PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

A NEW CHARACTERIZATION OF CHARACTERISTIC FUNCTIONS OF ABSOLUTELY CONTINUOUS DISTRIBUTIONS

SIMEON M. BERMAN

It is well known that if g belongs to L_2, then

$$\frac{\int g(x)\bar{g}(x+y)dx}{\int |g(x)|^2 dx}$$

is the characteristic function of an absolutely continuous distribution function. Conversely, every such characteristic function has the representation given above. Here we shown that if $R(s,t)$ is a covariance function such that $R(s,s)$ belongs to L_1, then

$$\frac{\int R(s,s+t)ds}{\int R(s,s)ds}$$

is the characteristic function of an absolutely continuous distribution. Conversely, every such characteristic function has the latter representation (put $R(s,t)=g(s)\bar{g}(t)$). The use of this new result is that certain functions are directly seen to be of the second form but not the first; hence, they can be identified as characteristic functions of absolutely continuous distributions.

1. The main theorem. Let $R(s,t)$, $-\infty < s,t < \infty$, be a complex-valued Borel function of two variables. It is a *covariance function* if for any positive integer n, and any set of pairs (s_i, u_i), $i = 1,\cdots,n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} R(s_i, s_j)u_i u_j \geqq 0.$$

By Kolmogorov's existence theorem and the well known moment properties of Gaussian processes, for every covariance function there exists a probability space and a complex Gaussian process $X(t)$, $-\infty < t < \infty$, on the space such that

$$EX(t) = 0 \text{ for all } t, \qquad EX(s)\bar{X}(t) = R(s,t) \text{ for all } s,t.$$

We say that X is *associated* with R.

The function $R(s,s)$ is nonnegative because it is equal to $E|X(s)|^2$. If

$$\int_{-\infty}^{\infty} R(s,s)ds < \infty, \tag{1.1}$$

then there exists an associated process X which is measurable and satisfies

$$E\int_{-\infty}^{\infty} |X(s)|^2 ds = \int_{-\infty}^{\infty} E|X(s)|^2 ds < \infty. \tag{1.2}$$

It also follows that $X(t)$ belongs to L_2 almost surely, and so there is a measurable version of the Fourier transform process

$$\hat{X}(u) = \int_{-\infty}^{\infty} e^{ius} X(s)ds, \quad -\infty < u < \infty. \tag{1.3}$$

By Parseval's Theorem we also have

$$\int_{-\infty}^{\infty} E|X(s)|^2 ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} E|\hat{X}(u)|^2 du. \tag{1.4}$$

THEOREM 1. *Let R be a covariance function satisfying* (1.1). *Then the function*

$$r(t) = \frac{\int_{-\infty}^{\infty} R(s,s+t)ds}{\int_{-\infty}^{\infty} R(s,s)ds} \tag{1.5}$$

is a characteristic function. The corresponding distribution function is absolutely continuous with the derivative

$$g(u) = \frac{E|\hat{X}(-u)|^2}{\int_{-\infty}^{\infty} E|\hat{X}(y)|^2 dy}, \tag{1.6}$$

where X is the associated process satisfying (1.2). *Conversely if $r(t)$ is the characteristic function of an absolutely continuous distribution, then there exists R satisfying* (1.1) *such that r is representable as* (1.5).

Proof. First we prove the direct assertion. It follows from the definition of X that

$$\int_{-\infty}^{\infty} R(s,s+t)ds = \int_{-\infty}^{\infty} EX(s)\bar{X}(s+t)ds.$$

By virtue of condition (1.1) and the Cauchy-Schwarz inequality, we can take the expectation outside of the integral, and then apply the Parseval theorem:

$$E\int_{-\infty}^{\infty} X(s)\bar{X}(s+t)ds = E\left\{(1/2\pi)\int_{-\infty}^{\infty} e^{-iut}\,|\hat{X}(u)|^2 du\right\}.$$

It follows that the ratio in (1.5) is equal to

$$\frac{\int_{-\infty}^{\infty} e^{iut} E\,|\hat{X}(-u)|^2 du}{\int_{-\infty}^{\infty} E\,|\hat{X}(y)|^2 dy}$$

This is exactly the Fourier transform of the function $g(u)$ in (1.6).

The converse is simple: it is given in the above abstract.

2. Factorable covariances. R is said to be factorable if there exists a monotone function $A(y)$ and a "kernel" function $\phi(t,y)$ such that

$$R(s,t) = \int_{-\infty}^{\infty} \phi(s,y)\bar{\phi}(t,y)dA(y).$$

The condition (1.1) becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(s,y)|^2 dA(y)ds < \infty.$$

It follows (by Fubini's theorem) that $\phi(\cdot,y)$ belongs to L_2 for almost all y (with repect to dA); thus

$$\hat{\phi}(u,y) = \int_{-\infty}^{\infty} e^{iuz}\phi(z,y)dz$$

exists for all such y. By virtue of the isometry $X(t)\rightarrow\phi(t,\cdot)$ the density (1.6) takes the form

$$g(u) = \frac{\int_{-\infty}^{\infty} |\hat{\phi}(-u,y)|^2 dA(y)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{\phi}(u,y)|^2 du dA(y)}. \tag{2.1}$$

As a characteristic function $r(t)$ is also the covariance function of a stationary Gaussian process. When the spectral distribution is absolutely continuous, the process has a well known representation as a moving average of "white noise" on the line (see [2], p. 533). We will show that when R is factorable the stationary process with covariance of the form (1.5) also has a representation as a moving average of "noise" in the plane. The latter representation is more informative and easiler to derive in certain special cases (see §4 below).

Let W be a real Gaussian random set function defined over the plane Borel sets, that is, $W(C)$ has a normal distribution for every plane Borel set C. Let W have the following moment structure:

$EW(C) = 0$ for all C

$EW(C)W(C') = 0$ if C and C' are disjoint (independent increments)

$EW^2(C) = \int_B dx \cdot \int_{B'} dA(y)$ if $C = B \times B'$ is a product of two linear sets.

Consider the stochastic integral with respect to W, divided by a positive constant:

$$Y(t) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x+t, y) W(dx \times dy)}{\left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi(x,y)|^2 \, dx dA(y) \right\}^{\frac{1}{2}}} . \tag{2.2}$$

By a direct calculation and by means of the fundamental properties of the stochastic integral we find that the process $Y(t)$ is stationary (and Gaussian) with covariance function (1.5).

3. Change of time parameter in the covariance.

Let $R(s,t)$ be a covariance, and $f(x)$ a real Borel function. Then the composite function $R(f(x), f(y))$ is also a covariance function (in x and y). According to Theorem 1, if

$$\int_{-\infty}^{\infty} R(f(x), f(x)) dx < \infty, \tag{3.1}$$

then

$$r(t) = \frac{\int_{-\infty}^{\infty} R(f(x), f(x+t)) dx}{\int_{-\infty}^{\infty} R(f(x), f(x)) dx} \tag{3.2}$$

is the characteristic function of an absolutely continuous distribution. By means of this result we can identify some general and interesting functions as such characteristic functions.

EXAMPLE. Let $\sigma^2(t)$, $-\infty < t < \infty$, be the incremental second moment function of a Gaussian process with mean 0 and stationary increments ($\sigma^2(t) = E|U(s+t) - U(s)|^2$ where U has stationary increments). Then the covariance function of the process is

$$R(s,t) = \tfrac{1}{2}[\sigma^2(s) + \sigma^2(t) - \sigma^2(t-s)].$$

Let $f(x)$ be a Borel function such that

$$\int_{-\infty}^{\infty} \sigma^2(f(x))dx < \infty;$$

then (3.1) is fulfilled, and so

$$r(t) = 1 - \frac{\int_{-\infty}^{\infty} \sigma^2(f(x+t) - f(x))dx}{2\int_{-\infty}^{\infty} \sigma^2(f(x))dx} \tag{3.3}$$

is the characteristic function of an absolutely continuous distribution.

Let $f(x)$ be a Borel function such that

$$\int_{-\infty}^{\infty} |f(x)|^\alpha dx = 1$$

for some α, $0 < \alpha \leqq 2$; then

$$r(t) = 1 - \tfrac{1}{2}\int_{-\infty}^{\infty} |f(x+t) - f(x)|^\alpha dx \tag{3.4}$$

is a characteristic function with an absolutely continuous distribution; indeed, it is a special case of (3.3) with $\sigma^2(t) = t^\alpha$. (The fact that (3.4) is a characteristic function was first proved by Lawrence Shepp in a private communication.)

This can be used to prove a general result about the space L_α. According to the classical representation of the characteristic function of an absolutely continuous distribution as a convolution, there exists a function $\tilde{f}$ such that

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^2 dx = 1$$

and $r(t)$ is representable as

$$1-\tfrac{1}{2}\int_{-\infty}^{\infty} |\tilde{f}(x+t)-\tilde{f}(x)|^2\,dx.$$

From (3.4) we then conclude that

$$\int_{-\infty}^{\infty} |f(x+t)-f(x)|^{\alpha}\,dx = \int_{-\infty}^{\infty} |\tilde{f}(x+t)-\tilde{f}(x)|^2\,dx, \text{ for all } t.$$

As far as I can determine the existence of such an $\tilde{f}$ in L_2 for each f in L_α is a result unknown up to now.

4. A new proof of Polya's theorem and related results. Polya described a class of characteristic functions, now called "Polya characteristic functions". [4] He showed that if $r(t)$ is a convex function such that $r(t) \geqq 0$, $r(t) = r(-t)$, $r(0) = 1$, and $\lim_{t\to\infty} r(t) = 0$, then $r(t)$ is the characteristic function of an absolutely continuous distribution. We will show that such a function is representable as in Theorem 1, and so provide a new proof of Polya's theorem; furthermore, we will get an explicit form of the density from the results of §2, and a stochastic integral representation.

As a convex function, $r(t)$ has an integral representation

$$r(t) = \int_{|t|}^{\infty} f(x)\,dx, \tag{4.1}$$

where $f(x)$, $x \geqq 0$, is nonnegative and nonincreasing, and

$$\int_0^{\infty} f(x)\,dx = 1.$$

Indeed, take f as the negative of the right hand derivative of r (see [3], p. 5). Extend f to all x by assigning it the value 0 on the negative axis. Then $r(t)$ is representable as

$$r(t) = \int_{-\infty}^{\infty} \min(f(x), f(x+t))\,dx.$$

This is of the form (1.5) with $R(s,t) = \min(s,t)$. Thus r is a characteristic function of the indicated type.

Let $I(t)$ be the indicator function of the positive t-axis. The covariance $\min(s,t)$ is factorable for positive s and t:

$$\min(s,t) = \int_0^\infty I(s-y)I(t-y)dy.$$

Put $\phi(x,y) = I(f(x)-y)$ and $A(y) = y$ for $y > 0$, and $A(y) = 0$ for $y \leqq 0$. It follows from (2.1) that the density function of the Polya characteristic function is

$$(1/2\pi)\int_0^\infty \left| \int_{-\infty}^\infty e^{iux} I(f(x)-y)dx \right|^2 dy.$$

The random set function W in (2.2) is the 2-dimensional Brownian motion with independent increments, and the stochastic integral becomes

$$Y(t) = \iint_{\{y>0\}} I(f(x+t)-y)W(dx \times dy).$$

Such a representation for the Polya covariance process was recently given by Cabana and Wschebor [1].

ACKNOWLEDGEMENT. I thank Lawrence Shepp for a valuable observation which led to a significant improvement of this work.

REFERENCES

1. E. M. Cabana, and M. Wschebor, *On the barrier problem for stationary Gaussian processes*, Univ. Repub. Fac. Ingen. Agrimens. Montevideo Publ. Mat. Estadist., **4** (1969), 123–128.

2. J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.

3. M. A. Krasnoselskii, and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, translated from the first Russian edition by Leo F. Boron, P. Noordhoff, Groningen, 1961.

4. G. Polya, *Remark on characteristic functions*, Proceedings First Berkeley Symposium Math. Statist. Probability, Edited by J. Neyman, Berkeley, 1949.

Received March 18, 1974. This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the National Science Foundation, Grant NSFGP-28651.

NEW YORK UNIVERSITY