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BONDED QUADRATIC DIVISION ALGEBRAS

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Osborn has shown that any quadratic algebra over a field of characteristic not two can be decomposed into a copy of the field and a skew-commutative algebra with a bilinear form. For any nonassociative algebra G over a field of characteristic not two, Albert and Oehmke have defined an algebra over the same vector space, which is bonded to G by a linear transformation T. In this paper this process is specialized to the class $\mathscr{A}$ of finite dimensional quadratic algebras A over fields of characteristic not two, which define a symmetric, nondegenerate bilinear form, to obtain quadratic algebras $B(A, T)$ bonded to A. In the main results T will be defined as a linear transformation on the skew-commutative algebra V defined by Osborn's decomposition of A. An algebra in $\mathscr{A}$ is called a division algebra if $A \neq 0$ and the equations $ax = b$ and $ya = b$, where $a \neq 0$ and b are elements in A, have unique solutions for x and y in A. Consequently, a finite dimensional algebra $A \neq 0$ is a division algebra if and only if A has no divisors of zero. A basis for V is said to be orthogonal, if it is orthogonal with respect to the above mentioned bilinear form. An algebra in $\mathscr{A}$ is weakly flexible if the ith component of the skew-commutative product of the ith and jth members of each orthogonal basis of V is 0. If $\mathscr{D}(\mathscr{A})$ denotes the class of division algebras in $\mathscr{A}$ and I denotes the identity transformation on V, then the main results are: (1) $A \in \mathscr{D}(\mathscr{A})$, T nonsingular and $B(A, T)$ flexible imply $B(A, T) \in \mathscr{D}(\mathscr{A})$, (2) if $A \in \mathscr{D}(\mathscr{A})$ and A is weakly flexible, then $B(A, T)$ is weakly flexible if and only if $T = \delta I$ for δ a scalar, and (3) if A is a Cayley-Dickson algebra in $\mathscr{D}(\mathscr{A})$, then $B(A, T)$ is a Cayley-Dickson algebra in $\mathscr{D}(\mathscr{A})$ if and only if $T = \pm I$. Finally, a class of nonflexible quadratic division algebras bonded to Cayley-Dickson division algebras will be exhibited.

1. Introduction. A finite dimensional algebra A with identity element 1 over a field F of characteristic not 2 is called a quadratic algebra in case 1, a, and a^2 are linearly dependent over F for all $a \in A$. Following the conventions used by Osborn [6] we shall identify the field F with the subalgebra $F1$ and refer to an element in $F1$ as a scalar. Furthermore, if an element $x \in A$ squares to a scalar but x is not a scalar, x is called a vector. If V is the set of all vectors in A, then A is a vector space direct sum of F and V. For x and $y \in A$, let (x, y) denote the scalar component of xy. Clearly (x, y) is a bilinear form from $A \times A$

to F. If x and $y \in V$, we define "$\times$" by $x \times y = xy - (x, y)$, V is closed under this product and Osborn [**6**, p. 203] shows it is skew-commutative. If $\alpha + x$ and $\beta + y \in A = F + V$, where α and $\beta \in F$ and x and $y \in V$, then

$$(\alpha + x)(\beta + y) = [\alpha\beta + (x, y)] + [\alpha y + \beta x + x \times y] \in F + V.$$

This decomposition of A into a copy of the field and a skew-commutative algebra with bilinear form makes it possible to restate questions about quadratic algebras in terms of questions about bilinear forms and skew-commutative algebras. For example, it is easy to show that A satisfies the flexible law if and only if the bilinear form (x, y) is symmetric and $(x, x \times y) = 0$ for all x and y in V, and that A is alternative if and only if A is flexible and $(y, x)x - (x, x)y + (y \times x) \times x = 0$ for all x and y in V.

Let $\mathscr{A}$ denote the class of algebras satisfying: A is a finite dimensional quadratic algebra over a field F of characteristic not two and A defines a symmetric, nondegenerate bilinear form (x, y). We call an algebra in $\mathscr{A}$ a division algebra if $A \neq 0$ and the equations $ax = b$ and $ya = b$, where $a \neq 0$ and b are elements in A, have unique solutions for x and y in A. Consequently, a finite dimensional algebra $A \neq 0$ is a division algebra if and only if A has no divisors of zero. Let $\mathscr{D}(\mathscr{A})$ denote the class of division algebras in $\mathscr{A}$. In the case that (x, y) is defined by a division algebra it will be nondegenerate, since otherwise there exists an element $\alpha + x \in A$ such that $(y, \alpha + x) = 0$ for all $y \in A$. But then $0 = (x, \alpha + x) = (x, \alpha) + (x, x) = x^2$, which contradicts the division property of A.

The assumptions of finite dimensionality of A and symmetry of (x, y) are sufficient to prove V has a basis $u_1, u_2, \cdots, u_n$ of mutually orthogonal vectors with respect to (x, y). Henceforth, when we speak of an orthogonal basis for V, we shall always mean orthogonal with respect to the bilinear form (x, y). Moreover, we will let $u_i^2 = \alpha_i \in F$ for $i = 1, \cdots, n$; and for $i \neq j$, let $u_i u_j = \sum_{k=1}^n \xi_{ijk} u_k$, so that the ξ_{ijk}'s are the multiplication constants of an orthogonal basis of V: Note that

$$\sum_{k=1}^n \xi_{ijk} u_k = u_i u_j = u_i \times u_j = -u_j \times u_i = -u_j u_i = -\sum_{k=1}^n \xi_{jik} u_k$$

for all $i, j, k = 1, \cdots, n$ and $i \neq j$. So

$$\xi_{ijk} = -\xi_{jik} \quad \text{for all nonzero } i, j, \text{ and } k. \tag{1.1}$$

If (x, y) is nondegenerate, then $\alpha_i \neq 0$ for $i = 1, \cdots, n$, since $\alpha_i = 0$ implies $0 = (u_i, u_i)$ which would imply $(u_i, y) = 0$ for all $y \in A$.

For $A \in \mathscr{A}$ let U be the subspace of A consisting of all finite linear combinations of vectors of the form $xy - yx$ for x and $y \in A$. Let T be a linear mapping from the subspace U into A and let $B(A, T)$ be an algebra with the same vector space as A and multiplication defined by

$$x \cdot y = \tfrac{1}{2}(xy + yx) + \tfrac{1}{2}(xy - yx)^T, \tag{1.2}$$

where xy denotes multiplication in A. T will be called a bonding mapping (**[2]** and **[5]**) of A and $B(A, T)$ will be said to be bonded to A. Using (1.2) it is seen that powers in $B(A, T)$ agree with those in A and that the identity of A is also the identity in $B(A, T)$. Thus $B(A, T)$ is also a quadratic algebra and we will let $(x, y)_T$ denote the bilinear form defined as the scalar component of $x \cdot y$ in $B(A, T)$ and let $x \times_T y = x \cdot y - (x, y)_T$, for all x and $y \in V$. V is closed under this skew-commutative product. Since (x, y) is assumed to be symmetric and $x \times y$ is skew-commutative, we have for all x and $y \in V$:

$$\begin{aligned} \tfrac{1}{2}(xy + yx) &= (x, y) \quad \text{and} \\ \tfrac{1}{2}(xy - yx) &= x \times y. \end{aligned} \tag{1.3}$$

So for all x and $y \in V$:

$$x \cdot y = (x, y)_T + x \times_T y = (x, y) + (x \times y)^T. \tag{1.4}$$

Clearly, for any basis $u_1, \cdots, u_n$ of V, the set of vectors $\{u_i \times u_j \mid i, j = 1, \cdots, n\}$ spans the space $U \subseteq V$. Since most of our knowledge is obtained under the assumption that T is a mapping into V, we will henceforth make the restriction

$$(u_i \times u_j)^T = \sum_{k=1}^{n} \beta_{ijk} u_k. \tag{1.5}$$

The β_{ijk}'s for $i, j, k, = 1, \cdots, n$ are then the corresponding multiplication constants for V in $B(A, T)$ and $(x, y)_T = (x, y)$.

2. LEMMA 2.1. *Let $A \in \mathscr{A}$ and let $u_1, \cdots, u_n$ be any orthogonal basis of V. Then A is flexible if and only if*

(a) $\xi_{iji} = 0$ *for all* $i, j = 1, \cdots, n$ and

(b) $\xi_{ijk}\alpha_k = \xi_{kij}\alpha_j = \xi_{jki}\alpha_i$ *for all i, j, k distinct in* $\{1, \cdots, n\}$.

Proof. By assumption (x, y) is symmetric, so it suffices to show that the condition $0 = (x, x \times y)$ for all x and y in V is equivalent to conditions (a) and (b). The condition $0 = (x, x \times y)$ is equivalent to the linearization $0=(x, z\times y)-(z, x\times y)$, and by the linearity of this relation it is equivalent to the set of equations

$$\begin{aligned} 0 &= (u_i, u_j \times u_k) + (u_j, u_i \times u_k) \\ &= (u_i, \xi_{jki}u_i) + (u_j, \xi_{ikj}u_j) \\ &= \xi_{jki}\alpha_i + \xi_{ikj}\alpha_j, \end{aligned}$$

for all $i, j, k \in \{1, \cdots, n\}$. The latter conditions are condition (a) of the theorem when $k = i$ or j, and condition (b) when i, j, and k are distinct.

We shall call $A \in \mathscr{A}$ *weakly flexible* if property (a) in Lemma 2.1 is satisfied for each orthogonal basis of V. Osborn [**6**, pp. 204–206] calls a skew-commutative algebra V *division-like* if there do not exist linearly independent u and $v \in V$ such that $u \times v = 0$ or $u \times v = u$ and he shows that $A = F + V$ is a division algebra if and only if V is division-like and a certain condition is satisfied by its bilinear form.

LEMMA 2.2. *Let $A \in \mathscr{D}(\mathscr{A})$. A is weakly flexible if and only if for x and $y \in V$ such that $(x, y) = 0$, there exists $z \in V$ such that $x = y \times z$.*

Proof. Suppose first that A is weakly flexible. Since $(x, y) = 0$, there exists an orthogonal basis $u_1 = x$, $u_2 = y$, $u_3, \cdots, u_n$ for V. Since $A \in \mathscr{D}(\mathscr{A})$, there exists $\alpha + z \in A$ such that

$$u_1 = u_2(\alpha + z) = \alpha u_2 + (u_2, z) + u_2 \times z = \alpha u_2 + u_2 \times z.$$

Let $z = \sum_{j=1}^{n} \gamma_j u_j$. Then

$$u_2 \times z = \sum_{j=1}^{n} \gamma_j (u_2 \times u_j) = \sum_{k=1}^{n} \sum_{j=1}^{n} \gamma_j \xi_{2jk} u_k.$$

Since A is weakly flexible, $\xi_{2j2} = 0$ for all $j = 1, \cdots, n$, so the coefficient of u_2 in $u_2 \times z$ is 0, which then implies $\alpha = 0$. Thus $x = y \times z$.

Conversely, let $u_1, \cdots, u_n$ be an orthogonal basis of V. Fix i and let $z^L = u_i \times z$ for $z \in V$. The assumption implies u_k is in the image of L for all $k \neq i$. V is division-like, so $u_i \neq u_i \times z$ for any $z \in V$, which implies the set of vectors $\{u_k \mid k \neq i\}$ spans the image of L. Hence $u_j^L = u_i \times u_j = \sum_{k \neq i} \xi_{ijk} u_k$, which implies $\xi_{iji} = 0$ for all $j = 1, \cdots, n$. The arbitrariness of i gives the desired conclusion.

We note that if $A \in \mathscr{A}$ is weakly flexible and $x, y \in V$ are such that $(x, y) \neq 0$, then $x = y \times z$ for $z \in V$ is impossible. There exists an orthogonal basis $y = u_1, u_2, \cdots, u_n$ of V and $x = \gamma u_1 + w$ for w in the span of $\{u_2, \cdots, u_n\}$ and $\gamma \neq 0$. Since A is weakly flexible, for any $z \in V$,

$y \times z$ is in the span of $\{u_2, \cdots, u_n\}$. Thus $x = \gamma u_1 + w = y \times z$ is not possible.

THEOREM 2.1. *Let $A \in \mathcal{D}(\mathcal{A})$, T nonsingular on U, and $B(A, T)$ flexible. Then $B(A, T) \in \mathcal{D}(\mathcal{A})$.*

Proof. Since $(x, y)_T = (x, y)$, $B(A, T)$ will be a division algebra, if V is division-like with respect to "$\times_T$". Suppose there exist linearly independent x and y in V such that $x \times_T y = x$. The flexibility of $B(A, T)$ implies $(x, x \times_T y)_T = 0$. Now

$$x^2 = x \cdot x = x \cdot (x \times_T y) = (x, x \times_T y)_T + x \times_T (x \times_T y) = 0 + x \times_T x = 0,$$

which contradicts the assumption that $A \in \mathcal{D}(\mathcal{A})$. Suppose there exist linearly independent x and y in V such that $x \times_T y = 0$. Then by (1.4), $0 = x \times_T y = (x \times y)^T$. But T is nonsingular, so $x \times y = 0$ which also contradicts $A \in \mathcal{D}(\mathcal{A})$.

If $1, u_1, u_2, \cdots, u_n$ is an orthogonal basis of $A \in \mathcal{D}(\mathcal{A})$, then $u_1 \times x \neq 0$ for x in the span of $\{u_2, \cdots, u_n\}$. Thus the $n-1$ vectors $u_1 \times u_2, u_1 \times u_3, \cdots, u_1 \times u_n$ are linearly independent. Moreover, since V is division-like, we cannot have

$$u_1 = \sum_{i=2}^{n} \beta_i (u_1 \times u_i) = u_1 \times \sum_{i=2}^{n} \beta_i u_i,$$

so the n vectors $u_1, u_1 \times u_2, u_1 \times u_3, \cdots, u_1 \times u_n$ are linearly independent. Let v be any vector such that $(u_1, v) = 0$. If A were weakly flexible, then by Lemma 2.2 there exists $z \in V$ such that $u_1 = v \times z$ which puts $u_1 \in U$. Thus U is a n-dimensional space contained in V, if $n > 1$. Hence it is plausible to assume T is a linear transformation from V into V.

COROLLARY 2.1. *Let A be flexible and in $\mathcal{A}$ but not in $\mathcal{D}(\mathcal{A})$. Then $B(A, T)$ is not in $\mathcal{D}(\mathcal{A})$ for any nonsingular T: $V \to V$.*

Proof. Since T is nonsingular on V, T^{-1}: $V \to V$ exists and it is easily checked that $A = B(B(A, T), T^{-1})$. So by Theorem 2.1, if $B(A, T)$ were in $\mathcal{D}(\mathcal{A})$, then A would have to also be in $\mathcal{D}(\mathcal{A})$.

THEOREM 2.2. *Let $A \in \mathcal{A}$ and suppose that for all $x \in U$, there exist y and $z \in V$ such that $x = y \times z$. If T is singular on U, then $B(A, T)$ is not a division algebra.*

Proof. T singular implies there exists $x \neq 0$ in U such that $x^T = 0$. Choose y and $z \in V$ such that $x = y \times z$. Then $0 = x^T =$

$(y \times z)^T = y \times_T z$, which implies $B(A, T)$ is not a division algebra since V is not division-like with respect to "$\times_T$".

By Lemma 2.2 the condition on V in Theorem 2.2 holds in particular if $A \in \mathcal{D}(\mathcal{A})$ is weakly flexible and $n > 1$. If $n = 1$, then $U = 0$. Thus for $A \in \mathcal{D}(\mathcal{A})$ weakly flexible and $B(A, T)$ flexible we have, by Theorems 2.1 and 2.2, $B(A, T) \in \mathcal{D}(\mathcal{A})$ if and only if T is nonsingular on V.

If we assume $A \in \mathcal{D}(\mathcal{A})$ is flexible and that T is a scalar δ times the identity transformation I on V, then for any orthogonal basis $u_1, \cdots, u_n$ of V we have

$$\sum_{k=1}^{n} \beta_{ijk} u_k = u_i \times_T u_j = (u_i \times u_j)^T = \sum_{k=1}^{n} \xi_{ijk} u_k^T = \sum_{k=1}^{n} \xi_{ijk} \delta u_k.$$

So $\beta_{ijk} = \xi_{ijk}\delta$ for all $i, j, k = 1, \cdots, n$. Since A is flexible, the β_{ijk} clearly satisfy the conditions in Lemma 2.1 which make $B(A, T)$ flexible and then by Theorem 2.1 $B(A, T) \in \mathcal{D}(\mathcal{A})$.

THEOREM 2.3. *Let A be weakly flexible in $\mathcal{D}(\mathcal{A})$ and let T: $V \to V$ be such that $B(A, T)$ is weakly flexible. Then T is a scalar multiple of I.*

Proof. Let $u_1, \cdots, u_n$ be an orthogonal basis of V. Pick u_r and u_s such that $r \neq s$. A is weakly flexible, so by Lemma 2.2 there exists $z \in V$ such that $u_s = u_r \times z$. Suppose $u_s^T = \sum_{l=1}^{n} {}_s\delta_l u_l$ and $z = \sum_{j=1}^{n} \gamma_j u_j$. Then

$$\sum_{l=1}^{n} {}_s\delta_l u_l = (u_r \times z)^T = \sum_{j=1}^{n} \gamma_j (u_r \times u_j)^T = \sum_{l=1}^{n} \sum_{j=1}^{n} \gamma_j \beta_{rjl} u_l.$$

So ${}_s\delta_r = \sum_{j=1}^{n} \gamma_j \beta_{rjr} = 0$, since $\beta_{rjr} = 0$ for all r and j. Thus $u_s^T = {}_s\delta_s u_s = \delta_s u_s$ for each $s = 1, \cdots, n$. The extra subscript is now dropped for the sake of simplicity. To show T is a scalar multiple of the identity let u_1 be any nonzero element of V. Then u_1 may be embedded in a basis $u_1, \cdots, u_n$ of V and we have $u_1^T = \delta_1 u_1$ for some $\delta_1 \in F$. Then also for any $v \in V$, $v^T = \delta_2 v$ for some $\delta_2 \in F$, and $\delta_1 u_1 + \delta_2 v = u_1^T + v^T = (u_1 + v)^T = \delta_3(u_1 + v) = \delta_3 u_1 + \delta_3 v$ for some $\delta_3 \in F$. Hence $\delta_1 = \delta_2 = \delta_3$ and $T = \delta_1 I$.

Since the Cayley-Dickson algebras are alternative, they are flexible. So for A a Cayley-Dickson algebra in $\mathcal{D}(\mathcal{A})$, $B(A, T)$ is flexible if and only if T is a scalar times I, the identity transformation on V.

COROLLARY 2.2. *Let A be a Cayley-Dickson algebra in $\mathcal{D}(\mathcal{A})$. $B(A, T)$ is a Cayley-Dickson algebra in $\mathcal{D}(\mathcal{A})$ if and only if $T = \pm I$.*

Proof. It is easily checked that a quadratic algebra is alternative if and only if it is flexible and $(y, x)x - (x, x)y + (y \times x) \times x = 0$ for all vectors x and y in the algebra. All that remains to be shown is that in $B(A, T)$, $(y, x)_T x - (x, x)_T y + (y \times_T x) \times_T x = 0$ if and only if $T = \pm I$. By (1.4) and Theorem 2.3

$$\begin{aligned}(y, x)_T x - (x, x)_T y + (y \times_T x) \times_T x &= (y, x)x - (x, x)y + [(y \times x)^T \times x]^T \\ &= (y, x)x - (x, x)y + \delta^2 (y \times x) \times x\end{aligned}$$

for some scalar δ. Since A is alternative, this expression is 0 if and only if $\delta^2 = 1$.

3. In this section the bonding mapping process is applied to a class of Cayley-Dickson division algebras over formally real fields to obtain nonflexible quadratic division algebras of dimension 8. We use the definition, as given by Kleinfeld **[4]**, of a Cayley-Dickson algebra in terms of its multiplication table with respect to a basis $1, u_1, \cdots, u_7$ and parameters α, β, and γ. Exact conditions on α, β, γ, and the field F which make the algebra a division algebra are given by Schafer **[7]**. We consider only the Cayley-Dickson division algebras over formally real fields with $\alpha = \beta = \gamma = -1$. (The Cayley numbers are in this class.) The multiplication table for the nonidentity basis elements in such an algebra $A = F + V$ is given in Table I. It is clear by Table I that for such a

TABLE I

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	-1	u_3	$-u_2$	u_5	$-u_4$	$-u_7$	u_6
u_2	$-u_3$	-1	u_1	u_6	u_7	$-u_4$	$-u_5$
u_3	u_2	$-u_1$	-1	u_7	$-u_6$	u_5	$-u_4$
u_4	$-u_5$	$-u_6$	$-u_7$	-1	u_1	u_2	u_3
u_5	u_4	$-u_7$	u_6	$-u_1$	-1	$-u_3$	u_2
u_6	u_7	u_4	$-u_5$	$-u_2$	u_3	-1	$-u_1$
u_7	$-u_6$	u_5	u_4	$-u_3$	$-u_2$	u_1	-1

Cayley-Dickson algebra $u_1, \cdots, u_7$ is an orthogonal basis for V, and that each u_i for $i = 1, \cdots, 7$ is equal to $u_j \times u_k$ for some $j, k \in \{1, \cdots, 7\}$, so that the subspace U as defined in §1 is equal to V. Moreover, $\alpha_i = (u_i, u_i) = -1$ for $i = 1, \cdots, 7$. A is the special case $\tau = 0$ of the class of division algebras we are about to define. Let T be a bonding mapping from V to V with matrix representation

$$\begin{bmatrix} 1 & \tau & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & & 0 \\ \vdots & & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & & 0 & 1 \end{bmatrix}, \quad \text{where} \quad \tau \neq 0 \tag{3.1}$$

is in F, with respect to the basis $u_1, \cdots, u_7$. By (1.4) the multiplication in $B(A, T)$ of two nonidentity basis elements is given by $u_i \cdot u_j = (u_i, u_j) + (u_i \times y_j)^T$, where (x, y) is the bilinear form determined by A and "$\times$" is the skew-commutative multiplication in V determined by A. So

$$\begin{aligned} u_i \cdot u_i &= (u_i, u_i) + (u_i \times u_i)^T \\ &= (u_i, u_i) = -1 \quad \text{for} \quad i = 1, \cdots, 7 \\ u_i \cdot u_j &= (u_i, u_j) + (u_i \times u_j)^T \\ &= (u_i \times u_j)^T \quad \text{for} \quad i \neq j; \quad i, j = 1, \cdots, 7. \end{aligned} \tag{3.2}$$

Using (3.2) one obtains the multiplication table for $B(A, T)$ given in Table II.

TABLE II

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	-1	u_3	$-u_2$	u_5	$-u_4$	$-u_7$	u_6
u_2	$-u_3$	-1	$u_1+\tau u_2$	u_6	u_7	$-u_4$	$-u_5$
u_3	u_2	$-u_1-\tau u_2$	-1	u_7	$-u_6$	u_5	$-u_4$
u_4	$-u_5$	$-u_6$	$-u_7$	-1	$u_1+\tau u_2$	u_2	u_3
u_5	u_4	$-u_7$	u_6	$-u_1-\tau u_2$	-1	$-u_3$	u_2
u_6	u_7	u_4	$-u_5$	$-u_2$	u_3	-1	$-u_1-\tau u_2$
u_7	$-u_6$	u_5	u_4	$-u_3$	$-u_2$	$u_1+\tau u_2$	-1

We shall prove that $B(A, T)$ is a division algebra for any T as in (3.1) such that $|\tau| < 2$, and we shall give examples of zero divisors when $\tau = 2$.

We take T in (3.1) such that $|\tau| < 2$ and let

$$x = au_0 + bu_1 + cu_2 + du_3 + fu_4 + gu_5 + hu_6 + ku_7$$

and

$$y = a'u_0 + b'u_1 + c'u_2 + d'u_3 + f'u_4 + g'u_5 + h'u_6 + k'u_7,$$

where $a, \cdots, k, a', \cdots, k' \in F$ be two aribtrary elements of $B(A, T)$. Using Table II we can express the relation $x \cdot y = 0$ in terms of the basis elements $1, u_1, \cdots, u_7$ for $B(A, T)$ in the following way:

$$\begin{aligned} 0 = &(aa' - bb' - cc' - dd' - ff' - gg' - hh' - kk')1 \\ &+ (ab' + ba' + cd' - dc' + fg' - gf' + kh' - hk')u_1 \\ &+ (ac' + ca' + db' - bd' + \tau cd' - \tau dc' + \tau fg' \\ &- \tau gf' + fh' - hf' + gk' - kg' + \tau kh' - \tau hk')u_2 \\ &+ (ad' + da' + bc' - cb' + fk' - kf' + hg' - gh')u_3 \\ &+ (af' + fa' + gb' - bg' + hc' - ch' + kd' - dk')u_4 \\ &+ (ag' + ga' + bf' - fb' + kc' - ck' + dh' - hd')u_5 \\ &+ (ah' + ha' + bk' - kb' + cf' - fc' + gd' - dg')u_6 \\ &+ (ak' + ka' + hb' - bh' + cg' - gc' + df' - fd')u_7. \end{aligned}$$

This gives eight homogeneous bilinear equations in the elements $a, \cdots, k, a', \cdots, k'$. The equation $x \cdot y = 0$ has a solution in $B(A, T)$ if and only if these eight equations can be made to equal zero simultaneously. We way think of the primed letters $a', \cdots, k'$ as variables and consider the coefficient matrix M_T of the set of eight equations. We have

$$M_T = \begin{bmatrix} a & -b & -c & -d & -f & -g & -h & -k \\ b & a & -d & c & -g & f & k & -h \\ c & d & a - \tau d & -b + \tau c & -h - \tau g & -k + \tau f & f + \tau k & g - \tau h \\ d & -c & b & a & -k & h & -g & f \\ f & g & h & k & a & -b & -c & -d \\ g & -f & k & -h & b & a & d & -c \\ h & -k & -f & g & c & -d & a & b \\ k & h & -g & -f & d & c & -b & a \end{bmatrix}.$$

It suffices to show this matrix is nonsingular for all choices of $a, \cdots, k$ not all zero. To show M_T is nonsingular for $|\tau| < 2$ we utilize a technique found in **[6]**. Let

$$M'_T = \begin{bmatrix} a & b & c & d & f & g & h & k \\ -b & a & d & -c & g & -f & -k & h \\ -c & -d & a & b & h & k & -f & -g \\ -d & c & -b & a & k & -h & g & -f \\ -f & -g & -h & -k & a & b & c & d \\ -g & f & -k & h & -b & a & -d & c \\ -h & k & f & -g & -c & d & a & -b \\ -k & -h & g & f & -d & -c & b & a \end{bmatrix}.$$

If we set $\Gamma = a^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2 + k^2$, then

$$M_T M'_T =$$

$$\begin{bmatrix} \Gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau(\Gamma - a^2 - b^2) & \Gamma - \tau(ad + bc) & \tau(ac - bd) & -\tau(ag + bf) & \tau(af - bg) & \tau(ak - bh) & \tau(ah - bh) \\ 0 & 0 & 0 & \Gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma \end{bmatrix}$$

Any choice of τ which makes $M_T M'_T$ nonsingular will clearly make M_T nonsingular. We have

$$\det M_T M'_T = \Gamma^7[\Gamma - \tau(ad + bc)]. \tag{3.3}$$

Since F is a formally real field, $\Gamma > 0$ in F unless $a = b = c = d = f = g = h = k = 0$. We expand the other factor of $\det M_T M'_T$ to obtain

$$\begin{aligned} \Gamma - \tau(ad + bc) = a^2 - \tau ad + d^2 + b^2 - \tau bc + c^2 \\ + f^2 + g^2 + h^2 + k^2. \end{aligned} \tag{3.4}$$

We want to show the expression in (3.4) is nonzero for any $\tau \in F$ such that $|\tau| < 2$. Consider the quadratic form $q = \lambda_1^2 + \tau\lambda_1\lambda_2 + \lambda_2^2$ and the nonsingular linear transformation given by $\lambda_1 = \mu_1 - \mu_2$ and $\lambda_2 = \mu_1 + \mu_2$. This transformation applied to q gives a new quadratic form $p = (2 + \tau)\mu_1^2 + (2 - \tau)\mu_2^2$. Since the transformation connecting them is nonsingular, q and p are congruent. Therefore, they have the same range of values when λ_1, λ_2 and μ_1, μ_2 assume all values in the formally real field F. But $|\tau| < 2$ implies $2+\tau > 0$ and $2-\tau > 0$. So $p > 0$ which implies $q > 0$ for $|\tau| < 2$. Applying this conclusion to (3.4) shows $\Gamma - \tau(ad+bc) > 0$ for $|\tau| < 2$. Thus $M_T M_T'$ and M_T are nonsingular and $B(A, T)$ has no nontrivial zero divisors.

Let T_0 be the nonsingular linear transformation obtained by setting $\tau = 2$ in (3.1). $B(A, T_0)$ will have divisors of zero. The multiplication table for $B(A, T_0)$ is Table II with $\tau = 2$. Let M_{T_0} be the matrix obtained from M_T by setting $\tau = 2$. It is easily seen that det $M_{T_0} = 0$ for $a = d$, $b = c$, and $f = g = h = k = 0$, so that nontrivial solutions to $x \cdot y = 0$ do exist in $B(A, T_0)$. (E.g. $x = 1 + u_1 + u_2 + u_3$ and $y = 1 + u_1 + u_2 - u_3$ have product 0 in $B(A, T_0)$.)

Albert **[1]**, Bruck **[3]**, and Osborn **[6]** have constructed classes of quadratic division algebras. A full determination of quadratic division algebras obtainable by this bonding mapping process has not been made even when A is taken to be a Cayley-Dickson algebra. The class of division algebras obtained above with $\tau \neq 0$ does not contain any flexible algebras, since $u_2 \cdot u_3 = u_1 + \tau u_2$ with $\tau \neq 0$ violates condition (a) of Lemma 2.1. Moreover, for T as in (3.1) with $\tau \neq 0$ one obtains $u_1 \times_T u_4 = u_5$, $u_4 \times_T u_5 = u_1 + \tau u_2$, $u_1 \times_T u_2 = u_3$, $u_2 \times_T u_4 = u_6$, and $u_2 \times_T u_5 = u_7$, so that the skew-commutative algebra generated in V by u_1 and u_4 is V itself. This shows that no $B(A, T)$ obtained as above with $\tau \neq 0$ is a division algebra of dimension 8 in the class discovered by Osborn **[6]**, since in his class of examples every two independent elements in V generate a subalgebra in V of dimension 3.

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