LINEAR PINCHERLE SEQUENCES

ED DUBINSKY

Dragilev's theory of regular bases in nuclear Fréchet spaces is applied to obtain necessary and sufficient conditions in terms of the zeros for a linear Pincherle sequence to be a basis for the space $\mathscr{F}_R$ of functions analytic on the interior of the disk of radius $R \leq \infty$. It is shown that a linear Pincherle basis is always proper. All possible phenomena for the basis radius of a linear Pincherle sequence are exhibited. In this connection it is shown that for any finite $R_0 > 0$ there is a sequence of analytic functions which is a basis for $\mathscr{F}_R$ if and only if $R \leq R_0$ or $R = \infty$.

A classical problem of fundamental interest is to study the representability of analytic functions as infinite series in a given sequence of functions. The purpose of this note is to point out one case in which recent developments in the theory of nuclear Fréchet spaces are directly applicable and lead to results which have not been obtained by other methods.

In order to describe the results it is necessary to explain certain concepts. A sequence (α_n) in a Fréchet space E is a *basis* if each $x \in E$ has a representation $x = \sum c_n \alpha_n$ where (c_n) is a sequence of scalars uniquely determined by x and the infinite series converges in the topology of E. Two bases, (α_n) and (β_n) are *equivalent* if $\sum c_n \alpha_n$ converges in E if and only if $\sum c_n \beta_n$ converges in E.

Let $0 < R \leq \infty$. We shall be interested in the nuclear Fréchet space $\mathscr{F}_R$ consisting of all functions analytic on the open disk of radius R, equipped with the topology of uniform convergence on compact sets. As is well known, the sequence (z^n) is a basis for $\mathscr{F}_R$. An arbitrary basis, (α_n) for $\mathscr{F}_R$ will be called *proper* if it is equivalent to (z^n).

The general question then can be viewed as the problem of determining when a sequence (α_n) is a basis for $\mathscr{F}_R$ and when it is a proper basis. Of course, one can only hope for conclusive results in special cases. M. Arsove, in a series of papers over the past 15 years (see [1] for a bibliography) has considered *Pincherle sequences* in which α_n has the form $\alpha_n(z) = z^n \psi_n(z)$, $n = 0, 1, 2, \cdots$ where each ψ_n is a function in $\mathscr{F}_R$ and $\psi_n(0) = 1$. Most recently, in [1], Arsove studied *linear Pincherle sequences* in which each ψ_n is a nonconstant linear function with its zero at z_n, so we can write,

$$\alpha_n = z^n \left(1 - \frac{z}{z_n} \right), \qquad n = 0, 1, 2, \cdots .$$

Our results can then be described as a complete determination, in terms of the sequence of zeros (z_n), of when a linear Pincherle sequence is a basis and when it is proper. Also we consider the question of how the result depends on the value of R. It is hoped that both our results and our method will be useful in studying general Pincherle sequences.

Our method is to apply a deep result of M. M. Dragilev ([3], Theorem 5), which we use in the formulation given by C. Bessaga [2]. In order to describe the method, we must introduce some further concepts. Let (x_n) be a basis for a nuclear Fréchet space E. We say that the basis is *regular* if there is a fundamental sequence of seminorms $(\|\cdot\|_k)$ for E such that

$$
\text{(R)} \qquad \frac{\|x_n\|_k}{\|x_n\|_{k+1}} \geqq \frac{\|x_{n+1}\|_k}{\|x_{n+1}\|_{k+1}} \qquad \text{for all } n \text{ and } k .
$$

We say that the basis is of type D_1 or D_2 if we have $(\|\cdot\|_k)$ such that (R) is satisfied and

$$
\text{(D}_1\text{)} \qquad \|x_n\|_1 = 1 \text{ for all } n \text{ and } \forall k \exists j \ni \sup_n \frac{\|x_n\|_k^2}{\|x_n\|_j} \leqq 1
$$

or

$$
\text{(D}_2\text{)} \qquad \lim_k \|x_n\|_k = 1 \text{ for all } n \text{ and } \forall k \exists j \ni \sup_n \frac{\|x_n\|_k}{\|x_n\|_j^2} \leqq 1 .
$$

Then the main result which we will use in this paper is due to Dragilev and has the following formulation:

LEMMA. *If (x_n) and (y_n) are regular basis in a nuclear Fréchet space and they are both of type D_1 or of type D_2 then they are equivalent.*

The proof of the lemma is immediate from [2], 1.10 formulas (5) and (6).

Now we explain some of the notation which will be used. For the topology in the space $\mathscr{F}_R$, we can take the seminorms $\|\cdot\|_\rho$, $0 < \rho < R$ given by

$$
\|f\|_\rho = \sum_{n=0}^\infty |c_n| \rho^n , \qquad f = \sum_{n=0}^\infty c_n z^n \in \mathscr{F}_R .
$$

We set, for $0 < R \leqq \infty$,

$$
\Lambda_R = \left\{ (\xi_n) \colon \exists f \in \mathscr{F}_R \ni f(z) = \sum_{n=0}^\infty \xi_n z^n, \ |z| < R \right\} .
$$

As is well known, $\xi \in \Lambda_R$ iff $\varlimsup |\xi_n|^{1/n} \leqq 1/R$. The map $J \colon \Lambda_R \to \mathscr{F}_R$

given by

$$J((\xi_n)) = \sum_{n=0}^{\infty} \xi_n z^n$$

is an algebraic isomorphism and we write $e^n = J^{-1}(z^n)$. We can use J to define a topology on Λ_R (carried over from $\mathscr{F}_R$). It is not hard to show that the topology is defined by the seminorms $\| \cdot \|_\rho, 0 < \rho < R$ where

$$\| (\xi_n) \|_\rho = \sup_n |\xi_n| \rho^n , \qquad (\xi_n) \in \Lambda_R .$$

Given a linear Pincherle sequence (α_n) with zeros (z_n) and $0 < R \leqq \infty$, we will use the labels $p = p(R)$ and $r_n = r_n(R)$ given by

$$p = \begin{cases} 1 & \text{if } R = \infty \\ R & \text{if } R < \infty \end{cases} , \quad r_n = \frac{|z_n|}{(p + |z_n|)p^n} , \quad n = 0, 1, 2, \cdots .$$

The symbol $z_m \cdots z_{n-1}$ is clear if $n > m$ and if $n = m$ we will take it to be 1.

An *automorphism* is a linear topological isomorphism of a space onto itself. We recall that two bases (x_n) and (y_n) in a Fréchet space E are equivalent iff there is an automorphism T of E such $Tx_n = y_n$. Also, it is obvious that if (x_n) is a basis and (t_n) is a sequence of nonzero scalars, then $(t_n x_n)$ is a basis.

THEOREM 1. *A linear Pincherle sequence (α_n) with zeros (z_n) is a basis for $\mathscr{F}_R, 0 < R \leqq \infty$ iff*

$$(\text{C}_R) \qquad \forall \rho < R \,\exists \bar\rho < R \ni \sup_n \sup_{n \geqq m} \frac{(1 + |z_n|)\rho^n}{|z_m \cdots z_n| \bar\rho^m} < \infty .$$

Proof. Suppose that (α_n) is a basis for $\mathscr{F}_R$.

The first step is to show that the basis $(r_n \alpha_n)$ is of type D_1 or D_2 according as $R = \infty$ or $R < \infty$. Consider the seminorms on $\mathscr{F}_R$ and compute, for $\rho < \bar\rho$,

$$\frac{\| \alpha_n \|_\rho \| \alpha_{n+1} \|_{\bar\rho}}{\| \alpha_n \|_{\bar\rho} \| \alpha_{n+1} \|_\rho} = \frac{\left(\rho^n + \dfrac{1}{|z_n|}\rho^{n+1} \right)\left(\bar\rho^{n+1} + \dfrac{1}{|z_{n+1}|}\bar\rho^{n+2} \right)}{\left(\bar\rho^n + \dfrac{1}{|z_n|}\bar\rho^{n+1} \right)\left(\rho^{n+1} + \dfrac{1}{|z_n|}\rho^{n+2} \right)}$$

$$= \frac{\bar\rho |z_n||z_{n+1}| + \bar\rho^2 |z_n| + \rho\bar\rho |z_{n+1}| + \rho\bar\rho^2}{\rho |z_n||z_{n+1}| + \rho^2 |z_n| + \rho\bar\rho |z_{n+1}| + \rho^2\bar\rho} > 1$$

whence the basis (α_n) is regular, so the basis $(r_n \alpha_n)$ is also regular.

If $R = \infty$, we have

$$\| r_n \alpha_n \|_1 = \frac{|z_n|}{1 + |z_n|}\left(1 + \frac{1}{|z_n|} \right) = 1$$

and, for $1 \leqq \rho^2 \leqq \bar\rho$,

$$\frac{||\, r_n \alpha_n\,||_\rho^2}{||\, r_n \alpha_n\,||_{\bar\rho}^2} = \frac{|\,z_n\,|\left(\rho^n + \frac{1}{|\,z_n\,|}\rho^{n+1}\right)^2}{(1 + |\,z_n\,|)\left(\bar\rho^n + \frac{1}{|\,z_n\,|}\bar\rho^{n+1}\right)} \leqq \frac{|z_n|^2 + 2\rho\,|\,z_n\,| + \rho^2}{|z_n|^2 + (\bar\rho + 1)\,|\,z_n\,| + \bar\rho} \leqq 1 \,,$$

so the property (D₁) is satisfied.

If $R < \infty$, we fix n and have

$$\lim_{\rho\to R} ||\, r_n \alpha_n\,||_\rho = \lim_{\rho\to R} \frac{\rho^n(\rho + |\,z_n\,|)}{R^n(R + |\,z_n\,|)} = 1 \,.$$

Also, for $0 < \rho < \bar\rho^2/R$,

$$\lim_{n\to\infty} \frac{||\, r_n \alpha_n\,||_\rho}{||\, r_n \alpha_n\,|_{\bar\rho}^2} = \lim_{n\to\infty} \frac{(R\rho)^n(\rho + |\,z_n\,|)}{\bar\rho^{2n}(\bar\rho + |\,z_n\,|)} = 0 \,,$$

so the property (D₂) is satisfied.

Next we observe that it is a simple matter to check that in $\mathscr{F}_R$, the basis $((1/p^n)z^n)_n$ is also of type D₁ or D₂ according as R is infinite or finite. Hence it follows from the lemma that the basis $(r_n \alpha_n)$ is equivalent in $\mathscr{F}_R$ to the basis $((1/p^n)z^n)$. This in turn implies that $(r_n p^n \alpha_n)$ is a proper basis in $\mathscr{F}_R$.

Thus we have an automorphism $T \colon \Lambda_R \to \Lambda_R$ given by the relation, $T\xi = \eta$ where ξ, η are determined by the relation,

$$(1) \qquad\qquad \sum_{n=0}^{\infty} \xi_n z^n = \sum_{n=0}^{\infty} \eta_n r_n p^n \alpha_n \,.$$

Using formula (3.3) of [1], we obtain that

$$Te^m = \sum_{n=m}^{\infty} \frac{1}{r_n p^n z_m \cdots z_{n-1}}\, e^n = \sum_{n=m}^{\infty} \frac{p + |\,z_n\,|}{z_m \cdots z_{n-1}\,|\,z_n\,|}\, e^n \,.$$

It is not hard to check, using the seminorms in Λ_R, that the continuity of T is equivalent to the statement,

$$\forall \rho < R\ \exists\, \bar\rho < R \ni \sup_m \frac{||\, Te^m\,||_\rho}{|\,e^m\,||_{\bar\rho}} < \infty \,.$$

Using the definitions of these seminorms in the explicit representation for T, we see that the continuity of T is equivalent to,

$$\forall \rho < R\ \exists\, \bar\rho < R \ni \sup_m \sup_{n \geqq m} \frac{(p + |\,z_n\,|)\rho^n}{|\,z_m \cdots z_n\,|\,\bar\rho^m} < \infty \,.$$

When $R = \infty$, this is exactly (C_∞). For $R < \infty$ this is equivalent to (C_R) in view of the fact that

$$\min(1, p) \leqq \frac{p + |\,z_n\,|}{1 + |\,z_n\,|} \leqq \max(1, p) \,, \qquad 0 < p < \infty \,.$$

This completes the necessity portion of the proof.

Now suppose that (C_R) holds. We can then immediately assert the existence and continuity of the map $T: \varLambda_R \to \varLambda_R$ defined by the relation (1). We can use this relation to compute,

$$T^{-1}e^m = r_m p^m \Big(e^m - \frac{1}{z_m} e^{m+1}\Big) .$$

This map is in any case continuous because

$$\sup_m \frac{\| T^{-1}e^m \|_\rho}{\| e^m \|_\rho} = \sup_m \max \Big\{ \frac{|z_m|}{p + |z_m|}, \ \frac{\rho}{p + |z_m|} \Big\} < \infty .$$

Hence T is an automorphism of $\varLambda_R$ so $JT^{-1}J^{-1}$ is an automorphism of $\mathscr{F}_R$ and since $JT^{-1}J^{-1}(z^n) = r_n p^n \alpha_n$ it follows that $(r_n p^n \alpha_n)$ is a basis for $\mathscr{F}_R$ and since $r_n p^n \ne 0$, it follows that (α_n) is a basis for $\mathscr{F}_R$. This completes the proof of the theorem.

COROLLARY. *A necessary condition for a linear Pincherle sequence* (α_n) *with zeros* (z_n) *to be a basis for* $\mathscr{F}_R$ *is*

$$\varlimsup_m \Big(\frac{1 + |z_m|}{|z_m|} \Big)^{1/m} \quad \begin{cases} < \infty & \text{if } R = \infty \\ \leqq 1 & \text{if } R < \infty . \end{cases}$$

Proof. We apply (C_R) and, in the second sup we always take $n = m$. The given condition is then easily derived.

REMARK. By taking other special values of m and n in (C_R) we can easily obtain other necessary conditions. For example, if we take $m = 0$ in the first sup, we immediately obtain condition (1.4) of [1]. These conditions will not be sufficient. For instance, the condition of the corollary is always satisfied if z_m is constant. However, by Lemma 2.2 of [1], if this is so and $|z_n| < R$, then (α_n) will not be a basis. On the other hand, the condition of the corollary is strong enough to settle the question of when a linear Pincherle basis is proper.

THEOREM 2. *Every linear Pincherle basis is proper.*

Proof. In the proof of Theorem 1 it was shown that if a linear Pincherle sequence (α_n) is a basis for $\mathscr{F}_R$, then $(r_n p^n \alpha_n)$ is a proper basis in $\mathscr{F}_R$. Hence $\sum_n \xi_n \alpha_n$ converges iff

$$\sum_{n=0}^{\infty} \frac{\xi_n}{r_n p^n} z^n \quad \text{converges in } \mathscr{F}_R ,$$

which is equivalent, by the Cauchy-Hadamard formula to

$$\varlimsup_n \Big(\frac{|\xi_n| (p + |z_n|)}{|z_n|} \Big)^{1/n} \leqq \frac{1}{R} ,$$

and, in view of the corollary, this is equivalent to $\overline{\lim}_n |\xi_n|^{1/n} \leq 1/R$, i.e., that $\sum_n \xi_n z^n$ converges in $\mathscr{F}_R$.

REMARK. The above result shows that the sufficient condition given in Lemma 3.4 of [1] is also necessary. Alternatively, that lemma can be used, along with Theorem 1 to obtain Theorem 2. This is done by observing that condition (C_R) directly implies condition (3.11) of [1].

We turn now to the question of basis radius. A Pincherle sequence is said to have *basis radius* R_0 provided that it is a basis for $\mathscr{F}_R$ if and only if $0 < R \leq R_0$. This is slightly different, on the face of it, from the definition given by Arsove [1] who requires that the sequence be a basis for $\mathscr{F}_R$ whenever $R < R_0$ and fail to be a basis when $R > R_0$. The two statements are equivalent because it follows from general considerations that if a sequence is a basis for $\mathscr{F}_R$ whenever $R < R_0$, then it is a basis for $\mathscr{F}_{R_0}$.

Arsove [1] points out that a Pincherle sequence may fail to have a basis radius. The next theorem gives more or less complete information in the linear case.

THEOREM 3. *Let (α_n) be a linear Pincherle sequence. Then (α_n) has a basis radius if and only if either (C_∞) fails or (C_R) holds for all $R \leq \infty$. In either case we have,*

$$R_0 = \sup \{R: 0 < R < \infty \text{ and } (C_R) \text{ holds}\}.$$

Proof. If (C_R) holds for all $R \leq \infty$, then obviously $R_0 = \infty$ and the given representation is valid. Suppose that (C_∞) fails and define R_0 as in the statement of the theorem. Then we have an increasing sequence (R_j) of positive numbers which converges to R_0 and $\ni (C_{R_j})$ holds for each j.

Suppose now that $R < R' < \infty$ and $(C_{R'})$ holds. Then given $\rho < R$, we consider $\rho(R'/R) < R'$ so we have $\bar{\rho}$ from $(C_{R'})$. But then $\bar{\rho}(R/R') < R$ and the inequality in (C_R) holds for ρ, $\bar{\rho}(R/R')$. Thus (C_R) holds.

Thus we may conclude that for each j and each $R < R_{j'}$, (C_R) holds. This implies that (C_R) holds for all $R < R_0$.

Finally, suppose $f \in \mathscr{F}_{R_0}$. For each $R < R_0$, (α_n) is a basis for $\mathscr{F}_R$ and $f \in \mathscr{F}_R$. Hence we have a unique sequence, $(c_n^R)_n$ of scalars $\ni f = \sum c_n^R \alpha_n$ and the convergence is uniform on each closed disk of radius less than R. By the uniqueness, (c_n^R) is independent of R and this shows that (α_n) is a basis for $\mathscr{F}_{R_0}$. (This last argument is a specific proof of the equivalence of Arsove's definition of basis radius and the one given above.)

This completes the proof in one direction. The converse is obvious

so the theorem is proved.

In Arsove's example showing that a linear Pincherle sequence can fail to have a basis radius, the sequence is a basis for $\mathscr{F}_\infty$ but not a basis for any $\mathscr{F}_R$, $0 < R < \infty$. We now give an example showing that for any R_0 with $0 < R_0 < \infty$, there is a linear Pincherle sequence which is a basis for $\mathscr{F}_R$ if and only if $R = \infty$ or $0 < R \leq R_0$.

EXAMPLE. Let (z_n) be defined by

$$z_n = \begin{cases} j^{4^j} & \text{for } n = 2^j, \ j = 1, 2 \cdots \\ 1 & \text{otherwise} . \end{cases}$$

Then the linear Pincherle sequence (α_n) with zeros (z_n) is a basis for $\mathscr{F}_R$ if and only if $R = \infty$ or $R \leq 1$.

Proof. First we consider $R = \infty$. Fix $\rho < \infty$ and choose $\bar{\rho} \geq \rho$. We may assume that $\rho > 1$. Then

$$\sup_{m \geq 2} \frac{1}{\bar{\rho}^m} \sup_{n \geq m} \frac{1 + |z_n|}{|z_m \cdots z_n|} \rho^n$$

$$\leq 2 \sup_{j \geq 0} \sup_{2^j < m \leq 2^{j+1}} \frac{1}{\bar{\rho}^m} \sup_{n \geq m} \frac{\rho^n}{|z_m \cdots z_{n-1}|}$$

$$= 2 \sup_{j \geq 0} \max \left\{ \sup_{2^j < m < 2^{j+1}} \frac{1}{\bar{\rho}^m} \sup_{n \geq m} \frac{\rho^n}{|z_m \cdots z_{n-1}|} , \right.$$
$$\left. \frac{1}{\bar{\rho}^{2^{j+1}}} \sup_{n \geq 2^{j+1}} \frac{\rho^n}{|z_{2^{j+1}} \cdots z_{n-1}|} \right\}$$

$$\leq 2 \sup_{j \geq 0} \max \left\{ \sup_{2^j < m < 2^{j+1}} \frac{1}{\bar{\rho}^m} \sup_{n \geq m} \frac{\rho^n}{|z_m \cdots z_{n-1}|} , \right.$$
$$\left. \frac{\rho}{\bar{\rho}^{2^{j+1}-1}} \sup_{n \geq 2^{j+1}-1} \frac{\rho^n}{|z_{2^{j+1}-1} \cdots z_{n-1}|} \right\}$$

$$\leq 2\rho \sup_{j \geq 0} \sup_{2^j < m < 2^{j+1}} \frac{1}{\bar{\rho}^m} \sup_{n \geq m} \frac{\rho^n}{|z_m \cdots z_{n-1}|}$$

$$= 2\rho \sup_{j \geq 0} \sup_{2^j < m < 2^{j+1}} \frac{1}{\bar{\rho}^m} \max \left\{ \sup_{m \leq n \leq 2^{j+1}} \rho^n , \right.$$
$$\left. \frac{\rho^{2^{j+1}+1}}{(j+1)^{4^{j+1}}}, \sup_{n > 2^{j+1}+1} \frac{\rho^n}{|z_m \cdots z_{n-1}|} \right\} .$$

Now, if $j \geq \rho - 1$, the expression after the first sup is dominated by

$$\sup_{2^j < m < 2^{j+1}} \frac{1}{\bar{\rho}^m} \max \left\{ \rho^{2^{j+1}}, \sup_{i > j} \frac{\rho^{4^{i+1}}}{i^{4^i}} \right\} \leq \sup_{2^j < m < 2^{j+1}} \frac{\rho^{2^{j+1}}}{\bar{\rho}^m} \leq 1 ,$$

for j sufficiently large. This shows that the quantity in (C_∞) is dominated by 2ρ for m sufficiently large and so it remains to show that for each m,

$$\sup_{n \geq m} \frac{\rho^n}{|z_m \cdots z_{n-1}|} < \infty .$$

But, given m, for n sufficiently large, we can take $2^j < n \leq 2^{j+1}$ and obtain,

$$\frac{\rho^n}{|z_m \cdots z_{n-1}|} \leq \frac{\rho^{2^{j+1}}}{j^{4j}} < 1 \quad \text{for } j \text{ sufficiently large} .$$

Hence by Theorem 1, (α_n) is a basis for $\mathscr{F}_\infty$.

For $R \leq 1$, we fix $\rho < R$ and choose $\bar{\rho}$ with $\rho \leq \bar{\rho} < R$ and we have

$$\sup_m \frac{1}{\bar{\rho}^m} \sup_{n \geq m} \frac{(1 + |z_n|)\rho^n}{|z_m \cdots z_{n-1}|} \leq 2 \sup_m \frac{1}{\bar{\rho}^m} \sup_{n \geq m} \frac{\rho^n}{|z_m \cdots z_{n-1}|}$$

$$= 2 \sup_m \frac{1}{\bar{\rho}^m} \sup_{n \geq m} \rho^n = 2 \sup_m \left(\frac{\rho}{\bar{\rho}}\right)^m = 2 ,$$

so (α_n) is a basis for $\mathscr{F}_R$, $R \leq 1$.

Finally, if $R > 1$ we can choose $\rho < R$, $\rho > R^{2/3}$. Then for any $\bar{\rho} < R$, we choose $m = 2^{j-1} + 1$, $n = 2^j$ to obtain,

$$\sup_m \sup_{n \geq m} \frac{1 + |z_n|}{|z_m \cdots z_n|} \rho^n \frac{1}{\bar{\rho}^m} \geq \sup_j \frac{1 + |z_{2^j}|}{|z_{2^{j-1}+1} \cdots z_{2^j}|} R^{2^j - 2^{j-1} - 1} \left(\frac{\rho}{R}\right)^{2^j}$$

$$\geq \sup_j \left[\left(\frac{\rho}{R}\right)^{2^j/2^{j-1}-1} R\right]^{2^{j-1}-1} \geq \sup_j \left[\left(\frac{\rho}{R}\right)^3 R\right]^{2^{j-1}-1} = \infty ,$$

so by Theorem 1, (α_n) is not a basis for $\mathscr{F}_R$.

This completes the proof of the example which is a special case of the statement we made before the example. For the general case, we simply observe that from Theorem 1 it follows immediately that if (α_n) is a linear Pincherle sequence which is a basis for $\mathscr{F}_R$, $0 < R < \infty$ with zeros, (z_n), then for any $a > 0$, the linear Pincherle sequence obtained by taking, as zeros, the sequence (z_n/a) is a basis for $\mathscr{F}_{aR}$.

Finally we observe that in the above example, if $n = 2^j$, then

$$|z_0 \cdots z_{n-1}|^{1/n} \geq j^{2^{j-1}}$$

so that for any positive number p we have

$$\varlimsup_n \frac{|z_0 \cdots z_{n-1}|^{1/n}}{n^p} = \infty .$$

This shows that the sufficient condition of Arsove ([1], Theorem 4.3) is not necessary for a linear Pincherle sequence to be a basis for $\mathscr{F}_\infty$.

Remarks added in proof.

1. It has recently been shown by L. Crone and W. B. Robinson
(Studia Math. 52 (1974) 203–207) that any two regular bases in a
nuclear Fréchet space are equivalent and thus we can drop the
requirement in the above Lemma that the two bases are both of
type D_1 or of type D_2. This considerably simplifies the proof of
Theorem 1.

2. Using a recent result of Nguyen Thanh Van (Ann. Inst.
Fourier, 22 : 2, 1972, pp. 187–190) it is possible to prove Theorem 2
without assuming that the basis is linear. There are many additional
applications of this fact including an improved version of Theorem 3.

REFERENCES

1. M. G. Arsove, *On the behavior of Pincherle basis functions*, Pacific J. Math., **44**
1, (1973), 13–31.
2. C. Bessaga, *Some remarks on Dragilev's theorem*, Studia Math., **31** (1968), 307–318.
3. M. M. Dragilev, *On regular bases in nuclear spaces*, Amer. Math. Soc. Transl.,
(2), **93** (1970), 61–82.

CLARKSON COLLEGE OF TECHNOLOGY