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INTEGRATION OF COMPACT SET-VALUED FUNCTIONS

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A theory of integration of compact set-valued functions is provided by applying the McShane $\mathscr{P}$-integral. This integral is a Riemann-type integral and includes the Bochner, Lebesgue and other types of integrals, and by using Riemann sums it avoids deep measure theory. Thus, the $\mathscr{P}$-integral of set-valued functions contains other types of integrals such as the Hukuhara and Debreu integrals. Generalizations of known results, including the convexity of the integral, are obtained, and the techniques do not require measure theory. Further, if a set-valued function is $\mathscr{P}$-integrable, then its integral equals the Aumann integral, where the latter is defined as the collection of integrals of selections.

1. Introduction. Considerations of summation and integration of set-valued functions go back to Minkowski. Recently, the calculus of set-valued functions was found to be very applicable in several mathematical fields, especially in control theory, mathematical economics, and statistics. Accordingly, many recent papers deal with the basic theory of integration of set-valued functions, and several approaches were established. One approach, due to Hukuhara [7], is to consider formal Riemann integration into the space of convex-compact sets. The Lebesgue integral is obtained by taking appropriate limits. A second approach was employed by Debreu [6], who used an embedding of the convex-compact sets into a Banach space and then considered the Bochner integral into this space. Yet another approach was developed by Aumann [3], which considered integration of selections of the set-valued functions. The Aumann integral is well suited for applications to various mathematical fields, and we shall give below the precise definition of this integral. In the three approaches outlined above, the main tools are measure theoretic techniques. Also, because of the usefulness of the Aumann integral, the relationships between the three types of integrals were investigated in [4], [5], and [6].

McShane [8] gave a definition of a Riemann type integral, which includes many former integrals such as the Lebesgue and Bochner integrals. Moreover, the definition has two very nice properties. First of all, the definition of McShane's integral (called the $\mathscr{P}$-integral) does not require measure theoretic concepts and is defined as a limit of Riemann-type sums. The second advantage of the McShane $\mathscr{P}$-integral is that it is defined for functions that take values in semi-groups, and there is no need to assume linearity of the range space. The main purpose of the present paper is to apply the McShane $\mathscr{P}$-integral to set-valued functions. We shall deal with functions defined on the real line with values in the set of compact subsets of R^q. With the Hausdorff metric on the compact subsets of R^q, the $\mathscr{P}$-integral will include the Debreu and Hukuhara integrals. (The latter are defined only for convex-compact valued functions.) Moreover, we obtain many properties of the integral without using measure theory. Thus, it is not necessary to consider an exceedingly elaborate measure theory to obtain a powerful integral, and even though measure theory is elegant, it sometimes fails to supply the simplest and most revealing proofs.

The main results are stated below. Some notations and preliminaries will be given in §2 and the proofs are given in §§3–6.

The first result, Theorem A, is similar to the first theorem in Aumann [1], but its proof does not require any measure theory. The space of compact subsets of R^q is denoted by $\mathscr{C}(R^q)$.

THEOREM A. *If $F\colon [a,b]\to\mathscr{C}(R^q)$ is $\mathscr{P}$-integrable, then its $\mathscr{P}$-integral is a convex set.*

The following result is similar to Aumann's third theorem.

THEOREM B. *The function $F\colon [a,b]\to\mathscr{C}(R^q)$ is $\mathscr{P}$-integrable if and only if the set-valued function* coF, *defined by* coF(t) *equals the convex hull of $F(t)$, is $\mathscr{P}$-integrable. Moreover, $(\mathscr{P})\int_a^b F$ and $(\mathscr{P})\int_a^b \mathrm{coF}$ are equal.*

For vector-valued functions the $\mathscr{P}$-integral is equivalent to the Lebesgue integral, and hence there are the obvious measure theoretic characterizations of the $\mathscr{P}$-integral. The following result provides a characterization of $\mathscr{P}$-integrable set-valued functions and indicates the connection between $\mathscr{P}$-integrability and measurability of set-valued functions. We say that a set-valued function F is *measurable* if for each closed set C in R^q, the set $\{t\colon F(t)\cap C\neq\varnothing\}$ is a Lebesgue measurable subset of $[a,b]$. A set-valued function F is said to be *Integrably bounded* if there exists a $\mathscr{P}$-integrable real-valued function g such that $x\in F(t)$ implies $\|x\|\leqq g(t)$.

THEOREM C. *The function $F\colon [a,b]\to\mathscr{C}(R^q)$ is $\mathscr{P}$-integrable if and only if it is integrably bounded and* coF *is measurable.*

In [2] it was shown that for convex and compact valued functions the same conditions characterize the Debreu integral.

If F is a set-valued function, then a *selection* of F is a vector-valued function $f\colon [a,b]\to R^q$ such that f is Lebesgue measurable and $f(t)\in F(t)$ almost everywhere in $[a,b]$. The *Aumann integral* of F, denoted by $(A)\int_a^b F$, is the set of all vectors of the form $\int_a^b f$, where f is a Lebesgue integrable selection of F. It is to be noted that the Aumann integral is defined for every set-valued function, and it may be empty. Since for scalar-valued functions the Lebesgue integral and the $\mathcal{P}$-integral are equal (see §2), it follows that the Aumann integral is also given by

$$(A)\int_a^b F=\{(\mathcal{P})\int_a^b f\colon f \text{ is } \mathcal{P}\text{-integrable and } f(t)\in F(t)\}.$$

For the case where F is a convex and compact set-valued function, it was shown in references **[5]** and **[6]** that the Hukuhara and the Debreu integrals equal the Aumann integral. We have the following similar result for the $\mathcal{P}$-integral, and again the proof does not use measure theoretic techniques.

THEOREM D. *If* $F\colon [a,b]\to \mathscr{C}(R^q)$ *is* $\mathcal{P}$*-integrable, then* $(\mathcal{P})\int_a^b F=(A)\int_a^b F.$

The Aumann integral exists for any compact set-valued function even if it is not $\mathcal{P}$-integrable. The following theorem clarifies the relationship between the two. It is also similar to a known result for a real-valued function, namely that if $g(t)$ is an integrably bounded real-valued function, then there exists an integrable function $f_0(t)$ such that $f_0(t)\leqq g(t)$ almost everywhere, and if f is any integrable function satisfying $f(t)\leqq g(t)$ a.e., then $f(t)\leqq f_0(t)$ a.e.

THEOREM E. *For every integrably bounded* $F\colon [a,b]\to \mathscr{C}(R^q)$ *such that* $(A)\int_a^b F\neq\varnothing$, *there is a* $\mathcal{P}$*-integrable function* $G\colon [a,b]\to \mathscr{C}(R^q)$ *such that* $G(t)\subseteq F(t)$ *a.e., and* G *is maximal with respect to this property. Moreover the* $\mathcal{P}$*-integral of* G *is equal to the Aumann integral of* F. *If* $(A)\int_a^b F=\varnothing$, *then there is no* $\mathcal{P}$*-integrable* G *such that* $G(t)\subseteq F(t)$ *for every* t.

2. Notations and preliminaries. Let A and B be subsets of R^q and $\lambda\in R$. The sum of A and B is the set $A+B$ given by $A+B=\{a+b\colon a\in A \text{ and } b\in B\}$, and λA is the set defined by

$\lambda A = \{\lambda a: a \in A\}$. The Hausdorff distance between A and B is defined by

$$h(A,B) = \inf\{\lambda > 0: A \subseteq B + \lambda U \quad \text{and} \quad B \subseteq A + \lambda U\},$$

where U is the closed unit ball in R^q. The set of all non-empty compact subsets of R^q, with the Hausdorff distance h, is a complete metric space which will be denoted by $\mathscr{C}(R^q)$. The convex hull of a set A will be denoted by coA, and $p \cdot x$ will represent the scalar product of $p \in R^q$ and $x \in R^q$. For $A \in \mathscr{C}(R^q)$, the "norm" of A is defined by $\|A\| = h(A,\{0\})$. It is easy to verify that $\mathscr{C}(R^q)$ is a topological semi-group and has many interesting properties. However, we shall only be concerned with the properties needed in the context of this paper.

The following definitions are found in greater generality in McShane's memoir **[8]**. However, the "simple" $\mathscr{P}$-integral is equivalent to the Lebesgue integral for a vector-valued function defined on an interval. (See Theorem 13.6 and the remarks in §14 of reference **[8]**.)

Let $a < b$ be real numbers. A finite collection $\Pi = \{(t_1, A_1), (t_2, A_2), \cdots, (t_n, A_n)\}$ is said to be a *partition* of $(a, b]$ if each $t_j \in [a, b]$, each A_j is either empty or an interval of the form $(a_j, b_j]$ with $a \leqq a_j < b_j \leqq b$, and each point of $(a, b]$ belongs to exactly one of the sets A_j. The *length* of A_j is given by $\Delta A_j = b_j - a_j$ if $A_j \neq \varnothing$ and $\Delta\varnothing = 0$. A *gauge* on $[a, b]$ is a real-valued function $\delta: [a, b] \to (0, +\infty)$. There is no requirement that δ be continuous or bounded away from zero. Given a gauge δ, a partition Π is said to be *δ-fine* if $A_j \subseteq (t_j - \delta(t_j), t_j + \delta(t_j))$ for each $j = 1, 2, \cdots, n$.

Digressing for a moment, it is not at all obvious that for a given gauge δ there exists any δ-fine partitions of $(a, b]$. However, this may be shown to be true by an indirect proof using only the completeness of the real line. Also, it should be noted that only slight modifications are needed to extend the definitions to include infinite intervals. This will not be pursued in the present paper.

If f is a function defined on $[a, b]$ with values in R (resp. $\mathscr{C}(R^q)$) and Π is a partition of $(a, b]$, then the *Riemann sum corresponding to f and Π* is given by

$$S(f; \Pi) = \sum_{j=1}^{n} f(t_j)\Delta A_j.$$

Such a function f is said to be *$\mathscr{P}$-integrable over* $[a, b]$ if there exists an element $I \in R$ ($I \in \mathscr{C}(R^q)$) with the property that for each $\epsilon > 0$, there exists a gauge δ such that if Π is any δ-fine partition of $(a, b]$, then

$$\|S(f; \Pi) - I\| < \epsilon \qquad (h(S(f; \Pi), I) < \epsilon).$$

The element I is said to be the *$\mathscr{P}$-integral* of f and we write $(\mathscr{P})\int_a^b f = I$.

At this point, we note that it is possible to derive the basic properties of the $\mathcal{P}$-integral directly from its definition, without mention of measure theory. However, since a primary objective of this paper is to compare the $\mathcal{P}$-integral and the Aumann integral of compact set-valued functions, we shall make double use of the equivalence of the $\mathcal{P}$-integral and the Lebesgue integral for vector-valued functions. That is, we shall use properties of the $\mathcal{P}$-integral of a R^q-valued function that are well known for the equivalent Lebesgue integral. In particular, we use the fact that a $\mathcal{P}$-integrable function must be Lebesgue measurable and conversely that a Lebesgue integrable function is $\mathcal{P}$-integrable.

3. *Proofs of Theorem A and B.* We shall need the following lemma which is due to L. Shapley and J. H. Folkman. A proof can be found in [**1**, Theorem 9. page 396]. It is to be noted that the proof is of combinatorial type and uses only simple properties of finite dimensional spaces.

LEMMA 3.1. (Shapley-Folkman). *If $A_1, \cdots, A_k$ is a finite family of sets in R^q such that $\| A_j \| \leqq L$ for a fixed L and for each $j = 1, 2, \cdots, K$, then*

$$h\left(\sum_{j=1}^{K} A_j, \mathrm{co} \sum_{j=1}^{K} A_j\right) \leqq \sqrt{q} L.$$

Proceeding to the proof of Theorem A we assume that $F\colon [a, b] \to \mathcal{C}(R^q)$ is $\mathcal{P}$-integrable. For a given $\epsilon > 0$ let δ be the gauge associated with ϵ. Define δ' by

$$\delta'(t) = \min\{\delta(t), \epsilon(\sqrt{q}(1 + \| F(t) \|))^{-1}\}.$$

Note that by this choice, if Π is a δ'-fine partition, then every element $F(t_j)\Delta A_j$ in the sum $S(F; \Pi) = \Sigma_{j=1}^{K} F(t_j)\Delta A_j$ has norm less than $\epsilon/\sqrt{q}$ and therefore Lemma 3.1 implies that

$$h(S(F; \Pi), \mathrm{co} S(F; \Pi)) \leqq \epsilon.$$

Since a δ'-fine parition is also δ-fine partition we know that $h(S(F; \Pi), (\mathcal{P})\int_a^b F) \leqq \epsilon$ and the latter together with the displayed inequality imply that the Hausdorff distance between $(\mathcal{P})\int_a^b F$ and the convex set $\mathrm{co} S(F; \Pi)$ is less than 2ϵ. Therefore $(\mathcal{P})\int_a^b F$ is the limit in the

Hausdorff metric of convex sets and hence is convex. This proves Theorem A.

In order to verify Theorem B notice that since the co operation is linear, and in particular $\mathrm{co}S(F;\Pi) = S(\mathrm{co}F;\Pi)$, we have shown that coF is $\mathcal{P}$-integrable and $(\mathcal{P})\int_a^b F = (\mathcal{P})\int_a^b \mathrm{co}F$. Moreover the inequality, $h(S(F;\Pi), \mathrm{co}S(F;\Pi)) \leqq \epsilon$ in our proof, did not depend on the integrability of F. Therefore the reverse implication may be proven by exactly the same method and this completes the proof of Theorem B.

4. *The Proof of Theorem C.* For a set A in R^q and a vector p in R^q, (R^q equals the dual of R^q) we define $s(p,A)$ by $s(p,A) = \sup\{p\cdot x : x \in A\}$. The function $s(\cdot,A)$ is known as the *support function* of A. It is a convex and positively homogeneous function and clearly real-valued if A is compact. If F is a set-valued function we shall use for simplicity $s(p,t)$ instead of $s(p,F(t))$. The following properties are easy to verify.

(i) $s(p,A+B) = s(p,A) + s(p,B)$.
(ii) If $\|p\| \leqq 1$, then $|s(p,A) - s(p,B)| \leqq h(A,B)$.
(iii) If A and B are convex, then

$$h(A,B) = \sup_{\|p\|=1} |s(p,A) - s(p,B)|.$$

(iv) If $p_1, p_2, \cdots$ is a dense sequence in R^q and A is compact, then

$$\mathrm{coA} = \bigcap_{j=1}^{\infty} \{x : p_j \cdot x \leqq s(p_j, A)\}.$$

Now suppose that F is $\mathcal{P}$-integrable. Properties (i) and (ii) directly imply that $s(p,t)$ is $\mathcal{P}$-integrable for each $p \in R^q$. Let $s(t) = \sum_{j=1}^{2q} |s(e_j,t)|$, where $e_1, e_2, \cdots, e_{2q}$ are the $2q$ vectors $(0,\cdots,0,\pm 1,0,\cdots,0)$. The function $s(t)$ is $\mathcal{P}$-integrable, (i.e. it is Lebesgue integrable) and the definition of the support function implies that $\|F(t)\| \leqq s(t)$. Thus F is integrably bounded. If $p_1, p_2, \cdots$ is a dense sequence in R^q, then property (iv) implies that $\mathrm{coF}(t) = \bigcap_{j=1}^{\infty}\{x : p_j \cdot x \leqq s(p_j,t)\}$. The measurability of $s(p_j,t)$ implies that the set-valued function $G_t(t) = \{x : p_j \cdot x \leqq s(p_j,t)\}$ is measurable. By Rockafellar [9; Corollary 1.3] it follows that $\mathrm{coF}(t) = \bigcap_{j=1}^{\infty} G_j(t)$ is measurable, and this completes the proof of the "only if" part of Theorem C.

Suppose that coF is measurable and F is integrably bounded. Then for each p the support function $s(p,t)$ is integrably bounded. Also, since $s(p, A) = s(p, \text{coA})$ for any set A, we have that $s(p,t)$ is measurable. To see this, notice that for each α the set

$$\{t : s(p,t) \geqq \alpha\} = \{t : \text{coF}(t) \cap \{x : p \cdot x \geqq \alpha\} \neq \varnothing\}$$

is a Lebesgue measurable subset of $[a, b]$.

Therefore, it follows that $s(p,t)$ is Lebesgue integrable, or equivalently, $\mathcal{P}$-integrable. Let $p_1, p_2, \cdots$ be a dense sequence in the unit sphere of R^q, and suppose $\epsilon > 0$ is given. Let δ_j, $j = 1, 2, \cdots$, be the gauge that established the $(\epsilon/4)$-approximation of the $\mathcal{P}$-integral of $s(p_j, t)$. Since F, and hence coF, is integrably bounded, there is a gauge δ and a compact set B such that if Π is any δ-fine partition, then $S(\text{coF}; \Pi) \subseteq B$. Moreover, it is easy to verify that the family of support functions $\{s(\cdot, K) : K$ is a compact subset $B\}$ is equicontinuous. Let $\gamma > 0$ be such that if K is a compact subset of B and $\| p - q \| < \gamma$, then $| s(p, K) - s(q, K) | < \epsilon/4$. Since the unit sphere is compact, there exists a finite collection, say $p_1, p_2, \cdots, p_{N(\gamma)}$, such that if $\| p \| = 1$, then $\| p - p_K \| < \gamma$ for some K, $1 \leqq K \leqq N(\gamma)$.

Let $\delta_\epsilon = \min\{\delta, \delta_1, \cdots, \delta_{N(\gamma)}\}$, and suppose that Π_1 and Π_2 are any two δ_ϵ-fine partitions. Note that $S_1 = S(\text{coF}; \Pi_1)$ and $S_2 = S(\text{coF}; \Pi_2)$ are compact subsets of B, and hence for each p in the unit sphere we have that

$$\begin{aligned} | s(p, S_1) - s(p, S_2) | &\leqq | s(p, S_1) - s(p_K, S_1) | \\ &\quad + | s(p_K, S_1) - s(p_K, S_2) | \\ &\quad + | s(p_K, S_2) - s(p, S_2) | \\ &\leqq | s(p_K, S_1) - s(p_K, S_2) | + \epsilon/2. \end{aligned}$$

But, $s(p, S(\text{coF}; \Pi)) = S(s(p, \cdot); \Pi)$, and hence it follows that $| s(p, S_1) - s(p, S_2) | < \epsilon$. Property (iii) implies that $h(S(\text{coF}; \Pi_1), S(\text{coF}; \Pi_2)) < \epsilon$, and we can conclude that the net of Riemann sums of coF over δ_ϵ-fine partitions is a Cauchy net. It is clear that this net converges to $(\mathcal{P}) \int_a^b \text{coF}$ and by Theorem B we have that F is $\mathcal{P}$-integrable.

REMARK 4.1. Notice that directly from the basic definitions it follows that

$$(\mathcal{P}) \int_a^b s(p, F(t))dt = s\left(p, (\mathcal{P}) \int_a^b F\right).$$

This equality is a useful tool in the study of integrals of set-valued functions.

5. *The Proof of Theorem D.* Recall that if F is a set-valued function then the Aumann integral of F is given by

$$(A)\int_a^b F = \left\{(\mathscr{P})\int_a^b f\colon f(t) \in F(t)\right\}.$$

We first show that if $F\colon [a,b] \to \mathscr{C}(R^q)$ is $\mathscr{P}$-integrable then $(A)\int_a^b F \subset (\mathscr{P})\int_a^b F$. Let f be a $\mathscr{P}$-integrable selection of F. For a given $\epsilon > 0$ let δ_1 and δ_2 be gauge functions that establish the ϵ-approximation of the $\mathscr{P}$-integrals of f and F respectively. Let $\delta = \min(\delta_1, \delta_2)$. If Π is a δ-fine partition then $S(f; \Pi)$ is an element of $S(F; \Pi)$. Since the sums are ϵ-approximations of the respective $\mathscr{P}$-integrals and since ϵ is arbitrarily small it follows that

$$(\mathscr{P})\int_a^b f \quad \text{belongs to} \quad (\mathscr{P})\int_a^b F.$$

We now show that the Aumann integral of F contains the $\mathscr{P}$-integral of F. Recall the e is an *exposed point* of the compact set A if e is the only point in the intersection of A and a certain support hyperplane of A. In terms of the support function the point e is an exposed point of A if there exists a $p \in R^q$ such that

$$p \cdot e = s(p, A) \quad \text{and} \quad p \cdot x < s(p, A) \quad \text{if} \quad e \neq x \in A.$$

The exposed points are dense in the extreme points of a set A. We shall show that if e is an exposed point of $(\mathscr{P})\int_a^b F$ then e belongs to the Aumann integral of F. Since the latter is convex and compact (Aumann [3, Theorems 1, 2]) it follows that the closure of the convex hull of the exposed points of $(\mathscr{P})\int_a^b F$, which is $(\mathscr{P})\int_a^b F$ itself, belongs to the Aumann integral of F.

Let e be an exposed point of $(\mathscr{P})\int_a^b F$. Let p be such that $e \neq x \in (\mathscr{P})\int_a^b F$ implies that $p \cdot x < p \cdot e$. For every set A in R^q denote $A_p = \{x \in A\colon p \cdot x = s(p, A)\}$. The operator $A \to A_p$ is linear, i.e., $A_p + B_p = (A + B)_p$. We shall prove that $F_p(t)$ is $\mathscr{P}$-integrable and that $(\mathscr{P})\int_a^b F_p = ((\mathscr{P})\int_a^b F)_p = \{e\}$. Let $K = (\mathscr{P})\int_a^b F$ and for $\eta > 0$ define $K_\eta = \{x \in K\colon p \cdot x \geqq s(p, K) - \eta\}$. In particular we have that $K_0 = \{e\}$. Since e is the unique point in K_0 it follows that the diameter of K_η tends to zero when η goes to zero. Let $\epsilon > 0$ be fixed. Choose η so small that the diameter of $K_{2\eta}$ will be less than ϵ. Let δ be a gauge function for F associated with η, and suppose Π is a δ-fine partition. Since

$$h\left(S(F;\Pi),(\mathscr{P})\int_a^b F\right) \leqq \eta, \tag{5.1}$$

it follows that $S(F;\Pi)\cap\{x: p\cdot x \geqq (p\cdot e)-\eta\}\neq\varnothing$. Therefore,

$$S(F;\Pi)_p = S(F_p;\Pi)\subset\{x: p\cdot x \geqq (p\cdot e)-\eta\}.$$

On the other hand, (5.1) implies that

$$S(F;\Pi)_p \subseteq \{x: \text{ there exists } y\in K_{2\eta} \text{ such that } \|y-x\|\leqq\eta\},$$

and hence $S(F;\Pi)_p$ is included within an $\epsilon+\eta$ neighborhood of $\{e\}$. This completes the proof of the equality

$$(\mathscr{P})\int_a^b F_p = \left((\mathscr{P})\int_a^b F\right)_p.$$

Since F_p has at most dimension $n-1$ we can use an induction argument in order to show that the Aumann integral of F_p equals $\{e\}$. The first step of the induction, i.e. for $n=0$, is obvious.

REMARK. We showed that $(\mathscr{P})\int_a^b F_p = \left((\mathscr{P})\int_a^b F\right)_p$ if p determines a hyperplane which supports at an exposed point. The proof does not use measure theoretic arguments. For the Aumann integral of measurable set-valued functions it is easy to verify the same equality for every p. Thus Theorems C and D imply that $(\mathscr{P})\int F_p = \left((\mathscr{P})\int F\right)_p$ for every p. We do not have a simple proof for this which will not use measure theory.

6. *Proof of Theorem E.* Let F be an integrably bounded set-valued function. Denoted by $\mathscr{F}$ the collection of equivalence classes of Lebesgue measurable selections of F. If $\mathscr{F}$ is empty then Theorem D implies that no integrable subfunction G of F exists. This proves the second statement of the theorem. Suppose now that $\mathscr{F}$ is not empty. For a vector p in R^q denote by $\rho(p,t)$ the supremum of the functions $p\cdot f(t)$, i.e. the smallest measurable function such that if $f\in\mathscr{F}$ then $p\cdot f(t)\leqq\rho(p,t)$ for almsot every t. The function $\rho(p,t)$ is defined up to a set of measure zero. If $H(p,t)=\{x: p\cdot x\leqq\rho(p,t)\}$, then $H(p,t)$ is measurable. Let $p_1, p_2 \cdots$ be a dense set in R^q. Define $G(t)\colon [a,b] \to \mathscr{C}(R^q)$ by

$$G(t) = \bigcap_{j=1}^{\infty} (H(p_j, t) \cap F(t)).$$

obviously, $G(t) \subset F(t)$. Moreover, if $p \in R^q$ and $f \in \mathscr{F}$, then f is also a selection of $F(t) \cap H(p,t)$. Consequently, if $f \in \mathscr{F}$ then f is a selection of G, and hence the Aumann integral of F equals that of G. In order to show that G is $\mathscr{P}$-integrable notice that $s(p, G(t)) = \rho(p,t)$. This follows from the fact that $\mathscr{F}$ is also the collection of selections of G. Therefore,

$$\mathrm{co}G(t) = \bigcap_{j=1}^{\infty} H(p_j, t)$$

and it is measurable as a denumerable intersection of measurable functions (see Rockafeller [9, Corollary 1.3]). Remarks 4.1 and Theorem D imply that G is maximal and this completes the proof.

7. *$\mathscr{P}$-integration into the Semigroup of all Bounded Sets.* As was noted in Section 1 the $\mathscr{P}$-integral is defined for functions which might take values in general topological semigroups. In particular, we may consider the semigroup of all bounded sets with the Hausdorff semi-metric. Obviously, such a set-valued function will be $\mathscr{P}$-integrable if and only if the pointwise closure of it will be $\mathscr{P}$-integrable and they will have the same $\mathscr{P}$-integral. In particular, the Riemann sums will converge in the Hausdorff semi-metric. Although the Aumann integral is defined for all set-valued functions, Theorem D cannot be generalized to the case where compactness is not assumed. Indeed, one can easily construct a set-valued function $F(t)$ such that the closure of $F(t)$ will be constant, but there is no measurable selection of F.

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