

COUNTABLE ORDINALS AND THE ANALYTICAL HIERARCHY, I

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The following results are proved, using the axiom of **Projective Determinacy**: (i) For $n \geq 1$, every Π^1_{2n+1} set of countable ordinals contains a Δ^1_{2n+1} ordinal, (ii) For $n \geq 1$, the set of reals Δ^1_{2n} in an ordinal is equal to the largest countable Σ^1_{2n} set and (iii) Every real is Δ^1_n inside some transitive model of set theory if and only if $n \geq 4$.

In general we shall use the terminology and notation of [3]. In particular letters i, j, k, \dots are used as variables over $\omega = \{0, 1, 2, \dots\}$ and $\alpha, \beta, \gamma, \dots$ as variables over ${}^\omega\omega$ (= the set of reals). For a collection of sets of reals Γ , *Determinacy* (Γ) abbreviates the statement that every set in Γ is determined and *Projective Determinacy* (PD) is the axiom that every projective set is determined.

1. An ordinal basis theorem. A well known boundedness result in recursion theory asserts that if $WO = \{\alpha: \leq_\alpha \text{ is a wellordering}\}$, where $\leq_\alpha = \{(m, n): \alpha(2^m \cdot 3^n) = 0\}$ and $A \subseteq WO$ is Σ^1_n , then $\sup\{|\alpha|: \alpha \in A\} < \delta^1_n$, where for $\alpha \in WO$, $|\alpha| = \text{length}(\leq_\alpha)$ and $\delta^1_n = \sup\{|\alpha|: \alpha \in \Delta^1_n \& \alpha \in WO\}$. We prove below a generalization of this fact to all odd levels of the analytical hierarchy.

THEOREM 1.1. Assume *Projective Determinacy*, when $n \geq 1$. If $A \subseteq WO$ is Σ^1_{2n+1} and $\sup\{|\alpha|: \alpha \in A\} < \aleph_1$, then $\sup\{|\alpha|: \alpha \in A\} < \delta^1_{2n+1}$.

Proof. For notational simplicity let us take $n = 1$ as a typical case. Thus let $A \subseteq WO$ be Σ^1_3 and assume $\sup\{|\alpha|: \alpha \in A\} < \aleph_1$. Let $B \subseteq {}^\omega\omega$ be Π^1_2 and $f: {}^\omega\omega \rightarrow {}^\omega\omega$ recursive such that $f[B] = A$. Consider then the following game: Player I plays β , player II plays γ and II wins iff $\gamma \in WO \& (\beta \in B \rightarrow |f(\beta)| \leq |\gamma|)$. Clearly player II has a winning strategy in this game. But his payoff set is Σ^1_2 , so by a result of Moschovakis [6] he has a winning strategy τ which is Δ^1_3 . Let $T = \{\beta * \tau: \beta \in {}^\omega\omega\}$, where $\beta * \tau$ is the result of II's moves following τ when I plays β . Then $T \subseteq WO$ and T is $\Sigma^1_1(\tau)$, so by the Boundedness Theorem

$$\sup\{|\gamma|: \gamma \in T\} < \delta^1_1(\tau) < \delta^1_1.$$

But clearly

$$\sup\{|\alpha| : \alpha \in A\} \leq \sup\{|\gamma| : \gamma \in T\}.$$

It is not hard now to reformulate Theorem 1.1. into a basis theorem. Call a set of ordinals $X \subseteq \aleph_1$ Π_n^1 if $\{\alpha \in WO : |\alpha| \in X\}$ is Π_n^1 . Recall that a countable ordinal ξ is called Δ_n^1 iff $\xi < \delta_n^1$.

THEOREM 1.2. *Assume Projective Determinacy and $n \geq 1$. Every nonempty Π_{2n+1}^1 set of ordinals contains a Δ_{2n+1}^1 ordinal.*

Proof. Take again $n = 1$. Let $X \subseteq \aleph_1$ be Π_1^1 and consider $A = \{\alpha \in WO : |\alpha| \leq \min X\}$. Then $\alpha \in A \Leftrightarrow \alpha \in WO \& \forall m (|\alpha_m| \notin X)$, where α_m is a real coding the restriction of \leq_α to its initial segment determined by m . Clearly A is Σ_3^1 and $A \subseteq WO$, so by Theorem 1.1. $\sup\{|\alpha| : \alpha \in A\} = \min X < \delta_3^1$, thus X contains a Δ_3^1 ordinal.

REMARK. If $X = \{\xi < \aleph_1 : \omega_0 < \xi \text{ is admissible}\}$ then X is Π_1^1 but contains no Δ_1^1 ordinal.

2. The set of reals Δ_n^1 in an ordinal. A real α is called Δ_n^1 in an ordinal $\xi < \aleph_1$ if α is Δ_n^1 in every real $\beta \in WO$ such that $|\beta| = \xi$. A simple argument shows that a real is Δ_1^1 in an ordinal iff it is Δ_2^1 in an ordinal iff it is constructible. Martin and Solovay [4] proved that under PD, the set of reals Δ_{2n+1}^1 in an ordinal, when $n \geq 1$, is exactly the set Q_{2n+1} (see [4] or [3].) [The set Q_{2n+1} can be defined in many equivalent ways. One of the most suggestive ones is the following: $Q_{2n+1} = \{\alpha : \alpha \text{ belongs to every model of } ZFC + PD \text{ for which } \Sigma_{2n}^1 \text{ formulas are absolute}\}$. We shall identify below the set of reals which are Δ_{2n}^1 in an ordinal. Before doing this though we shall give as an application of Theorem 1.1. a new proof of the result of Martin and Solovay. Their original proof used forcing.

THEOREM 2.1. (Martin-Solovay [4]). *Assume Projective Determinacy and $n \geq 1$. Then $Q_{2n+1} = \{\alpha : \alpha \text{ is } \Delta_{2n+1}^1 \text{ in an ordinal}\}$.*

Proof. Let $n = 1$ again. From the results of [3] (especially the Lemma before Theorem (3B-3)) we can easily see that every real in Q_3 is Δ_3^1 in an ordinal. Conversely, if α is Δ_3^1 in an ordinal, then by Theorem 1.2, α is Δ_3^1 in an ordinal ξ which is Δ_3^1 in α . Thus $A = \{\alpha : \alpha \text{ is } \Delta_3^1 \text{ in an ordinal}\}$ is Π_3^1 . By Theorem (3B-3) of [3] the set Q_3 is characterized as the largest Π_3^1 -bounded set, where a set $B \subseteq \omega^\omega$ is Π_3^1 -bounded if for all predicates $P(\alpha, \beta)$ in Π_3^1 the predicate $\exists \alpha \in B P(\alpha, \beta)$ is also Π_3^1 . To complete the proof it is thus enough to show that A is Π_3^1 bounded. For any Π_3^1 predicate $P(\alpha, \beta)$ put

$$X_\beta = \{\xi < \aleph_1 : \forall \gamma (\gamma \in WO \& |\gamma| = \xi \rightarrow \exists \alpha \in \Delta_3^1(\gamma) [P(\alpha, \beta) \& \alpha \in A])\}$$

Then X_β is Π_3^1 in β and $\exists \alpha \in AP(\alpha, \beta) \Rightarrow X_\beta \neq \emptyset$, so $\exists \alpha \in AP(\alpha, \beta) \Rightarrow X_\beta$ contains an ordinal Δ_3^1 in $\beta \Rightarrow \exists \alpha \in \Delta_3^1(\beta) [P(\alpha, \beta) \& \alpha \in A]$. Thus $\exists \alpha \in AP(\alpha, \beta) \Leftrightarrow \exists \alpha \in \Delta_3^1(\beta) [\alpha \in A \& P(\alpha, \beta)]$ and we are done.

We now proceed to identify the set of reals Δ_{2n}^1 in an ordinal. Our result generalizes the fact that $\{\alpha : \alpha \text{ is } \Delta_2^1 \text{ in an ordinal}\} = \{\alpha : \alpha \text{ is constructible}\} = C_2 =^{\text{def}} \text{the largest countable } \Sigma_2^1 \text{ set of reals (we assume here that there are only countably many constructible reals).}$

THEOREM 2.2. *Assume Projective Determinacy and let $C_{2n} =$ the largest countable Σ_{2n}^1 set of reals. Then $C_{2n} = \{\alpha : \alpha \text{ is } \Delta_{2n}^1 \text{ in an ordinal}\}$.*

Proof. For notational simplicity take $n = 2$. By Theorem (1C-3) of [3], $C_4 \subseteq \{\alpha : \alpha \text{ is } \Delta_4^1 \text{ in an ordinal}\} =^{\text{def}} S$. Since S is countable it is enough to prove that S is Σ_4^1 .

For each countable ordinal $\xi \geq \omega$ consider the space ${}^\omega \xi$ with the product topology, where ξ has the discrete topology. If $f \in {}^\omega \xi$ let \leq_f be the relation on ω given by $m \leq_f n \Leftrightarrow f(m) \leq f(n)$. Write $\alpha \in \Delta_4^1(f)$ iff $\alpha \in \Delta_4^1(\leq_f)$. We shall prove that if

$$P = \{\alpha : (\exists \xi)(\omega \leq \xi < \aleph_1 \& \{f \in {}^\omega \xi : \alpha \in \Delta_4^1(f)\} \text{ is not meager})\},$$

then $P \in \Sigma_4^1$ and $P = S$. Note first that

$$\alpha \in P \Leftrightarrow (\exists \sigma)(\sigma \in WO^* \& \{f \in {}^\omega |\sigma| : \alpha \in \Delta_4^1(f)\} \text{ is not meager}),$$

where $\alpha \in WO^* \Leftrightarrow \alpha \in WO \& |\alpha| \geq \omega \& \leq_\alpha$ has field ω . If $\sigma \in WO^*$ let $h_\sigma : \omega \rightarrow |\sigma|$ be the bijection such that $m \leq_\sigma n \Leftrightarrow h_\sigma(m) \leq h_\sigma(n)$ and let $h_\sigma^* : {}^\omega \omega \rightarrow {}^\omega |\sigma|$ be given by $h_\sigma^*(\alpha) = h_\sigma \circ \alpha$. Then h_σ^* is a homeomorphism of ${}^\omega \omega$ with ${}^\omega |\sigma|$. Thus

$$\alpha \in P \Leftrightarrow (\exists \sigma)(\sigma \in WO^* \& h_\sigma^{*-1}[\{f \in {}^\omega |\sigma| : \alpha \in \Delta_4^1(f)\}])$$

is not meager)

$$\Leftrightarrow (\exists \sigma)(\sigma \in WO^* \& \{\beta : \alpha \in \Delta_4^1(h_\sigma \circ \beta)\})$$

is not meager)

$$\Leftrightarrow (\exists \sigma)(\sigma \in WO^* \& \{\beta : \alpha \in \Delta_4^1(\leq_{h_\sigma \circ \beta})\})$$

is not meager).

But for $\sigma \in WO^*$,

$$\begin{aligned} m \leq_{h_\sigma \beta} n &\Leftrightarrow h_\sigma(\beta(m)) \leq h_\sigma(\beta(n)) \\ &\Leftrightarrow \beta(m) \leq_\sigma \beta(n) \\ &\Leftrightarrow H(\beta, \sigma)(2^m \cdot 3^n) = 0 \end{aligned}$$

where $H: {}^\omega\omega \times {}^\omega\omega \rightarrow {}^\omega 2$ is recursive and $H(\beta, \sigma)(k) = 1$, if $\forall m \forall n (k \neq 2^m \cdot 3^n)$. So

$$\alpha \in P \Leftrightarrow (\exists \sigma) (\sigma \in WO^* \& \{\beta: \alpha \in \Delta_4^1(H(\beta, \sigma))\} \text{ is not meager})$$

which by Theorem 2.2.5(b) of [2] shows that P is Σ_4^1 .

We prove now that $S = P$. Clearly $S \subseteq P$. For the converse let $\alpha \in P$ and find $\xi \geq \omega$ such that $\{f \in {}^\omega\xi: \alpha \in \Delta_4^1(f)\}$ is not meager. Let $\sigma \in WO$ be such that $|\sigma| = \xi$. We shall show that $\alpha \in \Delta_4^1(\sigma)$. Without loss of generality we can assume that \leq_σ has field all of ω (i.e., $\sigma \in WO^*$). Then as before $\{\beta: \alpha \in \Delta_4^1(H(\sigma, \beta))\}$ is not meager, thus $\{\beta: \alpha \in \Delta_4^1(\sigma, \beta)\}$ is not meager. But then by Theorem 3.1.2.(b) of [2], $\alpha \in \Delta_4^1(\sigma)$.

3. Ordinals and reals Δ_n^1 in models of set theory. Call a countable ordinal ξ *almost Δ_n^1* if there is a countable transitive model M of ZFC such that $\xi \in M$ and $M \models \xi$ is Δ_n^1 . For $n = 1, 2$ it is trivial to see that ξ is almost Δ_n^1 iff ξ is Δ_n^1 . By using a simple Solovay type game one can prove easily the following. (We call a transitive model M of ZFC Σ_n^1 -correct if Σ_n^1 formulas are absolute for M .)

PROPOSITION 3.1. *Assume Projective Determinacy. For any $n \geq 1$ and any countable ordinal ξ there is a model M of $ZFC + \text{Determinacy}(\Delta_{2n}^1)$ which is Σ_{2n}^1 -correct, $\xi \in M$ and $M \models \xi$ is Δ_{2n+1}^1 .*

Call now a real α *almost Δ_n^1* if there is a transitive countable model M of ZFC such that $\alpha \in M$ and $M \models \alpha$ is Δ_n^1 . Abbreviate by D_n the statement $\forall \alpha (\alpha \text{ is almost } \Delta_n^1)$. As with ordinals, α is almost Δ_n^1 iff α is Δ_n^1 , for $n = 1, 2$, so D_1, D_2 fail. Unlike the case of the ordinals D_3 also fails.

THEOREM 3.2. *Assume Projective Determinacy and let $D_n \Leftrightarrow \forall \alpha (\alpha \text{ is almost } \Delta_n^1)$. Then D_n holds iff $n \geq 4$.*

Proof. Clearly $\{\alpha: \alpha \text{ is not almost } \Delta_n^1\}$ is a Π_2^1 set, so if not empty it has a Δ_4^1 solution, a contradiction if $n \geq 4$. To complete the proof we show that D_3 fails. Let α_0 be a real which codes a countable transitive

model N of $ZFC + \text{Determinacy } (\Delta_2^1)$ which is Σ_2^1 -correct. We show that α_0 is not almost Δ_3^1 . If not, let M be a countable transitive model of ZFC such that $\alpha_0 \in M$ and $M \models \alpha_0$ is Δ_3^1 . Then $N \in M$ and Σ_2^1 formulas are absolute from M to N , so, since $N \models \text{Determinacy } (\Delta_2^1)$, clearly $M \models \text{Determinacy } (\Delta_2^1)$ (in general we cannot conclude that $M \models \text{Determinacy } (\Delta_3^1)$). By the arguments in [1] or [5] the class of Π_3^1 relations on ω has the prewellordering property, *in* M . But then by Moschovakis [7], $M \models$ "There is a Δ_2^1 game in which I has a winning strategy and α_0 is recursive in *every strategy* of this game". Since $N \models \text{Determinacy } (\Delta_2^1)$ at least one of these strategies is in N , a contradiction.

Theorem 3.2 exposes a weak phenomenon which for the first time happens in the fourth level of the analytical hierarchy. Hopefully the discovery of more and especially stronger such phenomena will result in the understanding of the structural differences between the third and the fifth level of the analytical hierarchy. For that purpose it seems that what is needed is a deeper understanding of the role of uncountable ordinals and models of set theory in the analytical hierarchy.

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