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REGULARITY AND QUOTIENTS IN RINGS WITH INVOLUTION

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Let R be a ring with involution. There exists a unique maximal nilpotent *-ideal N of R such that R/N with the induced involution satisfies the property that any regular element of $\bar{S}$, the subring generated by the symmetric elements of R/N is regular in R/N. When $N = 0$ we say that R satisfies the regularity condition. Assuming this condition, $(Q(R)) = R$ if and only if $Q(\bar{S}) = \bar{S}$. The existence of $Q(\bar{S})$ implies the existence of $Q(R)$, and the converse is shown in some special cases. If either S is commutative or R is a semi-prime Goldie ring, then the relation between $Q(R)$ and $Q(\bar{S})$ is explicitly described.

Recent work on rings with involution [8] has investigated the question of when a symmetric element which is regular with respect to all other symmetrics, is regular in the ring. Our first goal here is to examine the related situation concerning the subring generated by the symmetric elements. That is, when must an element, regular in this subring, be regular in the whole ring? We show that semi-primeness is a sufficient condition on the ring, and define a nilpotent ideal N of R so that R/N possesses this regularity property. In a slightly different direction, it has been shown [7] that for a semi-prime ring, the subring generated by the symmetric elements is a Goldie ring exactly when the whole ring is a Goldie ring. This result implies that each of these rings has a semi-simple Artinian classical ring of quotients when the other does. We determine here the relation between these quotient rings, and further, investigate more general conditions under which the existence of a quotient ring for one of these rings implies the existence of a quotient ring for the other. As one might expect, this latter problem is related to the one above concerning the regularity of elements in the subring generated by the symmetric elements.

Henceforth, R will denote a ring with involution, *; $S = S(R) = \{x \in R \mid x^* = x\}$, the set of symmetric elements of R; and $\bar{S} = \bar{S}(R)$, the subring generated by S. An important fact about S is that it is a Lie ideal of R (see [4] or [6]). Thus $xt - tx \in \bar{S}$ for every $x \in R$ and $t \in \bar{S}$. An ideal I of R is called a *-ideal if $I^* = I$. Before our first result, we recall the following

DEFINITION. If A is a nonempty subset of R, then $\ell(A) = \ell_R(A) =$

$\{x \in R \mid xa = 0 \text{ for all } a \in A\}$ and $r(A) = r_R(A) = \{x \in R \mid ax = 0 \text{ for all } a \in A\}$.

For any subring T of R and $t \in T$, we say that t is *regular* in T if $\ell_T(t) = r_T(t) = 0$. Clearly, $\ell_T(t) = \ell_R(t) \cap T$ and $r_T(t) = r_R(t) \cap T$.

LEMMA 1. *Let $s \in S$ be regular in $\bar{S}$. Then $r_R(s) = \ell_R(s)$ is a *-ideal of R which is nil of index 2, is 2-torsion free, and has cube zero.*

Proof. Let $x \in R$ with $sx = 0$. Since $\bar{S}$ is a Lie ideal of R, $xs = xs - sx \in \bar{S}$. But s is regular in $\bar{S}$ and $s(xs) = 0$, so $xs = 0$. Consequently, $r(s) \subset \ell(s)$, and similarly $\ell(s) \subset r(s)$. Note that $0 = sx$ implies that $0 = x^*s$, so that $r(s)^* \subset \ell(s) = r(s)$. Since $r(s)$ is a *-ideal of R, and $r(s) \cap \bar{S} = 0$ by hypothesis, it follows that $x + x^* = xx^* = 0$ for all $x \in r(s)$. Thus $x^2 = x(x + x^*) = 0$. Lastly, should $2x = 0$ for some $x \in r(s)$, we would have $x = x^* \in \bar{S} \cap r(s) = 0$. Therefore, $r(s)$ is 2-torsion free, so has cube zero.

Lemma 1 tells us that if a symmetric element is regular in $\bar{S}$, but not in R, then we can produce a nonzero nilpotent ideal of R. In fact, we need only consider symmetric elements since if $t \in \bar{S}$ is regular in $\bar{S}$, then tt^* is symmetric, is still regular in $\bar{S}$, and $\ell_R(tt^*) \supset \ell_R(t)$.

DEFINITION. R satisfies the *regularity condition* if each regular element of $\bar{S}$ is regular in R.

We next use Lemma 1 to define a nilpotent ideal of R which is, in a sense, a measure of how close R is to satisfying the regularity condition. A suprising fact about this ideal is that it will be the set theoretic union of the nilpotent ideals arising in Lemma 1.

DEFINITION. Let $V = \{s \in \bar{S} \mid s \in S \text{ and } s \text{ is regular in } \bar{S}\}$. Set $W = W(R) = \Sigma r(s)$ for all $s \in V$.

THEOREM 2. *The ideal W satisfies the following properties;*

(i) *W contains $r_R(t)$ and $\ell_R(t)$ for any t, regular in $\bar{S}$*
(ii) *W is a 2-torsion free *-ideal*
(iii) *W is nil of index 2*
(iv) $W^3 = 0$
(v) *$x \in W$ implies $x \in r(v)$ for some $v \in V$*
(vi) $W \cap \bar{S} = 0$.

Proof. That W is an ideal of R follows from its definition together with Lemma 1. Condition (ii) is also immediate from Lemma 1, and (iv) will hold once (iii) is shown. To see that (i) holds, we note again

that for t regular in $\bar{S}$, t^*t and tt^* are regular in $\bar{S}$, so are in V. Since $r(t) \subset r(t^*t)$, and $\ell(t) \subset \ell(tt^*) = r(tt^*)$, we have $r(t) \subset W$ and $\ell(t) \subset W$.

Next, choose s_1 and s_2 in V. It is clear that $s_1s_2s_1 \in V$ and that $s_1s_2s_1r(s_1) = 0$. Also, $s_1s_2s_1r(s_2) = 0$ since $r(s_2)$ is an ideal of R by Lemma 1. Therefore $r(s_1) + r(s_2) \subset r(s_1s_2s_1)$. An extension of this argument shows that for $s_1, s_2, \cdots, s_n \in V$, there is $v \in V$ with $r(s_1) + r(s_2) + \cdots + r(s_n) \subset r(v)$. Since $r(v)$ is nil of index 2, we have established (iii). Also, (v) is now immediate. Lastly, should $y \in W \cap \bar{S}$, then by (v) $y \in r(v)$ for some $v \in V$. But $r(v) \cap \bar{S} = 0$, so $y = 0$, (vi) holds, and the proof is complete.

COROLLARY 3. *If R is semi-prime then R satisfies the regularity condition.*

Proof. For any t, regular in $\bar{S}$, $\ell(t)$ and $r(t)$ are in W by Theorem 2-(i), and W is nilpotent by (iv). Since R is semi-prime, $W = 0$, so t is regular in R.

Of course, R may satisfy the regularity condition without being semi-prime. This is trivally true if $R = \bar{S}$ or if $\bar{S}$ has no regular elements. We present some additional and easy examples.

EXAMPLE 1. Let F be a field with char $F \neq 2$, and set $R = F[x, y]/(y^2)$. Define an involution on R by $x^* = x$ and $y^* = -y$. For $t \in R$, $t = f(x) + g(x)y$, and $t^* = f(x) - g(x)y$, so $t = t^*$ exactly when $g(x) = 0$. Since t is regular in R if and only if $f(x) \neq 0$, every nonzero symmetric element is regular in R. Clearly, R is not semi-prime since $(y)^2 = 0$.

EXAMPLE 2. A noncommutative example can be obtained from Example 1 by considering the subring $R_1 \subset M_2(R)$ of upper triangular matrices. Set $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c^* & -b^* \\ & a^* \end{pmatrix}$. One can easily check that for $A \in R_1$, A is regular exactly when $\det A = f(x) + g(x)y$ and $f(x) \neq 0$. Also, each symmetric element of R_1 is either regular in R_1 or is nilpotent. Hence $W(R_1) = 0$, so R_1 satisfies the regularity condition, although it is not a semi-prime ring.

The ideal $W(R)$ possesses a radical property with respect to the regularity condition in that $W(R/W) = 0$. Note that since W is a *-ideal of R, there is a natural involution induced on R/W via $(r + W)^* = r^* + W$.

THEOREM 4. *The ring R/W, with induced involution, satisfies the regularity condition.*

Proof. In view of Theorem 2, it is enough to consider $y + W \in S(R/W)$, which is regular in $\bar{S}(R/W)$. But if $y + W$ is regular in $\bar{S}(R/W)$, so is $y^2 + W$. Furthermore, if $y^2 + W$ is not regular in R/W, then neither is $y + W$. Since $y - y^* \in W$, we have $y^2 + W = yy^* + W$, and so, it suffices to consider $y + W \in S(R/W)$, regular in $\bar{S}(R/W)$, and with $y \in S(R)$.

Suppose that y is not regular in $\bar{S}$. Then $y(r(y) \cap \bar{S}) = 0$. But $\bar{S}(R) + W \subset \bar{S}(R/W)$ and $y + W$ is not regular in $\bar{S}(R/W)$, so $r(y) \cap \bar{S} \subset W$. By Theorem 2-(vi), $\bar{S} \cap W = 0$, implying that $r(y) \cap \bar{S} = 0$, a contradiction. Hence y must be regular in $\bar{S}$.

If $y + W$ were not regular in R/W, then there exists $t \notin W$ with $yt \in W$. Using Theorem 2-(v), we can find $v \in V$ with $vyt = 0$. Since vy is in $\bar{S}$ and is regular, we must have $t \in W$, again by Theorem 2. This contradiction shows that $y + W$ is regular in R/W, establishing the Theorem.

One can obtain a rather surprising description of W in the event that R satisfies the ascending chain condition on two-sided ideals which are right annihilators — so called annihilator ideals.

THEOREM 5. *If R satisfies the ascending chain condition on annihilator ideals, then $W = r(v)$ for some $v \in V$.*

Proof. Since $r(t)$ is an annihilator ideal for each $t \in V$, we may choose $t \in V$ so that $r(t)$ is maximal in $\{r(v) \mid v \in V\}$. As in the proof of Theorem 2, $r(t) + r(y) \subset r(yty)$ for any $y \in V$. Thus $r(t) \subset r(yty)$, and the maximality of $r(t)$ forces $r(t) = r(yty)$. Consequently, $r(y) \subset r(t)$ for all $y \in V$, and so, $W \subset r(t) \subset W$.

Observe that if R satisfies the ascending chain condition on right or left annihilators, or on annihilator ideals, then since W is an annihilator ideal by Theorem 5, R/W satisfies the same chain condition as R, using Lemma 3 of [5].

We turn our attention next to the consideration of quotient rings. Should a ring A possess a total (right) quotient ring, in the sense of Ore, we shall denote it by $Q(A)$. Recall that $Q(A)$ exists if and only if A satisfies the right Ore condition; namely, for $r, c \in A$ with $r \neq 0$ and c regular, there exist $r', c' \in A$ with c' regular and $rc' = cr'$. (See Chapter 4 of [4].)

Our first result on quotients deals with the special case when $Q(\bar{S}) = \bar{S}$, or equivalently, when each regular element of $\bar{S}$ is invertible.

THEOREM 6. *If $Q(\bar{S}) = \bar{S}$, then $Q(R) = R$. Furthermore, if $\bar{S}$ has regular elements, then*;

(i) $R = A \oplus B$ *for* A *and* B **-ideals of* R;

(ii) B *is 2-torsion free, nil of index* 2, *and* $b^* = -b$ *for each* $b \in B$;

(iii) A *has identity,* $\bar{S} \subset A$, *and* $Q(A) = A$;

(iv) A *satisfies the regularity condition.*

Proof. For c regular in R, cc^* is regular in both R and $\bar{S}$, so if $\bar{S}$ has no regular elements neither does R, and $Q(R) = R$. Assuming that $\bar{S}$ has regular elements, let e be the identity of $\bar{S}$. As mentioned above, $\bar{S}$ is a Lie ideal of R, so for $r \in R$, $er - re \in \bar{S}$. Consequently, $er - re = (er - re)e = ere - re$, and $er - re = e(er - re) = er - ere$. It follows that $er = re$, and so, e is a central idempotent of R. Also, $e^* = e$ since both are identity elements for $\bar{S}$. Hence, we may formally decompose R as the direct sum of the *-ideals eR and $(1 - e)R$, which proves (i).

Now $\bar{S} = e\bar{S} \subset eR$. Thus for $b \in (1 - e)R$, bb^* and $b + b^*$ belong to $(1 - e)R \cap \bar{S} = 0$. We conclude that $b^2 = b(b + b^*) = 0$, and since $2b = 0$ would imply that $b = b^* \epsilon (1 - e)R \cap \bar{S} = 0$, we also conclude that $(1 - e)R$ is 2-torsion free. This establishes (ii).

To prove (iii) we begin with the facts that e is the identity of $A = eR$, and that $\bar{S} \subset eR$. If x is regular in A, so is xx^*. But $Q(\bar{S}) = \bar{S}$ implies that $(xx^*)y = x(x^*y) = e$, for some $y \in \bar{S}$. Hence x is a unit in A and $Q(A) = A$. Condition (iv) now follows since regular elements of $\bar{S}$ are invertible in $\bar{S}$, and since $\bar{S}$ and A have the same identity.

In general, a converse to Theorem 6 is hopeless, for the fact that $Q(R) = R$ says nothing about $\bar{S}$. One may consider any ring A with $\bar{S}(A) = A$ and let $R = A \oplus N$, where N is a 2-torsion free ring with trivial multiplication. Then with the involution $(a, n)^* = (a^*, -n)$, $\bar{S}(R) = A$ and $Q(R) = R$, since R has no regular elements. Even if R has regular elements and $Q(R) = R$, we cannot conclude that $Q(\bar{S}) = \bar{S}$, or even that $Q(\bar{S})$ exists, as the next example shows.

EXAMPLE 3. Let F be a field with char $F \neq 2$, and $F\{\{x, y\}\}[z]$ the polynomial ring over z with coefficients in the ring of formal power series in the noncommuting indeterminates x and y. Denote by I, the ideal generated by z^2, zx, and zy. Set $R = F\{\{x, y\}\}[z]/I$, and let x, y and z denote their own images in R. Define an involution on R via $z^* = -z$, $x^* = x$, and $y^* = y$. For $r \in R$, if we write $r = a + \Sigma_{i=1}^{\infty} f_i(x, y) + bz$, where $a, b \in F$ and $\{f_i(x, y)\}$ are homogeneous polynomials of degree i, then $r^* = a + \Sigma f_i(x, y)^* - bz$. Note first that r is regular exactly when $a \neq 0$, in which case $r = (a + bz)(1 + \Sigma a^{-1} f_i(x, y))$ is invertible. Hence $Q(R) = R$. Clearly, each element in S may be written as $a + \Sigma s_i(x, y)$, for symmetric, homogeneous polynomials $s_i(x, y)$. But x and $y \in S$, and $\bar{S} \subset F\{\{x, y\}\}$. It follows

that $x\bar{S} \cap y\bar{S} = 0$, since $xF\{\{x, y\}\} \cap yF\{\{x, y\}\} = 0$. Consequently, the domain $\bar{S}$ does not satisfy the right Ore condition, so $\bar{S}$ cannot have a right quotient ring.

We note that the regularity condition fails to hold in Example 3. In fact, the converse to Theorem 6 is valid under this assumption on R.

THEOREM 7. *If R satisfies the regularity condition, and if $Q(R) = R$, then $Q(\bar{S}) = \bar{S}$.*

Proof. Once again, if $\bar{S}$ has no regular elements, there is nothing to prove. Suppose that $x \in S$ is regular in $\bar{S}$. Then x is regular in R by the regularity condition, so $x^{-1} \in R$. Now $(xx^{-1}) = 1 = 1^* = (x^{-1})^*x$, so $x^{-1} \in \bar{S}$. Let y be any regular element of $\bar{S}$. Since yy^* is also regular, with symmetric inverse t, we have $1 = (yy^*)t = y(y^*t)$, and $y^*t \in \bar{S}$, for $y \in \bar{S}$ implies $y^* \in \bar{S}$. Thus $Q(\bar{S}) = \bar{S}$, proving the theorem.

Having examined the situatation when either R or $\bar{S}$ is its own ring of quotients, we turn to the problem of the existence of quotient rings. Under the assumption of the regularity condition, one implication is straightforward.

THEOREM 8. *If R satisfies the regularity condition, and if $Q(S)$ exists, then $Q(R)$ exists.*

Proof. If R has no regular elements, there is nothing to prove, so choose $a, c \in R$ with $a \neq 0$ and c regular. Since c is regular in R, cc^* is regular in $\bar{S}$. Should $a(cc^*) = (cc^*)a = c(c^*a)$, then take $c' = cc^*$ and $a' = c^*a$ to obtain $ac' = ca' \neq 0$ with c' regular. If $acc^* - cc^*a \neq 0$, then this element is in $\bar{S}$, since $cc^* \in \bar{S}$ and $\bar{S}$ is a Lie ideal of R. By assumption, $Q(\bar{S})$ exists, so there are $w, y \in \bar{S}$ with w regular and $(acc^* - cc^*a)w = cc^*y \neq 0$. Re-arranging terms, we have $acc^*w = cc^*(y + aw)$. Let $c' = cc^*w$ and $a' = c^*(y + aw)$. Now c' is regular in R by the regularity condition, so $ac' \neq 0$. Therefore, R satisfies the right Ore condition, and so, $Q(R)$ exists.

Given that R satisfies the regularity conditon, it would be surprising if $Q(R)$ could exist without implying the existence of $Q(\bar{S})$. Although we have not been able to show that this holds in general it is true in some special cases. Of course, when $\bar{S}$ is commutative then $Q(\bar{S})$ always exists. If $\bar{S}$ is not commutative, then since it is a Lie ideal of R, $\bar{S}$ contains the ideal of R generated by all $xy - yx$ for $x, y \in \bar{S}$. This fact is well-known, and follows from the proof of Lemma 1.3 of [4]. When this ideal contains a regular element of R, we have the following result

THEOREM 9. *Let R satisfy the regularity condition, and suppose that I is an ideal of R contained in $\bar{S}$ such that I contains a regular element of R. Then $Q(S)$ exists if and only if $Q(R)$ exists.*

Proof. By Theorem 8, it is enough to prove that $Q(\bar{S})$ exists when $Q(R)$ exists. Let $a, t \in \bar{S}$ with $a \neq 0$ and t regular. Since t is regular in R, there exist $a', t' \in R$ with t' regular, and $at' = ta'$. If $w \in I$ is regular in R, then $at'w = ta'w$. Since $I \subset \bar{S}, t'w$ and $a'w$ are in $\bar{S}$. Also, $t'w$ is regular, so the right Ore condition holds in $\bar{S}$, and $Q(\bar{S})$ exists.

COROLLARY 10. *If R satisfies the regularity condition and if $\bar{S}$ is a domain, then $Q(R)$ exists if and only if $Q(\bar{S})$ exists.*

Another situation in which we can guarantee the existence of $Q(\bar{S})$, given that $Q(R)$ exists, is when essential ideals contain regular elements. Our next result is concerned with the existence of certain kinds of essential ideals in $\bar{S}$. First, we make the

DEFINITION. A right ideal T of R is called essential if $T \cap B \neq 0$ for each nonzero right ideal B of R.

LEMMA 11. *Let R be semi-prime and suppose that $d \in \bar{S}$ is regular. If dR is an essential right ideal of R, then $d\bar{S}$ is an essential right ideal of $\bar{S}$.*

Proof. Should $\bar{S}$ be commutative, the proof is trivial since $dA \subset A \cap d\bar{S}$ for any ideal A of $\bar{S}$. If $\bar{S}$ is not commutative, then as we observed above, $\bar{S}$ is a Lie ideal of R containing the nonzero *-ideal I of R, generated by all $xy - yx$ for $x, y \in \bar{S}$.

Let $A \neq 0$ be a right ideal of $\bar{S}$ and assume that $AI \neq 0$. Then AI is a right ideal of R, and so, $dR \cap AI \neq 0$. If $dx \in AI \subset \bar{S}$, then since $d \in \bar{S}$, we have $dx - xd \in \bar{S}$, which implies that $xd \in \bar{S}$. Thus $dxd \in d\bar{S} \cap AI$, and $dxd \neq 0$ since d is regular in R by Corollary 3. Consequently, $d\bar{S} \cap A \neq 0$.

Next assume that $AI = 0$. Then A is contained in Ann I, the annihilator of I, an ideal of R. Hence $\bar{S} \cap \text{Ann } I$ is a Lie ideal of R containing A. But $\bar{S} \cap \text{Ann } I$ is a commutative subring of R. To see this, note that for $x, y \in \bar{S} \cap \text{Ann } I$, $xy - yx \in I \cap \text{Ann } I = 0$ since R is semi-prime. Since $\bar{S} \cap \text{Ann } I$ is a commutative Lie ideal of R, it follows that for $y \in \bar{S} \cap \text{Ann } I$, either $2y = 0$ and $y^2 \in Z(R)$, the center of R, or $2y \neq 0$ and $2y \in Z(R)$ (see the proof of Lemma 1.3 of [4] or Lemma 2 of [7]). In particular, $A \cap Z(R) \neq 0$ unless A is nil of index 2. But this would force $\bar{S}$ to contain a nonzero nilpotent ideal, by

Levitzki's Theorem [4; Lemma 1.1], which is impossible since $\bar{S}$ is semi-prime [6; Theorem 3.5]. Therefore $A \cap Z(R) \neq 0$. Hence, for $y \in A \cap Z(R)$, we have $dy = yd \in d\bar{S} \cap A$, and $dy \neq 0$ for $y \neq 0$ since d is regular. Thus $d\bar{S} \cap A \neq 0$ when $A \neq 0$, so $d\bar{S}$ is an essential right ideal of $\bar{S}$.

When $Q(R)$ exists, then dR is an essential right ideal for d regular. Hence, when R is also semi-prime, then $d\bar{S}$ will be an essential right ideal of $\bar{S}$ when d is regular in $\bar{S}$. Thus $a^{-1}(d\bar{S}) = \{y \in \bar{S} \mid ay \in d\bar{S}\}$ is an essential right ideal of $\bar{S}$. To conclude that $\bar{S}$ has a quotient ring, it suffices to know that the essential ideals $a^{-1}(d\bar{S})$ contain regular elements.

THEOREM 12. *Let R be semi-prime and satisfy the ascending chain condition on right annihilators. If $Q(R)$ exists, then $Q(\bar{S})$ exists.*

Proof. Since R has an involution, R also satisfies the ascending chain condition on left annihilators. By a Theorem of Johnson and Levy [3], each essential right ideal of R has a regular element. The same holds true for $\bar{S}$ since the chain conditions on annihilators are inherited by subrings, and since $\bar{S}$ is also semi-prime. As in the discussion preceeding the Theorem, for $a, d \in \bar{S}$ with $a \neq 0$ and d regular, $d\bar{S}$ is essential in $\bar{S}$ by Lemma 11, so $a^{-1}(d\bar{S})$ contains a regular element. Thus $ad' = da'$ for $a', d' \in \bar{S}$ and d' regular, the right Ore condition holds in $\bar{S}$, and $Q(\bar{S})$ exists.

Lastly, we turn to a description of how $Q(\bar{S})$ and $Q(R)$ are related when they both exist. Given that R satisfies the regularity condition, then the natural injection of $\bar{S}$ into R extends to an isomorphism of $Q(\bar{S})$ into $Q(R)$ [1]. Thus, we can consider $Q(\bar{S})$ as a subring of $Q(R)$. We are able to give a precise description of how $Q(\bar{S})$ sits in $Q(R)$ in two special cases. The first of these is when $\bar{S}$ is commutative.

DEFINITION. If R is a ring, and T is a nonempty, multiplicatively closed subset of regular elements in $Z(R)$, the center of R, then RT^{-1} is the localization of R at T.

THEOREM 13. *Let R be semi-prime with either $2R = 0$ or R 2-torsion free. If $\bar{S}$ is commutative then $Q(R) = R$ or $Q(R) = RT^{-1}$ for $T = \{y \in Z \cap S \mid y \text{ is regular}\}$.*

Proof. If R has no regular elements then $Q(R) = R$. If R has regular elements, then so does S, and each such element is regular in R,

by Corollary 3. Since $S = S$ is a commutative Lie ideal of R, and R is semi-prime, as in the proof of Lemma 11, we have for each $s \in S$, that $s^2 \in Z$ if $2R = 0$ and $2s \in Z$ is R is 2-torsion free. In the latter case it easily follows that $s \in Z$. Hence, it is always true that $s^2 \in Z$ for each $s \in S$, and so, T is not empty.

Should $Q(S) = S$, then $Q(R) = R$ by Theorem 6. Otherwise, consider $R' = RT^{-1}$. R' has an involution extending *, given by $(rz^{-1})^* = r^*z^{-1}$. Note that if $s \in S$ is regular, then $s(s(s^2)^{-1}) = 1$ in R', so $Q(S(R')) = S(R')$. Applying Theorem 6 again yields $Q(R') = R'$. But it is clear that $R' = Q(R)$, since if $r \in R$ is regular, then in R', $r(r^*rr^*((r\sigma^*)^2)^{-1}) = 1$.

Another case in which we can describe the relation between $Q(R)$ and $Q(\bar{S})$ is when R is a semi-prime Goldie ring. $Q(R)$ exists by Goldie's Theorem [2, Theorem 4.1], and the existence of $Q(\bar{S})$ follows from Theorem 12. One can also obtain the existence of $Q(\bar{S})$ from the fact that $\bar{S}$ is itself a semi-prime Goldie ring [7, Theorem 1].

We recall two facts about quotient rings which are required below. Assume that R is a semi-prime Goldie ring and I is an ideal of R containing a regular element of R. Then I is also a semi-prime Goldie ring, both R and I have quotient rings, and these quotient rings are isomorphic via the map $r \to rc \cdot c^{-1}$, for $r \in R$ and $c \in I$, a regular element of R. The second fact, easily verified, is that given rings A and B with quotient rings, then $Q(A \oplus B) = Q(A) \oplus Q(B)$.

THEOREM 14. *Let R be a semi-prime Goldie ring with either $2R = 0$ or R 2-torsion free. Then either*

(i) $Q(R) \cong Q(\bar{S})$ *or*

(ii) *There exist *-ideals I and $J = \operatorname{Ann} I$ of R so that $Q(\bar{S}) = Q(I) \oplus Q(J \cap \bar{S})$ and $Q(R) = Q(I) \oplus Q(J)$. Furthermore, $J \cap \bar{S}$ is commutative, and if $P = \{x \in \bar{S} \cap Z(J) \mid x \text{ is regular in } J\}$, then $Q(J) = JP^{-1}$.*

Proof. If $\bar{S}$ is commutative, take $I = 0$ and apply Theorem 13. If $\bar{S}$ is not commutative, then it contains the *-ideal I of R generated by all $xy - yx$ for $x, y \in \bar{S}$. First assume that $\operatorname{Ann} I = 0$. Then for any right ideal $T \neq 0$ of R, $TI \neq 0$ and $TI \subset T \cap I$, so I is an essential right ideal of R. It follows that I contains a regular element of R [2, Theorem 3.9]. By the remark preceeding the Theorem, $Q(R) \cong Q(I)$, and also, $Q(I) \cong Q(\bar{S})$, since $\bar{S}$ is a Goldie ring [7]. Therefore (i) holds, so we may assume that $J = \operatorname{Ann} I \neq 0$.

It is easy to show that $I \oplus J$ has no left annihilator in R. Thus $I \oplus J$ is an essential right ideal of R, so contains a regular element of R, say $c = c_1 + c_2$, with c_1 regular in I and c_2 regular in J. Using the remarks above once again, we have that $Q(R) \cong Q(I \oplus J) \cong$

$Q(I) \oplus Q(J)$. The latter quotients exist since I and J are semi-prime Goldie rings. Also, cc^* is regular in R and is contained in $I \oplus (J \cap \bar{S})$, an ideal of $\bar{S}$. Consequently, $Q(\bar{S}) \cong Q(I) \oplus Q(J \cap \bar{S})$.

$J \cap \bar{S}$ is commutative, for if $x, y \in J \cap \bar{S}$, then $xy - yx \in J \cap I = 0$. As J is a *-ideal, $S(J) = \bar{S} \cap J$, so $Q(J) = JP^{-1}$ by Theorem 13.

COROLLARY 15. *If R is a prime Goldie ring, then $Q(R) \cong Q(\bar{S})$.*

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