 to the lines containing x_1 which are in $T(x_1) \cap x_{-2}^\perp$ we get $\langle T(x_1) \cap x_{-2}^\perp, x_{-1} \rangle \subseteq S(x_{-2})$. If $x_{-2} \notin T(x_1) \cap x_{-2}^\perp$, the geometry of $T(x_{-2})$ is different from the geometry of $T(x_1)$. Thus we may assume that $\langle x_1, x_{-2} \rangle$ is an axis line.

As in the previous section, if $n = \langle x_{-1} + u, x_2 + v \rangle$ is an opposite for $\langle x_1, x_{-2} \rangle$, then $\langle S(x_1), x_{-1} \rangle = \langle S(x_{-2}), x_2 \rangle = \langle S(x_{-1} + u), x_1 \rangle = \langle S(x_2 + v), x_{-2} \rangle$. Note that if $x_3 \in W(x_1)$, the singular part of the axis line space, then $\langle S(x_1), x_{-1} \rangle \subseteq x_3^\perp$. Then we see that $S(x_{-1} + u) \cap x_3^\perp = S(x_{-1} + u)$. Now apply Lemma 4 to $\langle x_1, x_3 \rangle$ and all the axis lines containing $x_{-1} + u$ to see x_3 is on axis lines with all of the points in $S(x_{-1} + u)$ and with x_1 . Thus the space $S(x_3)$ has larger dimension, producing a contradiction.

10. The elimination of other possibilities: II. Singular axis line spaces of large dimension. We again let $\langle x_1, x_2 \rangle$ be an axis line. We assume $\langle x_{-1}, x_{-2} \rangle$ is an opposite, and we let $S(x_1)$ be maximal dimension. In this section $T(x_1) = S(x_1) \cap x_{-1}^\perp$ and $T(x_1)$ is singular of dimension at least 3. We name the vectors spanning $T(x_1)$ " x_2, x_3, \dots, x_k " according to standard basis notation. We obtain $T(x_{-2}) = \langle x_{-1}, x_3, \dots, x_k \rangle$ by repeated applications of Lemma 4 with $\langle x_{-1}, x_{-2} \rangle$. Let $\langle x_{-3} + u, x_{-1} + v \rangle$ be an opposite to $\langle x_1, x_3 \rangle$ with $x_{-3} + u$ chosen in x_1^\perp , $x_{-1} + v$ chosen in x_3^\perp . Then $x_{-1} + v$ is on axis lines with all the points in $S(x_1) \cap (x_{-1} + v)^\perp \cap (x_{-3} + u)^\perp$ as well as with $x_{-3} + u$.

Now apply Lemma 4 with $\langle x_1, x_3 \rangle$ to see x_3 is on axis lines with $S(x_{-1} + v) \cap x_1^\perp \cap x_3^\perp = S(x_1) \cap (x_{-1} + v)^\perp \cap (x_{-3} + u)^\perp$ since $S(x_1) \subseteq \langle x_1, x_3 \rangle^\perp$, as well as with x_1 , and x_{-2} . But $R = \langle x_1, x_2, x_3 \rangle \cap \langle x_{-1} + v \rangle^\perp \cap \langle x_{-3} + u \rangle^\perp$ is nonempty. But $ax_1 + bx_3 \notin \langle x_{-1} + v \rangle^\perp \cap \langle x_{-3} + u \rangle^\perp$ for any a, b not both zero. Thus $x_2 + cx_1 + dx_3 \in R$. Thus $S(x_3)$ is not

singular, but has maximal dimension. This reduces us to the case of the previous section, providing a contradiction. This completes the main theorem.

REFERENCES

1. E. Artin, *Geometric Algebra*, Interscience Publishers, Inc., 1965.
2. M. Aschbacher, *A characterization of the unitary and symplectic groups over finite fields of characteristic at least 5*, to appear.
3. B. Fischer, *Finite groups generated by 3-transpositions I*, *Inventiones Math.*, **13** (1971), 232-246.
4. G. Glauberman, *Quadratic elements in unipotent linear groups*, *J. Algebra*, **20** (1972), 637-654.
5. D. Gorenstein, *Finite Groups*, Harper and Row, 1968.
6. B. Stark, *Some subgroups of $\Omega(V)$ generated by groups of root type 1*, *Illinois J. Math.*, **17** (1973), 584-607.
7. T. Tamagawa, *On the structure of orthogonal groups*, *Amer. J. Math.*, **80** (1958), 191-197.
8. J. Thompson, *Quadratic pairs*, to appear.

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