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ALMOST CHEBYSHEV SUBSPACES OF $L^1(\mu; E)$

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This paper studies the set of points which have a unique best approximation from a set M in a Banach space X. The ideal is for every element of X to have a unique best approximation (M is then called Chebyshev). Unfortunately, finite dimensional subspaces of $L^1[0, 1]$ fail to have this property. To remedy this problem and a similar situation in $C(T)$, A. L. Garkavi introduced almost Chebyshev subspaces as those for which the set of elements of X which do not have unique best approximations from M is of the first category.

A class of subsets is determined, containing all finite dimensional subspaces of $L^1(\mu; E)$ where μ is a non-atomic measure and E is a Banach space, which, though not Chebyshev, are almost Chebyshev.

Next characterizations are given of the finite dimensional almost Chebyshev subspaces of $L^1(\mu; R)$ when μ is arbitrary. Finally, these results are applied to $C(T)^*$, the Banach space of bounded Borel measures on a compact Hausdorff space T, determining the finite dimensional almost Chebyshev subspaces of $C(T)^*$. Scattered throughout the paper are results on the existence (or nonexistence, as the case may be) of continuous selections for the metric projections, including a characterization of the finite dimensional subspaces of $C(T)^*$ which support lower semi-continuous metric projections.

1. Introduction and definitions. Let M be a nonempty subset of a Banach space X. For any x in X, we say that y in M is a *best approximation to x from M if*

$$\| x - y \| = \inf \{\| x - m \|; m \text{ in } M\} .$$

We are interested in examining the uniqueness of the best approximation of functions in $X = L^1(\mu; E)$, the Bochner integrable functions from a measure space (Ω, Σ, μ) into a Banach space E. In particular, for a given subset, M, we are interested in the set of points which have a unique best approximation from M. It is well-known that if M is a finite dimensional subspace of $L^1(\mu; R)$ and μ is non-atomic, then there always exist functions which have more than one best approximation (see, e.g. [8]). The question arises whether this bad behavior is, in some sense, pathological.

A. L. Garkavi, aware of this problem and a similar one in $C(T)$, defined a subset M of a Banach space X to be *almost Chebyshev* if the set of points of X which do not have a unique best approximation in M is of first category. (A set is, of course, *Chebyshev* if every point of X has a unique best approximation.) In [1], he showed that there exist almost Chebyshev subspaces of any finite dimension in every separable Banach space. In [2], he characterized the finite dimensional almost Chebyshev subspaces of $C(T)$, where T is a compact metric space. S. B. Stechkin [13] had previously shown that any closed subset of a uniformly rotund Banach space is almost Chebyshev. See also [11], in which we show that c_0 has no finite dimensional almost Chebyshev subspaces except those which are actually Chebyshev.

The following lemma is very useful:

LEMMA A. *Let X be a Banach space, M a norm-separable, weak-sequentially boundedly compact subset of X. If the set of elements which have a unique best approximation is dense, then M is almost Chebyshev.*

Garkavi proves this result [1, p. 171] under the assumption that X is separable and M is a reflexive subspace of X, but a slight modification of his argument yields Lemma A.

Another problem of best approximation that we treat has to do with the continuity of the metric projection P_M, the set-valued map which assigns to each x in X, the set $P_M(x)$ of best approximations to x from M. The map is called lower semi-continuous (l.s.c.) if, for every open set U of X,

$$\{x \text{ in } X; P_M(x) \cap U \neq \varnothing\}$$

is open in X. Otherwise said, P_M is l.s.c. iff, for every closed set K of X,

$$\{x \text{ in } X; P_M(x) \subset K\}$$

is closed. A continuous selection for P_M is a continuous map $s: X \to M$ such that $s(x)$ is in $P_M(x)$ for every x in X. It is clear that if M is almost Chebyshev there is at most one continuous selection for the metric projection.

We first show that there is no finite dimensional Chebyshev subspace of $L^1(\mu; E)$ (μ non-atomic). We then show that every finite dimensional subspace is almost Chebyshev. We also show that no such subspace has a continuous selection, generalizing a result of Lazar, Wulbert, and Morris in $L^1(\mu; R)$ ([7]). Finally, we characterize the almost Chebyshev finite dimensional subspaces of $L^1(\mu; R)$ when μ is arbitrary. A few conclusions are also made concerning the space of Borel measures on some compact Hausdorff set.

We define $L^1(\mu; E)$ in the following way: Let (Ω, Σ, μ) be a measure space and E a Banach space with norm denoted by $|\cdot|$. The vector space of all Bochner integrable functions is denoted by

$\mathscr{L}^1(\mu; E)$; it is endowed with the topology induced by the semi-norm

$$\|f\| = \int_\Omega |f(s)|\, d\mu(s) .$$

The associated Hausdorff space is denoted by $L^1(\mu; E)$.

In order to realize the dual of $L^1(\mu; E)$, we define the space $L^\infty(\mu; E^*)$ in the following manner: A function $\varphi: \Omega \to E^*$ is called weakly-measurable if, for any z in E, the real-valued function defined on Ω by $s \mapsto \langle z, \varphi(s)\rangle$ is measurable. We denote by $\mathscr{L}^\infty(\mu; E^*)$ the vector space of weakly-measurable functions from Ω into E for which the essential supremum of $|\varphi(s)|$ on Ω is bounded; this quantity defines a semi-norm on $\mathscr{L}^\infty(\mu: E^*)$. Let $L^\infty(\mu; E^*)$ denote the associated Hausdorff space. It is well-known that $L^1(\mu; E)$ and $L^\infty(\mu; E^*)$ are complete.

In order to identify $L^\infty(\mu; E^*)$ with $L^1(\mu; E)^*$ we proceed as follows: For f in $L^1(\mu; E)$ and φ in $L^\infty(\mu; E^*)$ we define a duality $\langle \cdot, \circ \rangle$ by

$$\langle f, \varphi\rangle = \int \langle f(s), \varphi(s)\rangle\, d\mu(s) .$$

For fixed φ, the map from $L^1(\mu; E)$ into R defined by $f \mapsto \langle f, \varphi\rangle$ is a continuous linear functional. The norm of this functional can be shown to be the same as that of φ. Hence we have an isometry of $L^\infty(\mu; E^*)$ into $L^1(\mu; E)^*$. When μ is σ-finite, it can be shown that, for any Banach space E, the map given above is onto [3, p. 93]. For this reason, in the remainder of this paper, *we will assume that* $L^1(\mu; E)^* = L^\infty(\mu; E^*)$.

For any function f on Ω, set $Z(f) = \{s \text{ in } \Omega; f(s) = 0\}$, $\operatorname{supp}(f) = Z(f)^c$; if φ in $L^\infty(\mu; E^*)$, then

$$S(\varphi) = \{s \text{ in } \Omega; |\varphi(s)| < \|\varphi\|\} .$$

If f is in $L^1(\mu; E)$ and φ is in $L^\infty(\mu; E^*)$, we define f_φ by $f_\varphi(s) = \langle f(s), \varphi(s)\rangle$. (Note that f_φ is in $L^1(\mu; R)$.) If B is in Σ, then χ_B denotes the characteristic function of B. For any Banach space X, let $S(X) = \{x \text{ in } X; \|x\| = 1\}$ and θ be its zero element.

2. Non-Chebyshev subsets of $L^1(\mu; E)$.

The concept of a thin subset of $L^1(\mu; R)$ was introduced by J. F. C. Kingman and A. P. Robertson in their study of Lyapunov's theorem [4]. This idea leads us to a large class of sets in $L^1(\mu; E)$ which fail to be Chebyshev. We begin with the definition.

DEFINITION. A set $M \subset L^1(\mu; R)$ is said to be *thin* if, for every S in Σ with $\mu(S) > 0$, there exists a nonzero φ in $M^\perp$ whose support is in S. The set of all such φ will be denoted by $M^{\perp}(S)$.

REMARKS. The closed linear span of M is thin iff M is thin, and any subset of a thin set is thin. If there is an atom A in Σ on which some element g in M is nonzero, it is easy to see that $M^{\perp}(A)$ is empty; consequently, *in the remainder of this section*, (Ω, Σ, μ) *will be assumed non-atomic.*

EXAMPLE 1. Any subset of a finite dimensional subspace M of $L^1(\mu; R)$ is thin. This example was given and proved by Kingman and Robertson.

EXAMPLE 2. Kingman and Robertson give the following example of an infinite dimensional, thin subspace of $L^{\perp}(\mu; R)$: Let Ω be partitioned into a disjoint sequence of sets S_n of positive measure and let M be the subset of $L^1(\mu; R)$ of functions constant on each S_n. Then M is thin.

EXAMPLE 3. Since we are trying to connect thinness with approximation properties, it may be worthwhile to note that the above example leads us to a thin, closed, non-proximinal subspace: Since M in the above is not reflexive, there exists a non-proximinal hyperplane M_1 of M. Clearly M_1 is thin and closed.

The importance of thin subspaces is highlighted in the following version of the Lyapunov theorem.

THEOREM A (Kingman-Robertson). *If M is thin, then the map from Σ into R^M defined by* $S \mapsto \left(\int_S f\,d\mu;\ f \text{ in } M\right)$ *has convex and compact range in* R^M.

In order to make maximum use of this theorem, we introduce another related property in $L^1(\mu; E)$. It is clear that the definition of thin would be applicable in $L^1(\mu; E)$, but we find the following more useful.

DEFINITION. A subset M of $L^1(\mu; E)$ is called *pseudo-thin* if there exist distinct f and g in M and φ in $S(L^\infty(\mu; E^*))$ such that

(i) $\langle f - g, \varphi\rangle = \|f - g\|$ and

(ii) $M_\varphi = \{h_\varphi; h \text{ in } M\}$ is a thin subset of $L^1(\mu; R)$.

It is not difficult to verify the next example.

EXAMPLE 4. Any thin set M of $L^1(\mu; R)$ with more than one point is pseudo-thin.

It is also easy to give an example of a pseudo-thin set which is not thin.

EXAMPLE 5. Let S be a measurable subset of Ω such that $\mu(S) > 0$ and $\mu(S^c) > 0$. Let $L^1(S)$ be the set of all functions in $L^1(\mu; R)$ whose support is in S, and let N be a thin subspace of $L^1(S)$. Finally, set $M = N + L^1(S^c)$. Then $M^\perp(S^c) = \{\theta\}$ so that M is not thin, but it is pseudo-thin (for f_1, θ in N and $\varphi = \operatorname{sgn} f_1$ satisfy the conditions of the definition).

EXAMPLE 6. If M is a finite dimensional subspace of $L^1(\mu; E)$, then M is pseudo-thin.

Proof. Let f be a nonzero element of M and φ be in $S(L^\infty(\mu; E^*))$ with $\langle f, \varphi\rangle = \|f\|$ (here g of the definition is zero). It is clear than M_φ is finite dimensional and, therefore, thin. Consequently, M is pseudo-thin.

EXAMPLE 4. Let X be a subspace of E and suppose $\mu(\Omega) = 1$. Define M to be the set of all functions $f: \Omega \to E$ such that $f(s) = x$ for some x in X and all s in Ω. Then M is pseudo-thin.

Proof. For f in M, pick x^* in $S(E^*)$ such that $\langle f(s), x^*\rangle = \|x\|$. Then φ in $L^\infty(\mu; E^*)$ defined by $\varphi(s) = x^*$ is such that $\langle f, \varphi\rangle = \|f\|$ (again, g of the definition is θ). Since M_φ is just the set of constants in $L^1(\mu; R)$, it is thin. Thus M is pseudo-thin.

We can now prove our major negative result. The ideas are closely related to those of R. R. Phelps [8] (and his referee, Henry Dye) who proved that finite dimensional subspaces of $L^1(\mu; R)$ (μ non-atomic, of course) fail to be Chebyshev. The result we give enlarges both the type of subset treated and the range of the functions involved.

THEOREM 2.1. *Any pseudo-thin subset M of $L^1(\mu; E)$ fails to be Chebyshev.*

Proof. Pick f, g, and φ as in the definition of pseudo-thin. Since M_φ is thin, by Theorem A there is a B in Σ with

$$\int_B \langle h(s), \varphi(s)\rangle = \frac{1}{2}\int_\Omega \langle h(s), \varphi(s)\rangle \quad \text{for all } h \text{ in } M .$$

Define $a = \chi_B - \chi_{B^c}$ and $k = a(f - g) + f$. Note that $a\varphi$ is in $S(M^\perp)$. Also, since $\langle f - g, \varphi\rangle = \|f - g\|$, we may conclude that

$$\langle (f - g)(s), \varphi(s)\rangle = |(f - g)(s)| \text{ a.e.}$$

We now prove that $\{f, g\}$ is contained in $P_M(k)$. First, let h be in M. Then

$$\begin{aligned}\|k - h\| &\geqq \int \langle (k - h)(s), a(s)\varphi(s)\rangle \\ &= \int \langle k(s), a(s)\varphi(s)\rangle \\ &= \int \langle a(s)(f - g)(s) - f(s), a(s)\varphi(s)\rangle \\ &= \int \langle (f - g)(s), \varphi(s)\rangle \\ &= \|f - g\| .\end{aligned}$$

When we have shown that $\|f - g\| = \|k - f\| = \|k - g\|$ we will, therefore, be done. But clearly $\|k - f\| = \|f - g\|$. As for g, we have

$$\begin{aligned}\|k - g\| &= \|a(f - g) + (f - g)\| \\ &= 2\int_B |f - g| \\ &= 2\int_B \langle (f - g)(s), \varphi(s)\rangle \\ &= \int_\Omega \langle (f - g)(s), \varphi(s)\rangle = \|f - g\| .\end{aligned}$$

3. On EF subspaces.

Frequently proofs of properties concerning finite dimensional subspaces require only that the sets $P_M(f)$ be finite dimensional (and nonempty). For this reason the following definition has been made: A subset M of a normed linear space X is called an *EF subset* if $P_M(f)$ is nonempty and finite dimensional for each f in X. We now show that every thin, infinite dimensional subspace of $L^1(\mu; R)$ fails to be EF. We begin with a general result (in the spirit of Ivan Singer's characterizations of k-Chebyshev sets, i.e., sets such that $P_M(f)$ is nonempty and of dimension $\leqq k$ [12, p. 126]) and apply it to $L^1(\mu; R)$.

THEOREM 3.1. *Let M be a subset of a normed linear space X. If M is an EF set, then* (i) *M is proximinal and* (ii) *there does not exist a φ in $S(M^\perp)$, an f in X, and $\{g_i; i = 1, 2, \cdots\}$ linearly independent elements in M such that $\langle f, \varphi\rangle = \|f\| = \|f - g_i\|$, $i = 1, 2, \cdots$. If M is a subspace, then the converse holds.*

Proof. Suppose M is a subset of X and there exist elements $f, \varphi, g_1, g_2, \cdots$, as above. Let g be in M. Then $\|f - g\| \geqq \langle f - g, \varphi\rangle = \langle f, \varphi\rangle = \|f\|$ which shows $\{g_i\}$ is contained in $P_M(f)$, so that M is not EF.

On the other hand, suppose M is a proximinal subspace which is not EF. Then there is an f in X such that θ is in $P_M(f)$ and $P_M(f)$

is not finite dimensional, say $\{g_i; i = 1, 2, \cdots\}$ is an infinite set of linearly independent elements of $P_M(f)$. By the Hahn-Banach theorem, there exists a φ in $S(M^\perp)$ such that $\langle f, \varphi\rangle = \|f\| = \|f - g_i\|$ for all i.

THEOREM 3.2. *Let M be a subspace of $L^1(\mu; R)$. Then M is EF iff M is proximinal and there does not exist a φ in $S(L^\infty(\mu; R))$ and linearly independent elements $\{g_i; i = 1, 2, \cdots\}$ in M such that $\int \varphi g d\mu = 0$ for all g in M, and $|\varphi(s)| = 1$ almost everywhere on $Z(g_i)^c$ for each i.*

Proof. Suppose the conditions fail to hold; we show that M is not EF. By multiplying g_i by $(2^i\|g_i\|)^{-1}$, if necessary, we may assume that $\|g_i\| = 2^{-i}$. Define $f = \varphi\Sigma|g_i|$. Then $\|f\| \leqq \Sigma\|g_i\| < \infty$, so f is in $L^1(\mu; R)$. It is relatively easy to verify the violation of (ii) in the previous theorem. Hence M is not EF. Conversely suppose M is not EF, but is proximinal. Then by Theorem 3.1, there exists f in $L^1(\mu; R)$, φ in $L^\infty(\mu; R)$, φ in $S(M^\perp)$, and $\{g_i\}$ linearly independent elements of M such that $\langle f, \varphi\rangle = \|f\| = \|f - g_i\|$. So

$$\int (f - g_i)(s)\varphi(s) = \int |(f - g_i)(s)|$$

and

$$\int |f(s)| = \int f(s)\varphi(s) .$$

Consequently, up to a set of measure zero,

$$\{s \text{ in } \Omega; |\varphi(s)| = 1\} \supset Z(g_i)^c \cup Z(f - g_i)^c \supset Z(g_i)^c$$

which is what we wanted to show.

COROLLARY 3.1. *If M is a thin and infinite dimensional subspace of $L^1(\mu; R)$, then M is not EF.*

Proof. Since M is thin, there exists a subset B of Σ such that $\int_B f\,d\mu = 1/2\int_\Omega f\,d\mu$ for all f in M. Let $\varphi = \chi_B - \chi_{B^c}$. Then φ is in $S(M^\perp)$ and $|\varphi(s)| = 1$ a.e. on $Z(f)^c$ for all f in M. We may, therefore, apply the previous result.

4. Almost Chebyshev subspaces of $L^1(\mu; E)$. We are going to prove in this section that every finite dimensional subspace of $L^1(\mu; E)$ (μ non-atomic) is almost Chebyshev. In order to do this, we utilize a characterization of best approximations due to B. Kripke

and T. J. Rivlin [5] formulatd originally in $L^1(\mu; C)$. To formulate this result in the more general setting, we write, for x and y in E,

$$d(x, y) = \lim_{t \downarrow 0} \frac{\|x + ty\| - \|x\|}{t},$$

the right-hand derivative of the norm at x in the direction y (since the norm is a convex function, the existence of this limit is guaranteed). If f and g are in $L^1(\mu; E)$, we define $d(f, g)(s) = d(f(s), g(s))$, $|f|(s) = |f(s)|$, and $R(f) = Z(f)^c$. We may now state the generalized version of Kripke and Rivlin's result.

THEOREM 4.1. *Let M be a subset of $L^1(\mu; E)$ where μ is an arbitrary measure and E is any Banach space. Consider the following statements for f in $L^1(\mu; E)$:*

[1] $$g_0 \textit{ in } P_M(f)$$

[2] $$-\int_{R(f-g_0)} d(f - g_0, g) \leqq \int_{Z(f-g_0)} |g| \quad \textit{for all } g \textit{ in } g_0 - M.$$

We have the following conclusions:

A. For any set M, [2] implies [1]; if M is convex, then [2] is equivalent to [1].

B. For any set M, if strict inequality holds in [2] for all nonzero g in $g_0 - M$, then $\{g_0\} = P_M(f)$.

C. Suppose M is convex, g_0 and g in $P_M(f)$. Then

$$\int_{R(f-g_0)} d(f - g_0, g_0 - g) = -\int_{Z(f-g_0)} |g - g_0| .$$

The proof is similar to the one in [5], so we will omit it. We wish to write equation [2] in a simpler form for smooth Banach spaces E. In this case, it is well-known that for x in E, $x \neq \theta$, if $\partial(x)$ denotes the unique element of norm 1 in E^* such that $\langle x, \partial(x)\rangle = |x|$, then $d(x, y) = \langle y, \partial(x)\rangle$. For f in $L^1(\mu; E)$, choose φ of norm 1 in $L^\infty(\mu; E^*)$ for which $\langle f, \varphi\rangle = \|f\|$. It is easy to see that this implies $\langle f(s), \varphi(s)\rangle = |f(s)|$ almost everywhere, i.e., $\varphi(s) = \partial(f(s))$ for almost all s in Ω with $f(s) \neq 0$. We may therefore define $\partial f = \varphi\chi_{\mathrm{supp}(f)}$ as the unique element of norm 1 in $L^\infty(\mu; E^*)$ such that $\langle f, \partial f\rangle = \|f\|$ and $\mathrm{supp}\,(\partial f) = \mathrm{supp}\,(f)$.

COROLLARY 4.1. *If μ is an arbitrary measure, E a smooth Banach space, and M a convex subset of $L^1(\mu; E)$, then the following are equivalent for f in $L^1(\mu; E)$:*

$$[1] \qquad g_0 \text{ in } P_M(f)$$

$$[2] \qquad -\int_{R(f-g_0)} \langle g(s), \partial(f - g_0)(s)\rangle \leqq \int_{Z(f-g_0)} |g(s)| \quad \text{for every } g \text{ in } g_0 - M .$$

If strict inequality holds in [2] for every nonzero element of $g_0 - M$, then $\{g_0\} = P_M(f)$.

If $E = C$, then $\partial f(z) = \overline{\operatorname{sgn} f(z)} \equiv \overline{f(z)}/|f(z)|$ and we recover the result proved by Kripke and Rivlin.

We have found the following idea of some use in our study.

DEFINITION. A set M in $L^1(\mu; E)$ is called *svelte* if for every $\delta > 0$, there exists a measurable subset B of Ω for which $0 < \mu(B) < \delta$ and $\int_B |g| > 0$ for every nonzero element g in the linear span of M.

It was first revealed in [7] that finite dimensional subspaces of $L^1(\mu; R)$ are svelte whenever μ is non-atomic. Their (inductive) proof of this fact may be used without any changes for finite dimensional subspaces of $L^1(\mu; E)$ when, again, μ is non-atomic. Another example in non-atomic spaces are subspaces all of whose nonzero elements have the same support (e.g., smooth subspaces of $L^1(\mu; R)$ as shown in [14, p. 385]). We also mention that the Kingman-Robertson example of an infinite dimensional thin subspace of $L^1(\mu; R)$ (Example 2) can easily be shown to be svelte. As with thin subsets, any subset of a svelte set is svelte.

LEMMA 4.1. *Let μ and E be arbitrary and M a subset of $L^1(\mu; E)$. Suppose B is in Σ, f is in $L^1(\mu; E)$, and g_0 is in $P_M(f)$. Define $f^* = f\chi_{B^c} + g_0\chi_B$. Then $g_0 \in P_M(f^*) \subset P_M(f)$.*

Proof. For any g in M,

$$\begin{aligned} \|f^* - g_0\| &= \int_\Omega |f - g_0| - \int_B |f - g_0| \\ &\leqq \int_\Omega |f - g| - \int_B |f - g_0| \\ &\leqq \int_\Omega |f - g| + \left[\int_B |g - g_0| - \int_B |f - g|\right] \\ &= \|f^* - g\| . \end{aligned}$$

Hence g_0 is in $P_M(f^*)$. If g is not in $P_M(f)$, then the first inequality is strict, so $P_M(f^*) \subset P_M(f)$.

LEMMA 4.2. *Let μ and E be arbitrary, M a svelte subset of $L^1(\mu; E)$, f in $L^1(\mu; E)$, and g_0 in $P_M(f)$. For every $\varepsilon > 0$, there exists* *an* f_ε *such that* $\|f - f_\varepsilon\| < \varepsilon$, $g_0 \in P_M(f_\varepsilon) \subset P_M(f)$, *and* $\int_{Z(f-g_0)} |g - g_0| > 0$ *for all* g *in* M, $g \neq g_0$.

Proof. Using the absolute continuity of integrals, we have for every $\varepsilon > 0$, a $\delta > 0$ such that, if $\mu(B) < \delta$, then $\int_B |f - g_0| < \varepsilon$. Choose B as in the definition of svelte for the set $g_0 - M$; define f_ε as f^* in the above lemma. Then $B \subset \{s \in \Omega; f_\varepsilon(s) = g_0(s)\}$ a.e.; hence $\int_{Z(f_\varepsilon - g_0)} |g - g_0| > 0$. Also $\|f - f_\varepsilon\| = \int_B |f - g_0| < \varepsilon$. The rest of the lemma follows from the lemma above.

LEMMA 4.3. *Let* μ *be a non-atomic measure and* E *an arbitrary Banach space. Let* M *be a finite dimensional convex subset of* $L^1(\mu; E)$, f *in* $L^1(\mu; E)$, *and* g_0 *in* $P_M(f)$. *Then, for every* $\varepsilon > 0$, *there exists an* f^* *in* $L^1(\mu; E)$ *such that* $\|f - f^*\| < \varepsilon$ *and* $\{g_0\} = P_M(f^*)$.

Proof. Since the subset hypothesized are svelte, by Lemma 4.2 we may assume

$$\int_{Z(f-g_0)} |g - g_0| > 0 \tag{1}$$

for all g in $g_0 - M$, $g \neq g_0$. Since $P_M(f)$ is convex and finite dimensional, the set $P_M(f)$ and its relative interior have the same dimension, say m. (See, for example, [10, Theorem 6.2, p. 45].) If $m = 0$, then $\{g_0\} = P_M(f)$ and we are done. Let $\{g_i; i = 1, 2, \cdots\}$ be dense in $P_M(f)$. From Theorem 4.1.C and the assumption (1) above, it follows that $\Omega_i \equiv \{s \text{ in } \Omega; f(s) \neq g_0(s) \text{ and } d((f - g_0)(s), (g_0 - g_i)(s)) < 0\}$ has positive measure for each $i = 1, 2, \cdots$. By the absolute continuity of integrals, pick $\delta > 0$ so that $\mu(B) < \delta$ implies that

$$\int_B |f - g_0| < \varepsilon m^{-1}.$$

Now choose $B_i \subset \Omega_i$ with $0 < \mu(B_i) \leqq \delta 2^{-i}$. Set $B_* = \bigcup B_i$; then $\mu(B_*) < \delta$. Define $f_i = f\chi_{B_i^c} + g_0\chi_{B_i}$ for $i = *, 1, 2, \cdots$. From Lemma 4.1, it follows that $g_0 \in P_M(f_*) \subset P_M(f_i) \subset P_M(f)$ for $i = 1, 2, \cdots$, and $\|f - f_*\| < \varepsilon$. But from Theorem 4.1.A and the definition of B_i it follows that

$$\begin{aligned} &-\int_{R(f_i - g_0)} d(f_i - g_0, g_0 - g_i) \\ &= -\left[\int_{R(f-g_0)} d(f - g_0, g_0 - g_i) - \int_{B_i} d(f - g_0, g_0 - g_i)\right] \end{aligned}$$

$$= \int_{Z(f-g_0)} |g_0 - g_i| + \int_{B_i} d(f - g_0, g_0 - g_i) < \int_{Z(f-g_0)} |g_0 - g_i|$$
$$\leqq \int_{Z(f_i-g_0)} |g_0 - g_i| .$$

It follows, therefore, from Theorem 4.1.C that $g_i \notin P_M(f_i)$. Hence $g_i \notin P_M(f^*)$ for $i = 1, 2, \cdots$. Since $P_M(f^*) \subset P_M(f)$ and we have shown that $P_M(f^*)$ contains no relatively open subsets of $P_M(f)$, we may conclude that $\dim P_M(f^*) < \dim P_M(f)$. By applying the process given above to f^*, we may find an f^{**} such that

$$\| f^* - f^{**} \| < \varepsilon m^{-1}, \ g_0 \in P_M(f^{**}) \subset P_M(f^*)$$

and

$$\dim P_M(f^{**}) < \dim P_M(f^*) < \dim P_M(f) .$$

After at most m steps, therefore, we obtain a point, say f_0, so that $g_0 \in P_M(f_0)$, $\|f - f_0\| < \varepsilon$, and $\dim P_M(f_0) = 0$. We conclude that $\{g_0\} = P_M(f_0)$ and we are finished.

It is now easy to deduce the main theorems.

THEOREM 4.1. *Any finite dimensional convex subset of* $L^1(\mu; E)$ *is almost Chebyshev, provided that* μ *is non-atomic.*

Proof. The previous lemma shows that the set of elements with a unique best approximation is dense. Now apply Lemma A of the introduction.

THEOREM 4.2. *The metric projection onto a finite dimensional non-singleton convex subset of* $L^1(\mu; E)$ *admits no continuous selections if* μ *is non-atomic.*

Proof. Let $f \in L^1(\mu; E)$ have two best approximations, say g_0 and g_1 (such points exist by Theorem 2.1). Then by Lemma 4.3, there exist two sequences, $\{f_n\}$ and $\{h_n\}$, converging to f for which $\{g_0\} = P_M(f_n)$ and $\{g_1\} = P_M(h_n)$. Hence no selection can be continuous.

If we are willing to assume that E is a smooth Banach space, then it is possible to acheive somewhat stronger results. In particular, in Lemma 4.3, we need only assume that M is convex, svelte and quasi-thin (i.e., for every $f \in L^1(\mu; E)$, $M_{\partial f} = \{g_{\partial f}; g \in M\}$ is thin, where $g_{\partial f}(s) = \langle g(s), \partial f(s) \rangle$. If $E = R$, then any thin set is quasi-thin). Modifications of the two theorems are then made accordingly.

We now give a necessary condition for the lower semi-continuity of the metric projections onto subspaces of $L^1(\mu; E)$ where both μ and E are arbitrary. This result is necessarily only a partial generalization of the result in [7] (which presents necessary and sufficient conditions in $L^1(\mu; R)$) because, in the presence of an atom, the nature of the range space predominates. Our result is stronger, however, in that M is not assumed to be finite dimensional.

If M is a subspace of $L^1(\mu; E)$ we denote by $F(M)$ (or F when no confusion can arise) the subspace of M consisting of functions whose support is a finite union of atoms. Let M^θ be the set of all $x \in X$ such that $\theta \in P_M(x)$.

THEOREM 4.3. *Let E be a Banach space, and let μ satisfy, as always, $L^1(\mu; E)^* = L^\infty(\mu; E^*)$. Suppose M is a proximinal, non-Chebyshev subspace of $L^1(\mu; E)$ such that $F(M) = \{\theta\}$. Then P_M is not l.s.c.*

Proof. Since M is proximinal, convex, and non-Chebyshev, there exists an $f \in L^1(\mu; E)$ such that $P_M(f)$ contains a line segment. By translating the midpoint of this line segment to θ, if necessary, we may assume that $f \in M^\theta$ and there is a nonzero $g_1 \in M$ with g_1 and $-g_1$ in $P_M(f)$. By the Hahn-Banach theorem there is a $\varphi \in S(M^\perp)$ with $\langle f, \varphi\rangle = \|f\|$. Since $g_1 \notin F$, there exists a sequence of disjoint measurable subsets $B_i \subset \operatorname{supp}(g_1)$ such that $\mu(B_i) > 0$, $\sum_{j=i}^\infty \|f\chi_{B_j}\| \to 0$ and $\langle g_1(s), \varphi(s)\rangle$ has constant sign on B_i. For any t satisfying $0 \leqq t \leqq 1$, we have

$$\|f - tg_1\| = \|f\| = \int \langle f(s) - tg_1(s), \varphi(s)\rangle$$

which, in turn, implies

$$\langle f(s) - tg_1(s), \varphi(s)\rangle = |f(s) - tg_1(s)|$$

for almost all $s \in \Omega$. By passing to a subsequence, if necessary, we may assume the sign of $\langle g_1(s), \varphi(s)\rangle$ is either (a) $\geqq 0$ on each B_i or (b) $\leqq 0$ on each B_i. If (b) holds we replace g_1 by $-g_1$ so that we may assume (a). Define $A_i = \bigcup_{j=i}^\infty B_j$ and $f_i = f\chi_{A_i^c}$. Then $f_i \to f$ and, by Lemma 4.1, we see that

$$P_M(f_1) \supset \cdots \supset P_M(f_n) \supset \cdots \supset \{\theta\}\ .$$

We claim $P_M(f_1) \cap \{g;\, g = tg_1,\ 0 < t \leqq 1\} = \emptyset$ so that $P_M(f) \not\subset P_M(f_1)$. If this is so, then $P_M(f_1)$ is a closed set such that $D \equiv \{x \in L^1(\mu; E);\ P_M(x) \subset P_M(f_1)\}$ is not closed (since $f_i \in D$ for $i = 1, 2, \cdots$, but $f \notin D$), whence P_M is not l.s.c. So we must prove the claim. Put $A = A_1$ and $0 < t \leqq 1$. Then

$$\begin{aligned}
\| f_1 - tg_1 \| &= \int_{A^c} |f - tg_1| + \int_A |tg_1| \\
&= \int_{A^c} \langle f(s) - tg_1(s), \varphi(s) \rangle + \int_A |tg_1| \\
&= \int_{A^c} \langle f(s), \varphi(s) \rangle - \left[\int_\Omega - \int_A \langle tg_1(s), \varphi(s) \rangle \right] + \int_A |tg_1| \\
&= \int_{A^c} \langle f(s), \varphi(s) \rangle + t \int_A \langle g_1(s), \varphi(s) \rangle + t \int_A |g_1| \\
&> \int_{A^c} \langle f(s), \varphi(s) \rangle \\
&= \| f_1 \| .
\end{aligned}$$

5. **Almost Chebyshev subspaces of $L^1(\mu; R)$.** For the purposes of comparison, and because we do use it a few times, let us state the result of [7]. Recall that $S(\varphi) = \{s \in \Omega; |\varphi(s)| < \|\varphi\|\}$.

THEOREM 5.1. (Lazar, Wulbert, and Morris). *Let M be an n-dimensional subspace of $L^1(\mu; R)$. The metric projection onto M is l.s.c. iff there does not exist a nonzero $g \in M$ and nonzero $\varphi \in S(M^\perp)$ satisfying* (i) *$S(\varphi)$ is purely atomic and contains at most $n-1$ atoms,* (ii) *$Z(g) \supset S(\varphi)$, and* (iii) *$g \notin F$.*

We may restate this result as follows:

COROLLARY 5.1. *A finite dimensional subspace M of $L^1(\mu; R)$ supports a l.s.c. metric projection iff $P_M(f) \subset F$ for every $f \in M^\theta$.*

This corollary is for comparison purposes only, and so we shall not prove it (though it follows rather easily from the theorem).

The first characterization of almost Chebyshev subspaces of $L^1(\mu; R)$ is similar in style to the above corollary. It reduces the search for an open set to looking for a single element.

THEOREM 5.2. *A finite dimensional subspace M of $L^1(\mu; R)$ fails to be almost Chebyshev iff there exists an element $f_0 \in M^\theta$ with more than one best approximation such that $P_M(f_0) \subset F(M)$.*

Proof. First suppose that M is almost Chebyshev and f_0 is as in the theorem. Then there exists $f_n \to f_0$ and $x_n \in M$ such that $\{x_n\} = P_M(f_n)$ for $n \geqq 1$. Since $\{x_n\}$ is a bounded subset of M we may assume that $x_n \to x_0$ and $x_0 \in P_M(f_0)$. Since $x_n \in M$, it follows that $P_M(f_n - x_n) = P_M(f_n) - x_n \subset F(M)$ for $n = 0, 1, 2, \cdots$. Thus $P_M(f_n - x_n) = P_F(f_n - x_n)$ for $n = 0, 1, 2, \cdots$. However, by Theorem 5.1, P_F is l.s.c. Since each set $P_F(f_n - x_n)$ is a single point for $n \geqq 1$, it follows that the limit set $P_F(f_0 - x_0)$ is also a single point. But $P_F(f_0 - x_0) = P_M(f_0 - x_0)$ which contains more than one point by hypothesis.

To prove the converse, assume that for each $f \in M^\theta$, either $P_M(f)$ is a single point or there is a $g \in P_M(f)$ whose support does not consist of a finite number of atoms. We will show that M is almost Chebyshev. It is sufficient to show that for any $f \in M^\theta$, there exist $f_n \in L^1$, $f_n \to f$ such that $P_M(f_n)$ is a single point. If $P_M(f)$ is a single point, we are done. Otherwise, there is a $g_0 \in P_M(f)$ which has support not consisting of a finite number of atoms. Hence either $\{s \in \Omega; g_0(s) > 0\}$ or $\{s \in \Omega; g_0(s) < 0\}$ has this same property; call the relevant set C. Using the absolute continuity of the integral and the fact that $P_M(f)$ has finite dimension, say q, we can find, for any $\varepsilon > 0$, a set $B \subset C$ for which $\int_B |f - g| < \varepsilon q^{-1}$ for all $g \in P_M(f)$. It is an easy set-theoretic argument that there exists a subset A of B, $\mu(A) > 0$, for which $\int_A |f - rg_0|$ is not a constant for $0 \leqq r \leqq 1$. Hence $\Phi(g) = \int_A |f - g|$ is a convex function which attains its maximum only on the relative boundary of $P_M(f)$ (see [10, p. 342]). Let $g_1 \in P_M(f)$ be a point where the maximum of Φ is attained. Define $f^* = f\chi_{A^c} + g_1\chi_A$. Then $\|f - f^*\| < \varepsilon q^{-1}$ and $g_1 \in P_M(f^*) \subset P_M(f)$ by Lemma 4.1. Let g be in the relative interior of $P_M(f)$. The fact that $\Phi(g) < \Phi(g_1)$ can then by used to show that $\|f^* - g_1\| < \|f^* - g\|$. Hence $P_M(f^*)$ is contained in the relative boundary of $P_M(f)$. Since $P_M(f^*)$ is convex, it follows that the dimension of $P_M(f^*) < q$. Continuing in this way we find a point, say f_0, such that $\|f - f_0\| < \varepsilon$ and $\dim P_M(f_0) = 0$. Thus $P_M(f_0)$ consists of a single point and we are done.

The next characterization, though more complicated than the former, has the virtue of being more quantitative in character (like Theorem 5.1) and hence more readily applied to particular cases.

THEOREM 5.3. *An n-dimensional subspace M of $L^1(\mu; R)$ fails to be almost Chebyshev iff there exists a $\varphi \in S(M^\perp)$ and a nonzero $g_1 \in F(M)$ such that* (i) *$S(\varphi)$ is purely atomic and contains at most $n - 1$ atoms,* (ii) *$Z(g_1)$ contains $S(\varphi)$, and* (iii) *for every $g \in M \backslash F$, if $Z(g)$ contains $S(\varphi)$, then*

$$\mu(Z(g_1) \cap \{s; g(s)\varphi(s) > 0\}) > 0 .$$

Proof. Suppose there exists φ and g_1 as in the theorem. Set $f = \varphi |g_1|$; so $f \in L^1$ and it can be shown, as in Theorem 2.1, that θ and g_1 are in $P_M(f)$. If we can show $P_M(f) \subset F$ then, by Theorem 5.2, we will be finished. Suppose $g \in P_M(f)$, $g \notin F$. If $\mu(S(\varphi) \cap Z(g)^c) > 0$, then

$$\begin{aligned}
\|f-g\| &= \int_\Omega |\varphi|g_1| - g| \\
&= \int_{S(\varphi)} |g| + \int_{S(\varphi)^c} |\varphi|g_1| - g| \\
&> \int_{S(\varphi)} |\varphi||g| + \int_{S(\varphi)^c} |\varphi^2|g_1| - \varphi g| \\
&\geqq \int_\Omega |g_1| - \int_\Omega \varphi g \\
&= \int_\Omega |g_1| \\
&= \|f\|
\end{aligned}$$

and so $g \notin P_M(f)$. Thus $Z(g)$ contains $S(\varphi)$. Note that

$$\int_\Omega ||g_1| - \varphi g| > \int_\Omega |g_1| - \varphi g$$

for, if equality, then $|g_1| - \varphi g \geqq 0$ or $|g_1| \geqq \varphi g$ a.e. So, $\mu(Z(g_1) \cap \{s; \varphi(s)g(s) < 0\}) = 0$, contrary to hypothesis (iii). We therefore find that

$$\begin{aligned}
\|f-g\| &= \int_\Omega |\varphi|g_1| - g| \\
&= \int_\Omega ||g_1| - \varphi g| \\
&> \int |g_1| - \int \varphi g \\
&= \|f\|
\end{aligned}$$

so that $g \notin P_M(f)$.

To prove the converse, suppose M is not almost Chebyshev. Then there is an element f in M^θ with more than one best approximation such that $P_M(f) \subset F$. Pick $g_0 \in P_M(f)$, $g_0 \neq \theta$, and define $f_0 = f\chi_{\operatorname{supp}(g_0)}$. By Lemma 4.1 both θ and g_0 are in $P_M(f_0)$ which is in $P_M(f)$ which is in F, and $Z(f_0) \supset Z(g_0)$. Now we alter f_0 so that the zeros will be the same. Since $\operatorname{supp}(f_0)$ consists of a finite number of atoms, we may choose r, $0 < r < 1$, so that

$$r \max_{i \in \operatorname{supp}(f_0)} |g_0(i)| < \inf_{i \in \operatorname{supp}(f_0)} |f_0(i)| .$$

Define $f^* = f_0 - rg_0$ and $g_1 = -rg_0$. Then

(a) $$F \supset P_M(f^*) \supset \{\theta, g_1\}$$

(b) $$Z(f^*) = Z(g_1) .$$

Now via an argument of Phelps [9], there exists $\varphi \in S(M^\perp)$, such that $\langle f^*, \varphi\rangle = \|f^*\|$, $S(\varphi) \subset Z(g_1)$, and $S(\varphi)$ consists of at most $n-1$ atoms. Thus we have φ and g_1 as in (i) and (ii). It remains to prove (iii). Suppose $g \in M \backslash F$ and $S(\varphi) \subset Z(g)$. Since $\langle f^*, \varphi \rangle = \| f^* \|$ and $\| \varphi \| = 1$ it follows that $f^*(s)\varphi(s) = | f^*(s) |$; so $S(\varphi) \subset Z(f^*)$ and $S(\varphi) \subset Z(f^* - rg)$ for all real r. If (iii) fails to hold, then $Z(g_1) \subset \{s \in \Omega; g(s)\varphi(s) \leqq 0\}$. Choose $r > 0$ so small that

$$r \max_{i \in \operatorname{supp}(f^*)} | g(i) | < \inf_{i \in \operatorname{supp}(f^*)} | f^*(i) | .$$

Since $Z(f^*) = Z(g_1)$, then $r\varphi(s)g(s) \leqq 0$ whenever $f^*(s) = 0$; thus $| f^*(s) | \geqq r\varphi(s)g(s)$ for all $s \in \Omega$. Thus

$$\begin{aligned} \| f^* - rg \| &= \int | f^* - rg | \\ &= \int |\varphi| \, | f^* - rg | \\ &= \int | \varphi f^* - r\varphi g | \\ &= \int || f^* | - r\varphi g | \\ &= \int | f^* | - \int r\varphi g \\ &= \int | f^* | \\ &= \| f^* \| . \end{aligned}$$

Hence $rg \in P_M(f^*) \subset F$ which contradicts the assumption that $g \in M \backslash F$. Thus (iii) holds, and we are finished.

For non-atomic measure, we were also able to conclude that there were no continuous selections for the metric projection. Lazar [6], however, has characterized those onedimensional subspaces of l^1 which have continuous selections, and we therefore can exhibit an almost Chebyshev, non-Chebyshev subspace of l^1 which has a continuous selection; namely, the span of $(1/2, 1/4, 1/8, \cdots)$. We know this is almost Chebyshev by the above theorem; it fails to be Chebyshev by a theorem of Phelps [9, Theorem 1] and Lazar's result implies it has a continuous selection. We now know, of course, that there is at most one continuous selection.

The previous theorem can be used to verify the following examples in l^1. Let $g^1 = (1, 1, 0, 0, 0, \cdots)$, $g^2 = (1, 1, 1/2, 1/4, 1/8, \cdots)$, $g^3 = (1, 1/2, 1/4, 1/8, \cdots)$, and $g^4 = (1, -1, 1/2, 1/4, \cdots)$. Then (a) $M =$ span of $\{g^1, g^2\}$ is not almost Chebyshev, (b) $M =$ span of $\{g^1, g^3\}$ is almost Chebyshev but not Chebyshev, and (c) $M =$ span of $\{g^1, g^4\}$ is Chebyshev. We are also able to give an example of a three dimensional subspace of l^1 which is neither almost Chebyshev nor able to support a l.s.c. metric projection. Let $g^1 = (0, 1, 1, 0, 0, \cdots)$, $g^2 = (1/2, 1, 1, 0, \cdots)$, and $g^3 = (1, 1/4, 1/8, -1, 1, 1/16, 1/32, \cdots, 1/2^{n-2}, \cdots)$.

Let us sketch a proof of this last example: Apply Theorem 5.1 to show that $M =$ span of $\{g^1, g^2, g^3\}$ does not have a l.s.c. metric projection: Let $\varphi = (0, -1, 1, 1, \cdots)$ and $g = 2g^2 - g^3$. A simple check verifies the conditions of Theorem 5.1. To show M is not almost Chebyshev, let φ be as above and consider g^1. Clearly $g^1 \in F$, $\varphi \in S(M^\perp)$ and (i) and (ii) of Theorem 5.3 hold. For (iii), suppose $g \in M \backslash F$ and $S(\varphi) \subset Z(g)$. So $g(1) = 0$. We want to show $Z(g^1) \cap \{s; g(s)\varphi(s) > 0\} \neq \emptyset$. But $Z(g^1) = \{1, 4, 5, \cdots\}$ and $\{s\colon g(s)\varphi(s) \leqq 0\}$ contains 4 or 5 but not both. Thus M fails to be almost Chebyshev.

6. Applications to $C(T)^*$. We will now apply the results of the previous section on almost Chebyshev subspaces and l.s.c. metric projections to subspaces of $C(T)^*$ where T is a compact Hausdorff space. We will identify $C(T)^*$ with the Banach space of regular Borel measures on T. The set M will be an n-dimensional subspace genereated by $\mu_1, \cdots, \mu_n$. For $\nu \in C(T)^*$, put $\sigma_\nu = |\nu| + |\mu_1| + \cdots + |\mu_n|$ and let J_ν be the isometry of $L^1(\sigma_\nu; R)$ into $C(T)^*$ defined by $(J_\nu f)(E) = \int_E f d\sigma_\nu$. Note that J_ν maps onto the space to measures λ absolutely continuous with respect to σ_ν (written $\lambda \ll \sigma_\nu$). If $\lambda \ll \sigma_\nu$, then $J_\nu^{-1}(\lambda) = d\lambda/d\sigma_\nu$, the Radon-Nykodym derivative of λ with respect to σ_ν. Now set $M_\nu = [d\mu_1/d\sigma_\nu, \cdots, d\mu_n/d\sigma_\nu]$. Many of the ideas used in this section were used by R. R. Phelps in [9].

LEMMA 6.1. *$M \subset C(T)^*$ is almost Chebyshev iff $M_\nu \subset L^1(\sigma_\nu; R)$ is almost Chebyshev for every $\nu \in C(T)^*$.*

Proof. Suppose M fails to be almost Chebyshev. Let U be an open, nonempty, subset of $C(T)^*$ all of whose elements have more than one best approximation. Let $\nu \in U$. Clearly $J_\nu^{-1}(U)$ is an open, nonempty subset of $L^1(\sigma_\nu; R)$ each element of which has more than one best approximation out of M_ν.

Conversely, suppose M_ν is not almost Chebyshev for some ν in $C(T)^*$. For each $\lambda \in C(T)^*$, let $\lambda = \lambda_a + \lambda_s$ be the Lebesgue decomposition of λ with respect to σ_ν (so $\lambda_a \ll \sigma_\nu$ and $\lambda_s \perp \sigma_\nu$); then $\|\lambda\| = \|\lambda_a\| + \|\lambda_s\|$. Suppose U is an open nonempty subset of $L^1(\sigma_\nu; R)$ each element of which has more than one best approximation. Then $\{\lambda \in C(T)^*; J_\nu^{-1}(\lambda_a) \in U\}$ is a nonempty open subset of $C(T)^*$ each element of which has more than one best approximation.

This leads us to the following theorem.

THEOREM 6.1. *An n-dimensional subspace M of $C(T)^*$ fails to be almost Chebyshev iff there exists a Borel measurable function φ on*

T and a nonzero measure $\mu_0 \in M$ such that (i) $\mathrm{Supp}(\mu_0)$ *consists of a finite number of atoms $A_1, \cdots, A_p$,* (ii) $|\varphi(t)| \leqq 1$ *for every t in T,* (iii) $S(\varphi)$ *contains at most $n-1$ points $t_1, \cdots, t_m$,* (iv) $\int_T \varphi d\mu = 0$ *for all $\mu \in M$,* (v) $|\mu_0|(S(\varphi)) = 0$, *and* (vi) *if $\mu \in M$ has support not consisting of a finite number of atoms and $|\mu|(S(\varphi)) = 0$, then there is a Borel set $E \subset T \setminus \mathrm{supp}(\mu_0)$ with $\int_E \varphi d\mu > 0$.*

Before proving this rather formidable-looking theorem, we present a rather brief corollary.

COROLLARY. *If $\mu_1, \cdots, \mu_n$ are all non-atomic, then $M =$ span of $\{\mu_1, \cdots, \mu_n\}$ is almost Chebyshev.*

Proof. Since no nonzero measure in M can satisfy (i) of the above theorem, it follows that M is almost Chebyshev.

REMARK. Corollary 2 of [9] teaches us that such subspaces are not Chebyshev.

Proof of Theorem 6.1. Suppose that M is not almost Chebyshev. By the lemma, there exists a $\nu \in C(T)^*$ for which M_ν is a non-almost Chebyshev subspace of $L^1(\sigma_\nu; R)$. By Theorem 5.3, there exists a $\varphi \in L^\infty(\sigma_\nu; R)$ and a g_0 in M_ν, $g_0 \neq \theta$, such that (i′) $\mathrm{supp}(g_0)$ is a finite number of σ-atoms, (ii′) $|\varphi(t)| \leqq 1$ for $t \in T$, (iii′) $S(\varphi)$ contains at most $n-1$ points, (iv′) $\varphi \in M_\nu^\perp$, (v′) $Z(g_0) \supset S(\varphi)$ and (vi′) if $g \in M_\nu$ has support not consisting of a finite number of σ-atoms and $\sigma[S(\varphi) \cap Z(g)^c] = 0$ then $\sigma[Z(g_0) \cap \{s \in T; g(s)\varphi(s) > 0\}] \neq 0$. By defining $\mu_0 = J_\nu g_0$, conditions (i)–(v) of the theorem follow readily from the corresponding conditions (i′)–(v′). To prove (vi), let μ satisfy the conditions given in (vi) and set $g = J_\nu^{-1}(\mu)$. Then g satisfies the conditions of (vi′), and the conclusion of (vi′) may be used to get the conclusion of (vi).

To prove the converse, we suppose the conditions of the theorem hold, prove that $M_\theta \subset L^1(\Sigma|\mu_\iota|; R)$ fails to be almost Chebyshev, and apply Lemma 6.1. In fact, φ as given yields a $\varphi_0 \in M_\theta^\perp$ and μ_0 yields $g_\iota = d\mu_0/d\Sigma|\mu_i| \in M_\theta$ which have all the properties of Theorem 5.3.

Using the methods developed above, we can use the result of [7] on lower semi-continuity of the metric projection in $L^1(\mu; R)$ to prove results in $C(T)^*$. In fact, we have the following lemma and theorem.

LEMMA 6.2. *The metric projection onto $M =$ span of $\{\mu_1, \cdots, \mu_n\}$ is l.s.c. iff the metric projection of M_ν in $L^1(\sigma_\nu; R)$ is l.s.c. for each ν in $C(T)^*$.*

THEOREM 6.2. *The metric projection onto the n-dimensional subspace M of $C(T)^*$ fails to be l.s.c. iff there exists a Borel measurable function φ on T and a $\mu_0 \in M$ satisfying* (i) $|\varphi(t)| \leqq 1$ *for* $t \in T$, (ii) $S(\varphi)$ *contains at most* $n-1$ *points,* (iii) $\operatorname{supp}(\mu_0)$ *is not the union of a finite number of atoms,* (iv) $|\mu_0|(S(\varphi)) = 0$, *and* (v) $\int_T \varphi d\mu = 0$ *for each* $\mu \in M$.

The proofs of the above assertions are straightforward applications of the techniques already used in the previous theorem.

Note added in proof. In a recent paper S. Ja. Havinson and Z. S. Romanova have proven Theorem 4.1 in the special case of finite dimensional subspaces of $L^1(\mu; R)$. See *Approximation properties of finite dimensional subspaces in* L_1. Mat. Sbornik, **89** (131) (1972), 3-15 (translated in Math USSR Sbornik, **18** (1972), 1-14).

REFERENCES

1. A. L. Garkavi, *On Chebyshev and almost Chebyshev subspaces*, Izv. Akad. Nauk SSSR Ser. Mat., **28** (1964), 799-818. Translated in Amer. Math. Soc. Transl., (2) **96** (1970), 153-175.

2. ———, *Almost Chebyshev systems of continuous functions*, Izv. Vyssh. Uchebn. Zaved. Mat., **45** (1965), 36-44. Translated in Amer. Math. Soc. Transl., (2) **96** (1970), 177-187.

3. A. Ionescu Tulcea and C. Ionescu Tulces, *Topics in the Theory of Lifting*, Springer-Verlag New York Inc., New York, 1969.

4. J. F. C. Kingman and A. P. Robertson, *On a theorem of Lyapunov*, J. London Math. Soc., **43** (1968), 347-351.

5. B. R. Kripke and T. J. Rivlin, *Approximation in the metric* $L^1(X, \mu)$, Trans. Amer. Math. Soc., **119** (1965), 101-122.

6. Aldo J. Lazar, *Spaces of affine continuous functions*, Trans. Amer. Math. Soc., **134** (1968), 503-525.

7. Aldo J. Lazar, D. E. Wulbert, and P. D. Morris, *Continuous selections for metric projections*, J. of Funct. Anal., **3** (1969), 193-216.

8. R. R. Phelps, *Uniqueness of Hahn-Banach extensions and unique best approximation*, Trans. Amer. Math. Soc., **95** (1960), 238-255.

9. ———, *Chebyshev subspaces of finite dimension in* L_1, Proc. Amer. Math. Soc., **17** (1966), 646-652.

10. R. Tyrrell Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.

11. Edward Rozema, *Almost Chebyshev subspaces, lower semi-continuity, and unique Hahn-Banach extensions*, accepted by Proc. Amer. Math. Soc.

12. Ivan Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag New York Inc., New York, 1970.

13. S. B. Stechkin, *Approximative properties of sets in normed linear spaces*, Rev. Roum. Math. Pures et Appl., **8** (1963), 5-18 (Russian).

14. D. E. Wulbert, *Convergence of operators and Korovkin's theorem*, J. of Approx. Th., **1** (1968), 381-390.

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